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Weighted Sobolev spaces and ground state solutions for quasilinear elliptic problems with unbounded and decaying potentials

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Abstract

In this paper, we prove some continuous and compact embedding theorems for weighted Sobolev spaces, and consider both a general framework and spaces of radially symmetric functions. In particular, we obtain some *a priori* Strauss-type decay estimates. Based on these embedding results, we prove the existence of ground state solutions for a class of quasilinear elliptic problems with potentials unbounded, decaying and vanishing.

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Keywords: weighted Sobolev spaces; unbounded and decaying potentials; quasilinear elliptic problems

1 Introduction

In this paper, we consider the following quasilinear elliptic problems:

$$\begin{cases} -\triangle_p u + V(x)|u|^{p-2}u = K(x)|u|^{q-1}u, & x \in \mathbb{R}^N, \\ |u(x)| \to 0 \quad \text{as } |x| \to \infty, \end{cases} \tag{1.1}$$

where $N > 2$, $1 < p \le N$, $-\triangle_p u = -\operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $V(x)$ and $K(x)$ are nonnegative measurable functions, and may be unbounded, decaying and vanishing.

Recently, these type elliptic equations have been widely studied. As $p = 2$, if $V(x)$ and $K(x)$ satisfied

$$\sup_{x\in\mathbb{R}^N} K(x) < \infty, \qquad \inf_{x\in\mathbb{R}^N} V(x) > 0 \quad \text{and} \quad \lim_{|x|\to+\infty} V(x) = +\infty, \tag{1.2}$$

Rabinowitz [1] proved the existence of a ground state solution for problem (1.1). Further, when $V(x)$ has a positive lower bound and $K(x)$ is bounded, using critical point theory, del Pino and Felmer [2, 3] obtained that problem (1.1) might also not have a ground state solution. If $V(x)$ and $K(x)$ satisfied

$$\sup_{x\in\mathbb{R}^N} \left(1 + |x|\right)^{2-\alpha} V(x) > 0 \quad \text{and} \quad \sup_{x\in\mathbb{R}^N} \left(1 + |x|\right)^{\beta} K(x) < +\infty, \tag{1.3}$$

where $0 < \alpha < 1$, $\beta > (1-\alpha)(N - q(\frac{N}{2} - 1))$, Ambrosetti, Felli and Malchiodi [4], Ambrosetti, Malchiodi and Ruiz [5] obtained the ground and bound state solutions for problem (1.1).

In fact, condition (1.3) implies that $V(x)$ tends to zero at infinity. In particular, when the potentials $V(x)$ and $K(x)$ are neither bound away from zero nor bounded from above, Bonheure and Mercuri [7] proved the existence of the ground state solution for problem (1.1) and obtained the decay estimates by using the Moser iteration scheme. For the radially symmetric space $D^{1,2}_{rad}(\mathbb{R}^N) = \{u(x) = u(|x|), u \in L^{\frac{2N}{N-2}}(\mathbb{R}^N), \nabla u \in L^2(\mathbb{R}^N, \mathbb{R}^N)\}$, Strauss [8] obtained the famous Strauss inequality

$$\big|u(x)\big| \le C|x|^{-\frac{N-2}{2}} \|u\|_{D^{1,2}(\mathbb{R}^N)}, \tag{1.4}$$

for a.e. $x \in \mathbb{R}^N$ and $u \in D^{1,2}_{rad}(\mathbb{R}^N)$. Berestycki and Lions [9] proved the existence of a ground state solution for some scalar equation. In 2007, as the potentials $V(x)$ and $K(x)$ are radially symmetric, Su, Wang and Willem [10] obtained the existence of a ground state solution for problem (1.1) with $V(x)$ and $K(x)$ unbounded and decaying.

For $p \neq 2$, to the best of our knowledge, it seems to be little work done. do Ó and Medeiros [11] obtained the existence of a ground state solution for some p-Laplacian elliptic problems in $\mathbb{R}^N$. Zhang [12] considered a mountain pass characterization of the ground state solution for p-Laplacian elliptic problems with critical growth. When $V(x)$ and $K(x)$ are radially symmetric, Su, Wang and Willem [13] considered the following quasilinear elliptic problem:

$$\begin{cases} -\triangle_p u + V(|x|)|u|^{p-2}u = K(|x|)|u|^{q-1}u, & x \in \mathbb{R}^N, \\ |u(x)| \to 0 \quad \text{as } |x| \to \infty, \end{cases} \tag{1.5}$$

and proved some embedding results of a weighted Sobolev space for a radially symmetric function, and obtained the existence of ground and bound state solutions for problem (1.5).

In this paper, for the general potentials $V(x)$ and $K(x)$ allowing to be unbounded or vanish at infinity, we obtain some necessary and sufficient conditions about some continuous and compact embeddings for the weighted Sobolev space. Based on variational methods and some compact embedding results, we obtain the existence of ground and bounded state solutions for problem (1.1). On the other hand, for the radial potentials $V(|x|)$ and $K(|x|)$, in [13] various conditions have been considered for $V(|x|), K(|x|) \sim |x|^\alpha$ with $\alpha \in \mathbb{R}$. Our first purpose is to consider $V(|x|)$ and $K(|x|)$ whose behavior can be described by a more general class of functions. Furthermore, we obtain some *a priori* Strauss-type decay estimates and some continuous and compact embedding results for the radial symmetric weighted Sobolev space. The results then are used to obtain ground and bound state solutions for problem (1.5).

It is worth pointing out that we provide here a unified approach what conditions the potentials $V(x)$ and $K(x)$ should satisfy so that problem (1.1) and problem (1.5) have ground and bound state solutions, respectively. We extend the results in [13] to a large class of weighted Sobolev embeddings and obtain some new embedding theorems for the general potentials and radially symmetric potentials.

The paper is organized as follows. In Section 2, we collect some results. In Section 3, we obtain some embedding results for the general potentials. In Section 4, we focus on radially symmetric potentials and prove the continuous and compact embeddings. Section 5 is devoted to the existence of ground and bound state solutions for problem (1.1) and problem (1.5), respectively.

2 Preliminaries

In this section, let $\mathcal{C}_0^\infty(\mathbb{R}^N)$ denote the collection of smooth functions with compact support. Let $D^{1,p}(\mathbb{R}^N)$ be the completion of $\mathcal{C}_0^\infty(\mathbb{R}^N)$ under the norm

$$\|u\|_{D^{1,p}(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} |\nabla u|^p \,dx\right)^{\frac{1}{p}}. \tag{2.1}$$

We write $\mathcal{C}_{0,r}^\infty(\mathbb{R}^N) = \{u \in \mathcal{C}_0^\infty(\mathbb{R}^N) | u(x) = u(|x|)\}$ and $D_r^{1,p}(\mathbb{R}^N)$ is the corresponding subspace of a radial function for $D^{1,p}(\mathbb{R}^N)$.

Define, for $p \geq 1$ and $q \geq 1$,

$$L_V^p(\mathbb{R}^N) = \left\{u : \mathbb{R}^N \mapsto \mathbb{R} \;\middle|\; u \text{ is measurable, } \int_{\mathbb{R}^N} V(x)|u|^p \,dx < \infty\right\} \tag{2.2}$$

and

$$L_K^q(\mathbb{R}^N) = \left\{u : \mathbb{R}^N \mapsto \mathbb{R} \;\middle|\; u \text{ is measurable, } \int_{\mathbb{R}^N} K(x)|u|^q \,dx < \infty\right\}. \tag{2.3}$$

Then we have $W_V^{1,p}(\mathbb{R}^N) = D^{1,p}(\mathbb{R}^N) \cap L_V^p(\mathbb{R}^N)$, which is a Banach space under the uniformly convex norm

$$\|u\|_{W_V^{1,p}(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} \left(|\nabla u|^p + V(x)|u|^p\right) dx\right)^{\frac{1}{p}} = \left(\|u\|^p_{D^{1,p}(\mathbb{R}^N)} + \|u\|^p_{L_V^p(\mathbb{R}^N)}\right)^{\frac{1}{p}}, \tag{2.4}$$

where $\|u\|_{L_V^p(\mathbb{R}^N)} = (\int_{\mathbb{R}^N} V(x)|u|^p \,dx)^{\frac{1}{p}}$.

Now, we state some Hardy inequalities.

Lemma 2.1 [14] *If $N > 2$, $1 < p < N$, we have*

$$\int_{\mathbb{R}^N} |\nabla u|^p \,dx \geq \left(\frac{N-p}{p}\right)^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} \,dx \quad \textit{for every } u \in D^{1,p}(\mathbb{R}^N).$$

Lemma 2.2 [13] *If $N > 2$, $1 < p < N$, $p \leq q+1 < \infty$ and $q+1 = \frac{p(N+c)}{N-p}$ for some $c \in [-p, \infty)$, there exists $C > 0$ such that*

$$\left(\int_{\mathbb{R}^N} |\nabla u|^p \,dx\right) \geq C\left(\int_{\mathbb{R}^N} |x|^c |u|^{q+1} \,dx\right)^{\frac{p}{q+1}} \quad \textit{for every } u \in D_r^{1,p}(\mathbb{R}^N).$$

Lemma 2.3 [15] *If $N = p$, $\Omega \subset B_R(0)$ or $\Omega \subset B_R^c(0)$, then*

$$\int_\Omega |\nabla u|^N \,dx \geq \left(\frac{N-1}{N}\right)^N \int_\Omega \frac{|u|^N}{(|x| \log \frac{R}{|x|})^N} \,dx \quad \textit{for every } u \in D_0^{1,N}(\Omega),$$

where $B_R(0)$ is the ball in $\mathbb{R}^N$ centered at 0 with radius R, $B_R^c(0)$ denotes the complement of $B_R(0)$.

3 Embedding results for general potentials

In this section, we derive a tool giving the embedding results on a piece of the partition. We consider the possible relation between the behavior of $V(x)$ and $K(x)$.

Lemma 3.1 *Let $\Omega \subset \mathbb{R}^N$ be smooth possibly unbounded and*

$$\text{(H)} \quad 1 < p < N, \qquad p \le q+1 < p^*, \qquad p^* = \frac{Np}{N-p},$$

$V(x)$ and $K(x) : \mathbb{R}^N \mapsto \mathbb{R}$ be measure nonnegative functions. $V(x) > 0$ a.e. in $\mathbb{R}^N$.

(a) *If there exists $\alpha \in [0,1]$ such that*

$$\frac{Np}{(q+1)(N-(1-\alpha)p)} \ge 1 \quad \text{and}$$
$$\int_\Omega \bigl(K(x)\bigl(V(x)\bigr)^{-\frac{\alpha(q+1)}{p}}\bigr)^{\frac{Np}{Np-(q+1)(N-(1-\alpha)p)}}\,dx < \infty,$$

then the embedding

$$W_V^{1,p}(\Omega) \hookrightarrow L_K^{q+1}(\Omega)$$

is continuous;

(b) *If there exist $\alpha_1 \in [0,1]$ and $m \in (p,p^*)$ such that*

$$\frac{\alpha_1(q+1)}{p} + \frac{(1-\alpha_1)(q+1)}{m} \le 1 \quad \text{and}$$
$$K(x)\bigl(V(x)\bigr)^{-\frac{\alpha_1(q+1)}{p}} \in L^{\frac{pm}{pm-\alpha_1 m(q+1)-(1-\alpha_1)p(q+1)}}(\Omega)$$

and $\alpha_2 \in (0,1)$ such that

$$\frac{Np}{(q+1)(N-(1-\alpha_2)p)} \ge 1,$$

and $\forall \varepsilon > 0$, $\exists R_\varepsilon > 0$ such that

$$\int_{\Omega\setminus B_{R_\varepsilon}} \bigl[K(x)\bigl(V(x)\bigr)^{-\frac{\alpha_2(q+1)}{p}}\bigr]^{\frac{Np}{Np-(q+1)(N-(1-\alpha_2)p)}}\,dx < \varepsilon,$$

then the embedding

$$W_V^{1,p}(\Omega) \hookrightarrow L_K^{q+1}(\Omega)$$

is compact.

Proof (a) Since there exists $\alpha \in [0,1]$ such that $\frac{Np}{(q+1)(N-(1-\alpha)p)} \ge 1$, we have

$$\frac{Np}{Np-(q+1)(N-(1-\alpha)p)} \in [1,+\infty).$$

By Hölder's inequality and $\int_\Omega [K(x)(V(x))^{-\frac{\alpha(q+1)}{p}}]^{\frac{Np}{N-(q+1)(N-(1-\alpha)p)}}\,dx < +\infty$, we obtain

$$\begin{aligned}
&\int_\Omega K(x)|u|^{q+1}\,dx \\
&\quad= \int_\Omega K(x)\big(V(x)\big)^{-\frac{\alpha(q+1)}{p}}\big(V(x)\big)^{\frac{\alpha(q+1)}{p}}|u|^{\alpha(q+1)}|u|^{(1-\alpha)(q+1)}\,dx \\
&\quad\le \left\{\int_\Omega \big[K(x)\big(V(x)\big)^{-\frac{\alpha(q+1)}{p}}\big]^{\frac{Np}{N-(q+1)(N-(1-\alpha)p)}}\,dx\right\}^{\frac{N-(q+1)(N-(1-\alpha)p)}{Np}} \\
&\quad\quad\cdot\left(\int_\Omega V(x)|u|^p\,dx\right)^{\frac{\alpha(q+1)}{p}}\cdot\left(\int_\Omega |u|^{\frac{Np}{N-P}}\,dx\right)^{\frac{(N-p)(1-\alpha)(q+1)}{Np}} \\
&\quad\le C\|u\|^{\alpha(q+1)}_{L^p_V(\Omega)}\|u\|^{(1-\alpha)(q+1)}_{L^{p^*}(\Omega)},
\end{aligned} \tag{3.1}$$

where $\frac{(q+1)(N-(1-\alpha)p)}{Np} + \frac{Np-(q+1)(N-(1-\alpha)p)}{Np} = 1$. Since p^* is the critical Sobolev exponent, by the Sobolev embedding theorem, we have

$$\int_\Omega K(x)|u|^{q+1}\,dx \le C\|u\|^{(q+1)}_{W^{1,p}_V(\Omega)}.$$

Hence, we obtain that the embedding $W^{1,p}_V(\Omega) \hookrightarrow L^{q+1}_K(\Omega)$ is continuous.

(b) For any fixed $\varepsilon > 0$, let $B_{R_\varepsilon}(0)$ be the ball in Ω with R_ε. Since there exist $\alpha_1 \in (0,1)$ and $m \in (p,p^*)$ such that

$$\begin{aligned}
&\left(\frac{\alpha_1}{p}(q+1) + \frac{(1-\alpha_1)}{m}(q+1)\right)^{-1} \ge 1 \quad \text{and} \\
&K(x)\big(V(x)\big)^{-\frac{\alpha_1(q+1)}{p}} \in L^{\frac{pm}{pm-\alpha_1 m(q+1)-(1-\alpha_1)p(q+1)}}(\Omega),
\end{aligned}$$

arguing as in the proof of (3.1), by the compact embedding of $D^{1,p}_0(B_{R_\varepsilon})$ into $L^m(B_{R_\varepsilon})$, we have

$$\int_{B_{R_\varepsilon}} K(x)|u|^{q+1}\,dx \le C\|u\|^{\alpha(q+1)}_{L^p_V(B_{R_\varepsilon})}\|u\|^{(1-\alpha)(q+1)}_{L^m(B_{R_\varepsilon})} \le C\|u\|^{(q+1)}_{W^{1,p}_V(B_{R_\varepsilon})}.$$

Hence, we have

$$W^{1,p}_V(B_{R_\varepsilon}) \hookrightarrow L^{q+1}_K(B_{R_\varepsilon}) \quad \text{is compact.} \tag{3.2}$$

On the domain $\Omega\backslash B_{R_\varepsilon}$, since $\int_{\Omega\backslash B_{R_\varepsilon}} [K(x)(V(x))^{-\frac{\alpha_2(q+1)}{p}}]^{\frac{Np}{Np-(q+1)(N-(1-\alpha)p)}}\,dx < \varepsilon$, we have

$$\int_{\Omega\backslash B_{R_\varepsilon}} K(x)|u|^{q+1}\,dx \le C\|u\|^{\alpha_2(q+1)}_{L^p_V(\Omega)}\|u\|^{(1-\alpha_2)(q+1)}_{L^{p^*}(\Omega)}.$$

Assume $u_n \rightharpoonup 0$ (weakly) in $W^{1,p}_V(\Omega)$, then we have

$$\int_{\Omega\backslash B_{R_\varepsilon}} K(x)|u_n|^{q+1}\,dx \le c\varepsilon. \tag{3.3}$$

Combining (3.2) and (3.3), we have

$$\int_{\Omega} K(x)|u_n|^{q+1}\,dx \to 0 \quad \text{(strongly) as } n \to \infty.$$

Hence, we obtain that $W_V^{1,p}(\Omega) \hookrightarrow L_K^{q+1}(\Omega)$ is compact. □

Now, we state our main theorem in this section.

Consider a finite partition $\mathcal{M} = \sum_i \{\Omega_i\}$ of $\mathbb{R}^N$ and Ω_i is unbounded.

Theorem 3.2 *If condition* (H) *is satisfied for any $i \in \mathbb{N}$ and Ω_i, assume that*

(a) *there exist $\alpha_i \in [0,1]$ such that*

$$\frac{Np}{(q+1)(N-(1-\alpha_i)p)} \geq 1 \quad \textit{and} \quad \int_{\Omega_i} \bigl[K(x)\bigl(V(x)\bigr)^{-\frac{\alpha_i(q+1)}{p}}\bigr]^{\frac{Np}{Np-(q+1)(N-(1-\alpha_i)p)}}\,dx < +\infty,$$

then the embedding $W_V^{1,p}(\mathbb{R}^N) \hookrightarrow L_K^{q+1}(\mathbb{R}^N)$ is continuous;

(b) *there exist $\alpha_{1,i} \in [0,1]$ and $m_i \in (p,p^*)$ such that*

$$\frac{\alpha_{1,i}(q+1)}{p} + \frac{(1-\alpha_{1,i})(q+1)}{m_i} \leq 1 \quad \textit{and}$$

$$K(x)\bigl(V(x)\bigr)^{-\frac{\alpha_{1,i}}{p}(q+1)} \in L^{\frac{m_i p}{pm_i-\alpha_{1,i}m_i(q+1)-(1-\alpha_{1,i})p(q+1)}}(\Omega_i)$$

and $\alpha_{2,i} \in (0,1)$ such that

$$\frac{Np}{(q+1)(N-(1-\alpha_{2,i})p)} \geq 1$$

and

$$\forall \varepsilon > 0, \exists R_{i,\varepsilon} > 0, \quad \int_{\Omega_i \setminus B_{R_{i,\varepsilon}}} \bigl[K(x)\bigl(V(x)\bigr)^{-\frac{\alpha_{2,i}}{p}(q+1)}\bigr]^{\frac{Np}{Np-(q+1)(N-(1-\alpha_{2,i})p)}}\,dx < \varepsilon,$$

then the embedding $W_V^{1,p}(\mathbb{R}^N) \hookrightarrow L_K^{q+1}(\mathbb{R}^N)$ is compact.

Proof (a) Arguing as in the proof of (a) in Lemma 3.1, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} K(x)|u|^{q+1}\,dx &= \sum_i \int_{\Omega_i} K(x)|u|^{q+1}\,dx \\ &\leq C\sum_i \|u\|_{L_V^p(\Omega_i)}^{\alpha_i(q+1)} \|u\|_{L^{p^*}(\Omega_i)}^{(1-\alpha_i)(q+1)} \\ &\leq C\sum_i \|u\|_{W_V^{1,p}(\Omega_i)}^{(q+1)} \leq C\|u\|_{W_V^{1,p}(\mathbb{R}^N)}^{(q+1)}.\end{aligned}$$

Hence, we obtain that $W_V^{1,p}(\mathbb{R}^N) \hookrightarrow L_K^{q+1}(\mathbb{R}^N)$ is continuous.

(b) $\forall \varepsilon > 0$, $\exists R_{i,\varepsilon} > 0$ such that

$$\int_{\Omega_i \setminus B_{R_{i,\varepsilon}}} \bigl[K(x)\bigl(V(x)\bigr)^{-\frac{\alpha_{2,i}(q+1)}{p}}\bigr]^{\frac{Np}{Np-(q+1)(N-(1-\alpha_{2,i})p)}}\,dx < \varepsilon,$$

then we have

$$\sum_i \int_{\Omega_i \setminus B_{R_{i\varepsilon}}} \bigl[K(x)\bigl(V(x)\bigr)^{-\frac{\alpha_{2,i}(q+1)}{p}}\bigr]^{\frac{Np}{Np-(q+1)(N-(1-\alpha_{2,i})p)}}\,dx < \varepsilon. \tag{3.4}$$

Arguing as in the proof of (b) in Lemma 3.1, when $u_n \rightharpoonup 0$ (weakly) in $W_V^{1,p}(\mathbb{R}^N)$, we have

$$\sum_i \int_{\Omega_i \setminus B_{R_{i\varepsilon}}} K(x)|u|^{q+1}\,dx < \varepsilon \|u_n\|^{q+1}_{W_V^{1,p}(\mathbb{R}^N)}. \tag{3.5}$$

By (3.5) and the local compactness in $B_{R_{i,\varepsilon}}$, we obtain that

$$\int_{\mathbb{R}^N} K(x)|u|^{q+1}\,dx \to 0 \quad \text{as } n \to \infty.$$

Hence, we have $W_V^{1,p}(\mathbb{R}^N) \hookrightarrow L_K^{q+1}(\mathbb{R}^N)$ is compact. □

Remark 3.3 (1) Let $K(x) \in L^\infty(\mathbb{R}^N)$ and $q+1 \le p^*$, and $q+1 < p^*$, we obtain the standard local Sobolev embedding.

(2) Let $p = 2$, we obtain that if $K(x)[V(x)]^r \in L^\infty(\mathbb{R}^N)$ with $r = \frac{q+1}{2}(\frac{N}{2}-1) - \frac{N}{2}$, the embedding $W_V^{1,2}(\mathbb{R}^N) \hookrightarrow L_K^{q+1}(\mathbb{R}^N)$ is compact. This has already been obtained in [6].

4 Embedding theorem for a radially symmetric function space

Assume that $V(|x|)$ and $K(|x|)$ are radial weights. In [13], Su, Wang and Willem considered for potentials $V, K \sim r^\alpha$ with $\alpha \in \mathbb{R}$ and obtained some embedding theorems. In this section, we extend some results in [13] to a more general class of functions for $r \to +\infty, 0^+$. In particular, we also obtain some embedding theorems for the Sobolev space $W_{V,r}^{1,N}(\mathbb{R}^N)$. Theorem 4.5 and Theorem 4.6 are new embedding results.

Following [16, 17], we shall refer to this class as the Hardy-Dieudonne comparison class. Define

$$\mathcal{C}_1(+\infty) = \bigl\{1; x^\alpha (\alpha \neq 0); [\underbrace{\log\log\cdots\log x}_{k \text{ times products}}]^\alpha (\alpha \neq 0, k \in \mathbb{N}); e^{cx^\alpha} (c \neq 0; \alpha > 0)\bigr\}.$$

Then we take the set of all the finite products

$$\mathcal{C}_1'(+\infty) = \left\{\prod_{k=1}^n f_k : f_k \in \mathcal{C}_1(+\infty), n \in \mathbb{N}\right\}.$$

Since $\mathcal{C}_1'(+\infty)$ is not closed with respect to the operation $f \mapsto \exp f$, we consider

$$\mathcal{C}_1''(+\infty) = \bigl\{\exp cf(x) : f \in \mathcal{C}_1'(+\infty), f(+\infty) = +\infty, c \neq 0\bigr\}.$$

Then we have $\mathcal{C}_2(+\infty) = \mathcal{C}_1' \cup \mathcal{C}_1''$ and $\mathcal{C}_2'(+\infty) = \{\prod_{k=1}^n f_k : f_k \in \mathcal{C}_2(+\infty), n \in \mathbb{N}\}$. The process can be iterated, we have the following.

Definition 4.1 The set $\mathcal{C}(+\infty) = \bigcup_{n\in\mathbb{N}} \mathcal{C}_n'$ is called Hardy-Dieudonne class of functions at $+\infty$. $\mathcal{C}(x_0^+) = \{f(x) = g((x-x_0)^{-1}), g \in \mathcal{C}(+\infty)\}$ is called Hardy-Dieudonne class of functions at x_0^+.

Now, let $V(|x|)$, $K(|x|)$ be continuous nonnegative functions in $(0,\infty)$, and

(V) $\liminf_{r\to\infty}\frac{V(r)}{V_\infty(r)}>0$ and $\liminf_{r\to 0^+}\frac{V(r)}{V_0(r)}>0$;

(K) $\limsup_{r\to\infty}\frac{K(r)}{K\infty(r)}<\infty$ and $\limsup_{r\to 0^+}\frac{K(r)}{K_0(r)}<\infty$,

where $V_0,K_0\in\mathcal{C}(0^+)$ and $V_\infty,K_\infty\in\mathcal{C}(+\infty)$.

By conditions (V) and (K), we obtain that there exist positive constants r_0, r_∞, a_0, a_∞, b_0, b_∞ such that

$$a_0V_0(r)\le V(r),\quad \forall r\le r_0,\quad \text{and}\quad a_\infty V_\infty(r)\le V(r),\quad \forall r>r_\infty, \tag{4.1}$$

$$b_0K_0(r)\le K(r),\quad \forall r\le r_0,\quad \text{and}\quad b_\infty K_\infty(r)\le K(r),\quad \forall r>r_\infty. \tag{4.2}$$

Now, we define the following two radially symmetric Sobolev spaces:

$$L^{q+1}_{K,r}(\mathbb{R}^N)=\left\{u:\int_{\mathbb{R}^N}K(|x|)|u|^{q+1}\,dx<\infty, u\text{ is radial}\right\}$$

and $W^{1,p}_{V,r}(\mathbb{R}^N)=\{u:u\in D^{1,p}_r(\mathbb{R}^N),\int_{\mathbb{R}^N}V(|x|)|u|^p\,dx<\infty, u\text{is radial}\}$ under the uniformly convex norm

$$\|u\|_{W^{1,p}_{V,r}(\mathbb{R}^N)}=\left(\int_{\mathbb{R}^N}|\nabla u|^p\,dx+\int_{\mathbb{R}^N}V(|x|)|u|^p\,dx\right)^{\frac{1}{p}}.$$

Lemma 4.2 *Assume that $u\in W^{1,p}_{V,r}(\mathbb{R}^N)$ and $V(x)$ satisfies condition* (V), $1<p<N$. *If*

(a) (1) $p(N-1)(V_0(r))+(p-1)rV_0'(r)>0$ *for* $0<r<r_0$, *then there exists* $C>0$ *such that*

$$|u(r)|\le C\left[r^{1-N}(V_0(r))^{\frac{1-p}{p}}\right]^{\frac{1}{p}}\|u\|_{W^{1,p}_{V,r}(\mathbb{R}^N)}\quad \textit{a.e. in } (0,r_0). \tag{4.3}$$

(2) $p(N-1)(V_\infty(r))+(p-1)rV_\infty'(r)\ge 0$ *for* $r>r_\infty$, *then there exists* $C>0$ *such that*

$$|u(r)|\le C\left[r^{1-N}(V_\infty(r))^{\frac{1-p}{p}}\right]^{\frac{1}{p}}\|u\|_{W^{1,p}_{V,r}(\mathbb{R}^N)}\quad \textit{a.e. in } (r_\infty,+\infty). \tag{4.4}$$

(b) (1) $p(N-1)(V_0(r))+(p-1)rV_0'(r)<0$ *for* $0<r<r_0$ *and* $V_0^{-1}(r)$, $V_0'(r)$, $r\in L^\infty(0,r_0)$, *then* (4.3) *holds*.

(2) $p(N-1)(V_0(r))+(p-1)rV_0'(r)<0$ *for* $r>r_\infty$ *and* $V_\infty^{-1}(r)$, $V_\infty(r)$, $r\in L^\infty(r_\infty,\infty)$, *then* (4.4) *holds*.

Proof (a)(1) By density, it is enough to prove it for $u\in D(\mathbb{R}^N)$ with support in B_{γ_0}. We have

$$\begin{aligned}
&|u(r)|^p(V_0(r))^{\frac{p-1}{p}}r^{N-1}\\
&\quad\le p\int_r^{r_0}|u(s)|^{p-1}|u'(s)|(V_0(s))^{\frac{p-1}{p}}s^{N-1}\,ds\\
&\qquad-(N-1)\int_r^{r_0}|u(s)|^p(V_0(s))^{\frac{p-1}{p}}s^{N-2}\,ds\\
&\qquad-\frac{p-1}{p}\int_r^{r_0}(V_0(s))^{-\frac{1}{p}}|u(s)|^p(V_0'(s))s^{N-1}\,ds.
\end{aligned}$$

By Hölder's inequality and (4.1), there exists $C > 0$ such that

$$\begin{aligned} &p\int_r^{r_0} |u(s)|^{p-1}|u'(s)|(V_0(s))^{\frac{p-1}{p}} s^{N-1}\,ds \\ &\quad\le \left(\int_r^{r_0} |u'(s)|^p s^{N-1}\,ds\right)^{\frac{1}{p}} \left(\int_r^{r_0} (V_0(s))|u(s)|^p s^{N-1}\,ds\right)^{\frac{p-1}{p}} \\ &\quad\le \omega_N^{-1}\left(\int_{B_{r_0(0)}\setminus B_r(0)} |\nabla u|^p\,dx\right)^{\frac{1}{p}} \left(\int_{B_{r_0(0)}\setminus B_r(0)} [V_0(s)]|u|^p\,dx\right)^{\frac{p-1}{p}} \\ &\quad\le \omega_N^{-1} a_0^{-\frac{p-1}{p}} \int_{\mathbb{R}^N} (|\nabla u|^p + V(x))|u|^p\,dx \\ &\quad\le C\|u\|^p_{W^{1,p}_{V,r}(\mathbb{R}^N)}, \end{aligned} \tag{4.5}$$

where ω_N is the volume of the unit sphere in $\mathbb{R}^N$.

On the other hand, $p(N-1)(V_0(r)) + (p-1)rV_0'(r) > 0$ for $0 < r < r_0$. By a simple computation, we obtain

$$\begin{aligned} &(N-1)\int_r^{r_0} |u(s)|^p (V_0(s))^{\frac{p-1}{p}} s^{N-1}\,ds \\ &\quad - \frac{p-1}{p}\int_r^{r_0} (V_0(s))^{-\frac{1}{p}} |u(s)|^p (V_0'(s)) s^{N-1}\,ds > 0. \end{aligned} \tag{4.6}$$

Combining (4.5) and (4.6), we have

$$|u(r)| \le C\left[r^{1-N}(V_0(r))^{\frac{1-p}{p}}\right]^{\frac{1}{p}} \|u\|_{W^{1,p}_{V,r}(\mathbb{R}^N)} \quad \text{a.e. in } (0, r_0).$$

(2) By density, it is enough to prove it for $u \in D(\mathbb{R}^N)$ with support in $B^c_{R_\infty}$, we have

$$\begin{aligned} |u(r)|^p (V_\infty(r))^{\frac{p-1}{p}} r^{N-1} \le{}& p\int_{r_\infty}^{+\infty} |u(s)|^{p-1}|u'(s)|(V_\infty(s))^{\frac{p-1}{p}} s^{N-1}\,ds \\ &- (N-1)\int_{r_\infty}^{+\infty} |u(s)|^p (V_\infty(s))^{\frac{p-1}{p}} s^{N-1}\,ds \\ &- \frac{p-1}{p}\int_{r_\infty}^{+\infty} (V_\infty(s))^{-\frac{1}{p}} |u(s)|^p (V_0'(s)) s^{N-2}\,ds. \end{aligned}$$

By a similar computation as for (4.5) and(4.6), we have

$$|u(r)|^p (V_\infty(r))^{\frac{p-1}{p}} r^{N-1} \le C\|u\|^p_{W^{1,p}_{V,r}(\mathbb{R}^N)},$$

and this yields (4.4).

(b)(1) If $p(N-1)(V_0(r)) + (p-1)rV_0'(r) > 0$ for $0 < r < r_0$.

By Hölder's inequality and Lemma 2.1, we have

$$\begin{aligned} &(N-1)\int_r^{r_0} |u(s)|^p (V_0(s))^{\frac{p-1}{p}} s^{N-2}\,ds \\ &\quad = (N-1)\int_r^{r_0} |u(s)|^{p-1} (V_0(s))^{\frac{p-1}{p}} \frac{|u(s)|}{|s|} \cdot s^{N-1}\,ds \end{aligned}$$

$$\begin{aligned}
&\le (N-1)\left(\int_r^{r_0} \left|u(s)\right|^p \left(V_0(s)\right) s^{N-1}\,ds\right)^{\frac{p-1}{p}} \cdot \left(\int_r^{r_0} \frac{|u(s)|^p}{|s|^p} \cdot s^{N-1}\,ds\right)^{\frac{1}{p}} \\
&\le \omega_N^{-1}(N-1)\left(\int_{B_{r_0(0)}\setminus B_r(0)} V_0\left(|x|\right)|u|^p\,dx\right)^{\frac{p-1}{p}} \cdot \left(\int_{\mathbb{R}^N} |\nabla u|^p\,dx\right)^{(\frac{p}{N-p})p} \\
&\le (N-1)\omega_N^{-1} a_0^{-\frac{p-1}{p}} \int_{\mathbb{R}^N} \left(|\nabla u|^p + V\left(|x|\right)\right)|u|^p\,dx \\
&\le C_2 \|u\|^p_{W^{1,p}_{V,r}(\mathbb{R}^N)}.
\end{aligned} \tag{4.7}$$

Similarly, we have

$$\begin{aligned}
&\frac{p-1}{p}\int_r^{r_0} \left|u(s)\right|^p \left(V_0(s)\right)^{-\frac{1}{p}} \left(V_0'(s)\right) s^{N-1}\,ds \\
&\quad = \frac{p-1}{p}\int_r^{r_0} \left|u(s)\right|^{p-1} \left(V_0(s)\right)^{\frac{p-1}{p}} \cdot V_0^{-1}(s)V_0'(s) \cdot s\frac{|u(s)|}{s} s^{N-1}\,ds \\
&\quad \le C_3' \left\| V_0^{-1}(s)V_0'(s)s\right\|_{L^\infty(0,r_0)} \|u\|^p_{W^{1,p}_{V,r}(\mathbb{R}^N)}.
\end{aligned} \tag{4.8}$$

Combining (4.5), (4.7) and (4.8), we obtain that (4.3) holds.

(2) If $p(N-1)(V_0(r)) + (p-1)rV_0'(r) < 0$ for $r > r_\infty$ and $V_\infty^{-1}(r)V_\infty(r)r \in L^\infty(r_\infty,\infty)$, we can argue as in the above proof. □

Let $p = N$, we consider the Sobolev space $W^{1,N}_{V,r}(\mathbb{R}^N)$.

Lemma 4.3 *Assume that $u \in W^{1,N}_{V,r}(\mathbb{R}^N)$ and $V(x)$ satisfies condition* (V). *If*

(a) $NV_0(r) + rV_0'(r) < 0$ *for* $0 < r < r_0$ *and* $V_0'(r)V_0^{-1}(r)r \in L^\infty(0,r_0)$, *then there exists* $C > 0$ *such that*

$$\left|u(r)\right| \le C\left(r^{1-N}\left(V_0(r)\right)^{\frac{N-1}{N}}\left|\log\frac{1}{r}\right|\right)^{\frac{1}{N}} \|u\|_{W^{1,N}_{V,r}(\mathbb{R}^N)} \quad \text{a.e. in } (0,r_0).$$

(b) $NV_\infty(r) + rV_\infty'(r) < 0$ *for* $r > r_\infty$ *and* $V_\infty'(r)V_\infty^{-1}(r)r \in L^\infty(r_\infty,+\infty)$, *then there exists* $C > 0$ *such that*

$$\left|u(r)\right| \le C\left(r^{1-N}\left(V_\infty(r)\right)^{\frac{N-1}{N}}\left|\log\frac{1}{r}\right|\right)^{\frac{1}{N}} \|u\|_{W^{1,N}_{V,r}(\mathbb{R}^N)} \quad \text{a.e. in } (r_\infty,\infty).$$

Proof (a) Arguing as in the proof of (2) of (a) in Lemma 4.2, by Hölder's inequality and Lemma 2.3, we have

$$\begin{aligned}
&(N-1)\int_r^{r_0} \left|u(s)\right|\left(V_0(s)\right)^{\frac{N-1}{N}} s^{N-2}\,ds \\
&\quad = (N-1)\left|\log\frac{1}{r}\right|\int_r^{r_0} \left|u(s)\right|^N \left(V_0(s)\right)^{\frac{N-1}{N}} \frac{|u(s)|}{s|\log\frac{1}{s}|} s^{N-1}\,ds \\
&\quad \le (N-1)\left|\log\frac{1}{r}\right|\left(\int_r^{r_0} \left|u(s)\right|^N \left(V_0(s)\right) s^{N-1}\,ds\right)^{\frac{N-1}{N}} \\
&\qquad \cdot \left(\int_r^{r_0} \frac{|u(s)|^N}{(s|\log\frac{1}{s}|)^N} s^{N-1}\,ds\right)^{\frac{1}{N}}
\end{aligned}$$

$$\begin{aligned}
&\le (N-1)\left|\log\frac{1}{r}\right|\omega_N^{-1}\left(\int_{B_{r_0}(0)\setminus B_r(0)}|u|^N\big(V_0\big(|x|\big)\big)\,ds\right)^{\frac{N-1}{N}}\cdot\left(\int_{\mathbb{R}^N}|\nabla u|^N\,dx\right)^{(\frac{N}{N-1})^N}\\
&\le C\left|\log\frac{1}{r}\right|\|u\|^N_{W^{1,N}_{V,r}(\mathbb{R}^N)}.
\end{aligned}\tag{4.9}$$

On the other hand, we have

$$\begin{aligned}
&\frac{(N-1)}{N}\int_r^{r_0}\big|u(s)\big|^N\big(V_0(s)\big)^{-\frac{1}{N}}\big(V_0's)\big)s^{N-1}\,ds\\
&\quad\le C\big\|V_0^{-1}(s)V_0'(s)s\big\|_{L^\infty(0,r_0)}\left|\log\frac{1}{r}\right|\|u\|_{W^{1,N}_{V,r}(\mathbb{R}^N)}.
\end{aligned}\tag{4.10}$$

Combining (4.9) and (4.10), we obtain

$$\big|u(r)\big|\le C\left(r^{1-N}\big(V_0(r)\big)^{\frac{N-1}{N}}\left|\log\frac{1}{r}\right|\right)^{\frac{1}{N}}\|u\|_{W^{1,N}_{V,r}(\mathbb{R}^N)}\quad\text{a.e. in }(0,r_0).$$

(b) Similarly, we can argue as in the proof of (a) in Lemma 4.3. □

Remark 4.4

(1) The previous estimates should be compared with Lemma 1 in [13],

$$\big|u(x)\big|\le C|x|^{-\frac{(N-p)}{p}}\|u\|_{D^{1,p}(\mathbb{R}^N)}\quad\text{for a.e. }x\in\mathbb{R}^N,$$

for every $u\in D^{1,p}_{rad}(\mathbb{R}^N)$;

(2) Our results extend Lemma 4 and Lemma 5 in [13], and we obtain the general Strauss-type decay estimates;

(3) Under the conditions of Lemma 4.2 and Lemma 4.3, we obtain that there exist two comparison functions $g_1, g_2\in\mathcal{C}(0,+\infty)$ such that

$$\begin{aligned}
&\big|u(r)\big|\le Cg_1\|u\|_{W^{1,p}_{V,r}(\mathbb{R}^N)}\quad\text{a.e. in }(0,r_0)\quad\text{and}\\
&\big|u(r)\big|\le Cg_2\|u\|_{W^{1,p}_{V,r}(\mathbb{R}^N)}\quad\text{a.e. in }(r_\infty,\infty).
\end{aligned}\tag{4.11}$$

Now, we state our main embedding theorems in this section.

Theorem 4.5 *If $V(x)$ and $K(x)$ satisfy* (V) *and* (K), $\mathbb{R}^N\setminus\{\text{supp}(V(x))\}$ *is relatively compact,* $K\in L^\infty_{loc}(\mathbb{R}^N\setminus\{0\})$.

(a) *If $1<p<N$ and $p\le q+1<\infty$,*
  (1) $K(r)r^{\frac{Np+(N-p)(q+1)}{p}}\in L^\infty(0,r_0)$ *and* $K(r)r^{\frac{Np+(N-p)(q+1)}{p}}\in L^\infty(r_\infty,\infty)$, *or*
  (2) $K_0(r)V_0^{-1}(r)(g_1(r))^{(q+1)-p}\in L^\infty(0,r_0)$ *and* $K_\infty(r)V_\infty^{-1}(r)(g_2(r))^{(q+1)-p}\in L^\infty(r_\infty,\infty)$,
  then the embedding $W^{1,p}_{V,r}(\mathbb{R}^N)\hookrightarrow L^{q+1}_K(\mathbb{R}^N)$ *is continuous.*

(b) *If $p=N$, $q+1\ge N$,* $K_0(r)(\frac{1}{r|\log\frac{1}{r}|})^N(g_1(r))^{(q+1)-N}\in L^\infty(0,r_0)$ *and* $K_\infty(r)(\frac{1}{r|\log\frac{1}{r}|})^N(g_2(r))^{(q+1)-N}\in L^\infty(r_\infty,\infty)$, *then the embedding* $W^{1,N}_{V,r}(\mathbb{R}^N)\hookrightarrow L^{q+1}_K(\mathbb{R}^N)$ *is continuous.*

Proof (a)(1) If $K(r)r^{\frac{Np+(N-p)(q+1)}{p}} \in L^\infty(0,r_0)$, we obtain

$$\int_0^{r_0} K(s)|u|^{q+1}s^{N-1}\,ds \le \omega_N^{-1}\left\|K(r)r^{\frac{Np+(N-p)(q+1)}{p}}\right\|_{L^\infty(0,r_0)}\int_{\mathbb{R}^N}|x|^{\frac{Np-(N-1)(q+1)}{p}}|u|^{q+1}\,dx.$$

By Lemma 2.2, we obtain

$$\int_{\mathbb{R}^N}|x|^{\frac{Np+(N-1)(q+1)}{p}}|u|^{q+1}\,dx \le C\left(\int_{\mathbb{R}^N}|\nabla u|^p\,dx\right)^{\frac{(q+1)}{p}}.$$

Hence, we have

$$\int_0^{r_0} K(s)|u|^{q+1}s^{N-1}\,ds \le C\|u\|^{q+1}_{W^{1,p}_{V,r}(\mathbb{R}^N)}. \tag{4.12}$$

If $K(r)r^{\frac{Np+(N-p)(q+1)}{p}} \in L^\infty(r_\infty,\infty)$, arguing as previously, similarly we obtain

$$\int_{r_\infty}^{\infty} K(s)|u|^{q+1}s^{N-1}\,ds \le C\left(\int_{\mathbb{R}^N}|\nabla u|^p\,dx\right)^{\frac{(q+1)}{p}} \le C\|u\|^{q+1}_{W^{1,p}_{V,r}(\mathbb{R}^N)}. \tag{4.13}$$

(2) If $K_0(r)V_0^{-1}(r)(g_1(r))^{(q+1)-p} \in L^\infty(0,r_0)$, by (4.11) and conditions (V) and (K), we obtain

$$\begin{aligned}
&\int_0^{r_0} K(s)|u|^{q+1}s^{N-1}\,ds \le \omega_N^{-1}\left\|K_0(r)V_0^{-1}(r)\big(g_1(r)\big)^{(q+1)-p}\right\|_{L^\infty(0,r_0)},\\
&\left(\int_0^{r_0} K_0(s)u^p s^{N-1}\,ds\right)\|u\|^{(q+1)-p}_{W^{1,p}_{V,r}(\mathbb{R}^N)} \le \omega_N^{-1}\|u\|^{(q+1)-p}_{W^{1,p}_{V,r}(\mathbb{R}^N)}\int_0^{r_0} K_0(s)u^p s^{N-1}\,ds\\
&\qquad \le C\|u\|^{q+1}_{W^{1,p}_{V,r}(\mathbb{R}^N)}.
\end{aligned} \tag{4.14}$$

If $K_\infty(r)V_\infty^{-1}(r)(g_2(r))^{(q+1)-p} \in L^\infty(r_\infty,\infty)$, we obtain similarly

$$\int_{r_\infty}^{\infty} K(s)|u|^{q+1}s^{N-1}\,ds \le C\|u\|^{q+1}_{W^{1,p}_{V,r}(\mathbb{R}^N)}. \tag{4.15}$$

(b) If $p = N$, $K_0(r)(\frac{1}{r|\ln\frac{1}{r}|})^{N-(q+1)}(g_1(r))^{(q+1)-N} \in L^\infty(0,r_0)$. By Hölder's inequality and Lemma 2.3, we have

$$\begin{aligned}
&\int_0^{r_0} K(s)|u|^{q+1}\,ds\\
&\quad \le C\left\|K_0(r)\left(r\left|\log\frac{1}{r}\right|\right)^N\big(g_1(r)\big)^{q+1-N}\right\|_{L^\infty(0,r_0)}\cdot\int_0^{r_0}\frac{|u|^N}{(|x|\log\frac{1}{x})^N}s^{N-1}\,ds\|u\|^{(q+1)-N}_{W^{1,N}_{V,r}(\mathbb{R}^N)}\\
&\quad \le C\|u\|^{(q+1)-N}_{W^{1,N}_{V,r}(\mathbb{R}^N)}\left(\int_0^{r_0}\frac{|u|^N}{(|x|\log\frac{1}{x})^N}s^{N-1}\,ds\right)\\
&\quad \le C\|u\|^{(q+1)-N}_{W^{1,N}_{V,r}(\mathbb{R}^N)}\left(\int_{\mathbb{R}^N}|\nabla u|^N\,dx\right)^N\\
&\quad \le C\|u\|^{q+1}_{W^{1,N}_{V,r}(\mathbb{R}^N)}.
\end{aligned} \tag{4.16}$$

If $K_0(r)(\frac{1}{r|\ln\frac{1}{r}|})^N(g_2(r))^{(q+1)-N} \in L^\infty(r_\infty,\infty)$, we obtain similarly

$$\int_{r_\infty}^{\infty} K(s)|u|^{q+1}\,ds \le C\|u\|^{q+1}_{W^{1,N}_{V,r}(\mathbb{R}^N)}. \tag{4.17}$$

From Lemma 6 in [13], we have the following.

Under the conditions of Theorem 4.5, for $0 < r < R < \infty$ and $R \gg 1$, the embedding

$$W^{1,p}_V(B_R\backslash B_r) \hookrightarrow L^{q+1}_K(B_R\backslash B_r) \quad \text{is compact.} \tag{4.18}$$

Now, we prove that the embedding $W^{1,p}_{V,r}(\mathbb{R}^N) \hookrightarrow L^{q+1}_K(\mathbb{R}^N)$ for $1 < p \le N$ is continuous. It suffices to show

$$\mathcal{S}_r(V,K) = \inf_{u\in W^{1,p}_{V,r}(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N}(|\nabla u|^p + V(|x|)|u|^p)\,dx}{(\int_{\mathbb{R}^N} K(|x|)|u|^{q+1}\,dx)^{\frac{p}{q+1}}} > 0.$$

Assume to the contrary that $\mathcal{S}(V,K) = 0$, then there exists $\{u_n\} \subset W^{1,p}_{V,r}(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} K\big(|x|\big)|u_n|^{q+1}\,dx = 1 \quad \text{and} \quad \int_{\mathbb{R}^N} \big(|\nabla u_n|^p + V\big(|x|\big)|u_n|^p\big)\,dx \to 0 \quad \text{as } n \to \infty.$$

But from (4.12) and (4.13), or (4.14) and (4.15), or (4.16) and (4.17), and (4.18), let r_∞ be large enough, we obtain

$$\begin{aligned} 1 &= \int_{\mathbb{R}^N} K\big(|x|\big)|u_n|^{q+1}\,dx \\ &= \int_0^{r_0} K\big(|x|\big)|u_n|^{q+1}\,dx + \int_{r_\infty}^{\infty} K\big(|x|\big)|u_n|^{q+1}\,dx + \int_{r_0}^{r_\infty} K\big(|x|\big)|u_n|^{q+1}\,dx \\ &\le C\|u_n\|^{q+1}_{W^{1,N}_{V,r}(\mathbb{R}^N)} \to 0 \quad \text{as } n \to \infty, \end{aligned}$$

which yields a contradiction. □

Theorem 4.6 *If $V(x)$ and $K(x)$ are nonnegative measurable functions satisfying* (V) *and* (K). $K(x) \in L^\infty_{loc}(\mathbb{R}^N\backslash\{0\})$ *and* $\mathbb{R}^N\backslash\{\operatorname{supp} V(x)\}$ *is relatively compact.*

(a) *If $1 < p < N$, $p \le q+1 < \infty$.*
  (1) $K(r)r^{\frac{Np+(N-p)(q+1)}{p}} = o(1)$ *as* $r \to 0^+$ *and* $K(r)r^{\frac{Np+(N-p)(q+1)}{p}} = o(1)$ *as* $r \to \infty$,
  or
  (2) $K_0(r)V_0^{-1}(r)[g_1(r)]^{q+1-p} = o(1)$ *as* $r \to 0^+$ *and* $K_\infty(r)V_\infty^{-1}(r)[g_2(r)]^{q+1-p} = o(1)$ *as* $r \to \infty$,
  then the embedding $W^{1,p}_{V,r}(\mathbb{R}^N) \hookrightarrow L^{q+1}_K(\mathbb{R}^N)$ *is compact.*

(b) *If* $p = N$, $q+1 \ge N$, $K_0(r)(\frac{1}{r|\log\frac{1}{r}|})^{N-(q+1)} \cdot (g_1(r))^{q+1-N} = o(1)$ *as* $r \to 0^+$ *and* $K_\infty(r)(\frac{1}{r|\log\frac{1}{r}|})^{p-(q+1)} \cdot (g_2(r))^{q+1-p} = o(1)$ *as* $r \to \infty$, *then the embedding* $W^{1,N}_{V,r}(\mathbb{R}^N) \hookrightarrow L^{q+1}_K(\mathbb{R}^N)$ *is compact.*

Proof (a) Arguing as in the proof of (a) and (b) of Theorem 4.5, we obtain that there exists $\varepsilon > 0$,

$$\int_0^{r_0} K(|x|)|u|^{q+1}\,dx \le \varepsilon \|u\|^{q+1}_{W^{1,p}_{V,r}(\mathbb{R}^N)}$$

and $\int_{r_\infty}^{\infty} K(|x|)|u|^{q+1}\,dx \le \varepsilon \|u\|^{q+1}_{W^{1,p}_{V,r}(\mathbb{R}^N)}$.

Assume that $u_n \rightharpoonup 0$ (weakly) in $W^{1,p}_{V,r}(\mathbb{R}^N)$ $(1 < p \le N)$, we obtain

$$\int_{\mathbb{R}^N} K(|x|)|u_n|^{q+1}\,dx \le C\varepsilon \|u_n\|^{q+1}_{W^{1,p}_{V,r}(\mathbb{R}^N)} \le C\varepsilon,$$

then we have $\|u_n\|_{L^{q+1}_{K,r}(\mathbb{R}^N)} \to 0$ (strongly). Hence the embedding is compact. □

5 Ground and bound state solutions

Now, consider problem (1.1) with general potentials

$$\begin{cases} -\triangle_p u + V(x)|u|^{p-2}u = K(x)|u|^{q-1}u, & x \in \mathbb{R}^N, \\ |u(x)| \to 0 \quad \text{as } |x| \to \infty. \end{cases}$$

Theorem 5.1 *Under the assumptions of Theorem* 3.2, *i.e.*, $W^{1,p}_V(\mathbb{R}^N)$ *is compact embedded into* $L^{q+1}_K(\mathbb{R}^N)$, *then problem* (1.1) *has a ground state solution*.

Proof Now, we define the functional $I(u)$ on the Sobolev space $W^{1,p}_V(\mathbb{R}^N)$,

$$\begin{aligned} I(u) &= \frac{1}{p}\int_{\mathbb{R}^N} \bigl(|\nabla u|^p + V(x)|u|^p\bigr)\,dx - \frac{1}{q+1}\int_{\mathbb{R}^N} K(x)|u|^{q+1}\,dx \\ &= \frac{1}{p}\|u\|^p_{W^{1,p}_V(\mathbb{R}^N)} - \frac{1}{q+1}\|u\|^{q+1}_{L^{q+1}_K(\mathbb{R}^N)}. \end{aligned}$$

It is obvious that the critical point of the functional $I(u)$ is exactly the weak solution of problem (1.1). The existence of a ground state solution follows from the compact embedding $W^{1,p}_V(\mathbb{R}^N) \hookrightarrow L^{q+1}_K(\mathbb{R}^N)$ immediately.

Further, consider problem (1.5) with radially symmetric potentials

$$\begin{cases} -\triangle_p u + V(|x|)|u|^{p-2}u = K(|x|)|u|^{q-1}u, & x \in \mathbb{R}^N, \\ |u(x)| \to 0 \quad \text{as } |x| \to \infty, \end{cases}$$

where $1 < p \le N$. Similarly to Theorem 5.1, we obtain the following theorem. □

Theorem 5.2 *Under the assumptions of Theorem* 4.6, *i.e.*, $W^{1,p}_{V,r}(\mathbb{R}^N)$ *is a compact embedding into* $L^{q+1}_{K,r}(\mathbb{R}^N)$, *then problem* (1.5) *has a ground state solution*.

For a more general equation than (1.5),

$$\begin{cases} -\triangle_p u + V(|x|)|u|^{p-2}u = K(|x|)f(u), & x \in \mathbb{R}^N, \\ |u(x)| \to 0 \quad \textit{as } |x| \to \infty. \end{cases} \tag{5.1}$$

If $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ and $f(0) = 0$, $|f(u)| \leq C(|u|^{p-1} + |u|^q)$ and there exists $\mu > p$ such that

$$0 < \mu F(u) = \mu \int_0^u f(s)\,ds \leq uf(u), \quad \forall u \in \mathbb{R},$$

then we have the following theorem.

Theorem 5.3 *Under the above conditions and assumptions of Theorem* 4.6, *problem* (5.1) *has a positive solution. If, in addition, f is odd in u, then problem* (5.1) *has infinitely many solutions in $W^{1,p}_{V,r}(\mathbb{R}^N)$.*

Competing interests
The author declares that he has no competing interests.

Authors' contributions
The author read and approved the final manuscript.

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