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Existence of positive solutions for a kind of periodic boundary value problem at resonance

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Abstract

In the paper we provide sufficient conditions for the existence of positive solutions for some second-order differential equation subject to periodic boundary conditions. Our method employs a Leggett-Williams norm-type theorem for coincidences due to O'Regan and Zima. Two examples are given to illustrate the main result of the paper.

Keywords: periodic boundary value problem; positive solution; coincidence equation

1 Introduction

In the paper we are interested in the existence of positive solutions for the periodic boundary value problem (PBVP)

$$\begin{cases} x''(t) + h(t)x'(t) + f(t, x(t), x'(t)) = 0, & t \in [0, T], \\ x(0) = x(T), \qquad x'(0) = x'(T), \end{cases} \tag{1}$$

where $f : [0, T] \times [0, \infty) \times \mathbb{R} \to \mathbb{R}$ and $h : [0, T] \to (0, \infty)$ are continuous functions. Our study is motivated by current activity of many researchers in the area of theory and applications of PVBPs; see, for example, [1–4] and references therein. In particular, in a recent paper [1], Chu, Fan and Torres have studied the existence of positive periodic solutions for the singular damped differential equation

$$x''(t) + h(t)x'(t) + a(t)x(t) = f\big(t, x(t), x'(t)\big)$$

by combining the properties of the Green's function of the PBVP

$$\begin{cases} x''(t) + h(t)x'(t) + a(t)x(t) = 0, & t \in [0, T], \\ x(0) = x(T), \qquad x'(0) = x'(T), \end{cases} \tag{2}$$

with a nonlinear alternative of Leray-Schauder type (see, for example, [5]). It should be noted that $a \not\equiv 0$ was the key assumption used in [1]. If $a \equiv 0$, then PBVP (2) has nontrivial solutions, which means that the problem we are concerned with here, that is, PBVP (1), is at resonance. There are several methods to deal with the resonant PBVPs. For example, in [6], Torres studied the existence of a positive solution for the PBVP

$$\begin{cases} x''(t) = f(t, x(t)), & t \in (0, 2\pi), \\ x(0) = x(2\pi), & x'(0) = x'(2\pi), \end{cases}$$

by considering the equivalent problem

$$\begin{cases} x''(t) + a(t)x(t) = f(t, x(t)) + a(t)x(t), & t \in (0, 2\pi), \\ x(0) = x(2\pi), & x'(0) = x'(2\pi), \end{cases}$$

via Krasnoselskii's theorem on cone expansion and compression. Further results in this direction can be found in [7] and [8]. In [9] Rachůnková, Tvrdý and Vrkoč applied the method of upper and lower solutions and topological degree arguments to establish the existence of nonnegative and nonpositive solutions for the PBVP

$$\begin{cases} x''(t) = f(t, x(t)), & t \in (0, 1), \\ x(0) = x(1), & x'(0) = x'(1). \end{cases} \tag{3}$$

The same PBVP was studied by Wang, Zhang and Wang in [10]. Their existence and multiplicity results on positive solutions are based on the theory of a fixed point index for A-proper semilinear operators on cones developed by Cremins [11].

The goal of our paper is to provide sufficient conditions that ensure the existence of positive solutions of (1) with the function h positive on $[0, T]$. Our general result is illustrated by two examples. The method we use in the paper is to rewrite BVP (1) as a coincidence equation $Lx = Nx$, where L is a Fredholm operator of index zero and N is a nonlinear operator, and to apply the Leggett-Williams norm-type theorem for coincidences obtained by O'Regan and Zima [12]. We would like to emphasize that the idea of results of [11] and [12], as well as these of [13–15], goes back to the celebrated Mawhin's coincidence degree theory [16]. For more details on this significant tool, its modifications and wide applications, we refer the reader to [17–22] and references therein.

In this paper, for the first time, the existence theorem from [12] is used for studying the boundary value problem with the nonlinearity f depending also on the derivative. In general, the presence of x' in f makes the problem much harder to handle. We point out that, to the best of our knowledge, there are only a few papers on PBVPs that discuss such a nonlinearity; we refer the reader to [15, 23–25] for some results of that type. We also complement several results in the literature, for example, in [1, 26] and [27]. It is evident that the existence theorems for PBVP (1) can be established by the shift method used in [6], that is, one can employ the results of [1] to the periodic problem we study here. However, the conditions imposed on f in [1] are not comparable with ours.

2 Coincidence equation

For the convenience of the reader, we begin this section by providing some background on cone theory and Fredholm operators in Banach spaces.

Definition 1 A nonempty subset C, $C \neq \{0\}$, of a real Banach space X is called a cone if C is closed, convex and

(i) $\lambda x \in C$ for all $x \in C$ and $\lambda \geq 0$,
(ii) $x, -x \in C$ implies $x = 0$.

Every cone induces a partial ordering in X as follows: for $x, y \in X$, we say that

$$x \preceq y \quad \text{if and only if} \quad y - x \in C.$$

The following property holds for every cone in a Banach space.

Lemma 1 [28] *For every $u \in C \setminus \{0\}$, there exists a positive number $\sigma(u)$ such that*

$$\|x + u\| \geq \sigma(u)\|x\|$$

for all $x \in C$.

Consider a linear mapping $L : \operatorname{dom} L \subset X \to Y$ and a nonlinear operator $N : X \to Y$, where X and Y are Banach spaces. If L is a Fredholm operator of index zero, that is, $\operatorname{Im} L$ is closed and $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < \infty$, then there exist continuous projections $P : X \to X$ and $Q : Y \to Y$ such that $\operatorname{Im} P = \operatorname{Ker} L$ and $\operatorname{Ker} Q = \operatorname{Im} L$ (see, for example, [14, 16]). Moreover, since $\dim \operatorname{Im} Q = \operatorname{codim} \operatorname{Im} L$, there exists an isomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} L$. Denote by L_P the restriction of L to $\operatorname{Ker} P \cap \operatorname{dom} L$. Then L_P is an isomorphism from $\operatorname{Ker} P \cap \operatorname{dom} L$ to $\operatorname{Im} L$ and its inverse

$$K_P : \operatorname{Im} L \to \operatorname{Ker} P \cap \operatorname{dom} L$$

is defined.

As a result, the coincidence equation $Lx = Nx$ is equivalent to $x = \Psi x$, where

$$\Psi = P + JQN + K_P(I - Q)N.$$

Let $\rho : X \to C$ be a retraction, that is, a continuous mapping such that $\rho(x) = x$ for all $x \in C$. Put

$$\Psi_\rho = \Psi \circ \rho.$$

Let Ω_1, Ω_2 be open bounded subsets of X with $\overline{\Omega}_1 \subset \Omega_2$ and $C \cap (\overline{\Omega}_2 \setminus \Omega_1) \neq \emptyset$. Assume that

1° L is a Fredholm operator of index zero,
2° $QN : X \to Y$ is continuous and bounded and $K_P(I - Q)N : X \to X$ is compact on every bounded subset of X,
3° $Lx \neq \lambda Nx$ for all $x \in C \cap \partial\Omega_2 \cap \operatorname{dom} L$ and $\lambda \in (0, 1)$,
4° ρ maps subsets of $\overline{\Omega}_2$ into bounded subsets of C,
5° $d_B([I - (P + JQN)\rho]|_{\operatorname{Ker} L}, \operatorname{Ker} L \cap \Omega_2, 0) \neq 0$, where d_B stands for the Brouwer degree,

6° there exists $u_0 \in C \setminus \{0\}$ such that $\|x\| \leq \sigma(u_0)\|\Psi x\|$ for $x \in C(u_0) \cap \partial\Omega_1$, where

$$C(u_0) = \{x \in C : \mu u_0 \preceq x \text{ for some } \mu > 0\}$$

and $\sigma(u_0)$ is such that $\|x + u_0\| \geq \sigma(u_0)\|x\|$ for every $x \in C$,

7° $(P + JQN)\rho(\partial\Omega_2) \subset C$ and $\Psi_\rho(\overline{\Omega}_2 \setminus \Omega_1) \subset C$.

Theorem 1 [12] *Under the assumptions* 1°-7° *the equation* $Lx = Nx$ *has a solution in the set* $C \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

In the next section, we use Theorem 1 to prove the existence of a positive solution for PBVP (1). For applications of Theorem 1 to nonlocal boundary value problems at resonance, we refer the reader to [22], [29] and [30].

3 Periodic boundary value problem

We now provide sufficient conditions for the existence of positive solutions for PBVP (1). For convenience and ease of exposition, we make use of the following notation:

$$e(t) = \exp\left(-\int_0^t h(\tau)\,d\tau\right), \qquad \varphi(t) = \int_0^t e(\tau)\,d\tau, \qquad \Phi(t) = \int_0^t \varphi(\tau)\,d\tau, \tag{4}$$

and

$$\psi(t) = \frac{1}{e(t)}\left(\frac{1}{1 - e(T)} - \frac{\varphi(t)}{\varphi(T)}\right), \quad t \in [0, T]. \tag{5}$$

We observe that $0 < \psi(t) < \frac{1}{e(T)(1-e(T))}$ on $[0, T]$. Moreover, we put

$$k(t,s) = \frac{1}{Te(s)} \begin{cases} \frac{\varphi(s)}{\varphi(T)}[\varphi(T)s - T\varphi(t) + \Phi(T)] - \Phi(s), & 0 \leq s \leq t \leq T, \\ \frac{\varphi(s)}{\varphi(T)}[\varphi(T)(s - T) - T\varphi(t) + \Phi(T)] + T\varphi(t) - \Phi(s), \\ \quad 0 \leq t \leq s \leq T, \end{cases} \tag{6}$$

and

$$K(t,s) = k(t,s) + \frac{M - \int_0^T k(t,\tau)\,d\tau}{\int_0^T \psi(\tau)\,d\tau}\psi(s), \quad t,s \in [0, T], \tag{7}$$

where M is a positive constant.

We assume that

(H1) $f : [0, T] \times [0, \infty) \times \mathbb{R} \to \mathbb{R}$ and $h : [0, T] \to (0, \infty)$ are continuous functions.

We also assume that there exist $R > 0$, $0 < \alpha \leq \beta$, $0 < M \leq \frac{e(T)(1-e(T)) \int_0^T \psi(\tau)\,d\tau}{\alpha T}$, $r \in (0, R)$, $m \in (0, 1)$, $\eta \in [0, T]$ and a continuous function $g : [0, T] \to [0, \infty)$ such that

(H2) $f(t, x, y) > -\alpha x + \beta|y|$ for $(t, x, y) \in [0, T] \times [0, R] \times [-R, R]$,

(H3) $f(t, R, 0) < 0$ for $t \in [0, T]$,

(H4) $f(0, x, R) = f(T, x, R)$ and $f(0, x, -R) = f(T, x, -R)$ for $x \in [0, R]$,

(H5) $f(t, x, -R) \leq h(t)R$ for $t \in [0, T]$ and $x \in [0, R)$,

(H6) $f(t, x, y) \geq g(t)(x + |y|)$ for $(t, x, y) \in [0, T] \times (0, r] \times [-r, r]$,

(H7) $\frac{1}{\alpha T} \geq K(t, s) \geq 0$ for $t, s \in [0, T]$ and $m \int_0^T K(\eta, s)g(s)\,ds \geq 1$.

Theorem 2 *Under the assumptions* (H1)-(H7), *PBVP* (1) *has a positive solution on* $[0,T]$.

Proof Let $\|\cdot\|_\infty$ denote the supremum norm in the space $C[0,T]$, that is, $\|x\|_\infty = \sup_{t\in[0,T]}|x(t)|$. Consider the Banach spaces $X = C^1[0,T]$ with the norm $\|x\| = \max\{\|x\|_\infty, \|x'\|_\infty\}$, and $Y = C[0,T]$ with the norm $\|\cdot\|_\infty$.

We write problem (1) as a coincidence equation

$$Lx = Nx,$$

where

$$Lx(t) = -x''(t) - h(t)x'(t), \quad t \in [0,T],$$

and

$$Nx(t) = f\bigl(t, x(t), x'(t)\bigr), \quad t \in [0,T],$$

with $\operatorname{dom} L = \{x \in X : x'' \in C[0,T], x(0) = x(T), x'(0) = x'(T)\}$. Then

$$\operatorname{Ker} L = \bigl\{x \in X : x(t) = c, t \in [0,T], c \in \mathbb{R}\bigr\}$$

and

$$\operatorname{Im} L = \left\{y \in Y : \int_0^T \psi(s)y(s)\,ds = 0\right\},$$

where ψ is given by (5).

Clearly, $\operatorname{Im} L$ is closed and $Y = Y_1 + \operatorname{Im} L$ with

$$Y_1 = \left\{y_1 \in Y : y_1 = \frac{1}{\int_0^T \psi(s)\,ds}\int_0^T \psi(s)y(s)\,ds, y \in Y\right\}.$$

Since $Y_1 \cap \operatorname{Im} L = \{0\}$, we have $Y = Y_1 \oplus \operatorname{Im} L$. Moreover, $\dim Y_1 = 1$, which gives $\operatorname{codim} \operatorname{Im} L = 1$. Consequently, L is Fredholm of index zero, and the assumption 1° is satisfied.

Define the projections $P : X \to X$ by

$$Px(t) = \frac{1}{T}\int_0^T x(s)\,ds, \quad t \in [0,T],$$

and $Q : Y \to Y$ by

$$Qy(t) = \frac{1}{\int_0^T \psi(s)\,ds}\int_0^T \psi(s)y(s)\,ds, \quad t \in [0,T].$$

It is a routine matter to show that for $y \in \operatorname{Im} L$, the inverse K_P of L_P is given by

$$(K_P y)(t) = \int_0^T k(t,s)y(s)\,ds, \quad t \in [0,T],$$

with the kernel k defined by (6). Clearly, the assumption 2° is satisfied. For $y \in \operatorname{Im} Q$, define

$$J(y) = My.$$

Then J is an isomorphism from $\operatorname{Im} Q$ to $\operatorname{Ker} L$. Next, consider a cone

$$C = \{x \in X : x(t) \geq 0 \text{ on } [0,T]\}.$$

For $u_0(t) \equiv 1$, we have $\sigma(u_0) = 1$ and

$$C(u_0) = \{x \in C : x(t) > 0 \text{ on } [0,T]\}.$$

Let

$$\Omega_1 = \{x \in X : \|x\| < r, |x(t)| > m\|x\|_\infty \text{ and } |x'(t)| > m\|x'\|_\infty \text{ on } [0,T]\},$$

and

$$\Omega_2 = \{x \in X : \|x\| < R\}.$$

Obviously, Ω_1 and Ω_2 are open bounded subsets of X, and $\overline{\Omega_1} \subset \Omega_2$.

To verify 3°, suppose that there exist $x_0 \in C \cap \partial\Omega_2 \cap \operatorname{dom} L$ and $\lambda_0 \in (0,1)$ such that $Lx_0 = \lambda_0 N x_0$. Then $x(t) \geq 0$ on $[0,T]$, $\|x_0\| = R$,

$$-x_0''(t) - h(t)x_0'(t) = \lambda_0 f(t, x_0(t), x_0'(t)), \quad t \in [0,T], \tag{8}$$

and

$$x_0(0) = x_0(T), \qquad x_0'(0) = x_0'(T). \tag{9}$$

There are two cases to consider.

1. If $\|x_0\| = \|x_0\|_\infty$, then there exists $t_0 \in [0,T]$ such that $x(t_0) = R$. For $t_0 \in (0,T)$, we get $0 \leq -x''(t_0) = \lambda_0 f(t_0, R, 0)$, contrary to the assumption (H3). Similarly, if $t_0 = 0$ or $t_0 = T$, BCs (9) imply $x'(0) = x'(T) = 0$. Hence, $0 \leq -x''(t_0) = \lambda_0 f(t_0, R, 0)$ which contradicts (H3) again.

2. If $\|x_0\| = \|x_0'\|_\infty > \|x_0\|_\infty$, then there exists $t_0 \in [0,T]$ such that $|x'(t_0)| = R$. Observe that (H2) implies $f(t, x, \pm R) > 0$ for $t \in [0,T]$ and $x \in [0,R]$. Suppose that $t_0 \in (0,T)$. If $x'(t_0) = R$, we get from (8)

$$-h(t_0)R = \lambda_0 f(t_0, x_0(t_0), R), \tag{10}$$

a contradiction. For $x'(t_0) = -R$, we have

$$h(t_0)R = \lambda_0 f(t_0, x_0(t_0), -R) < f(t_0, x_0(t_0), -R), \tag{11}$$

contrary to (H5). By similar arguments, if $t_0 = 0$ or $t_0 = T$, BCs (9) and (H4) imply either (10) or (11). Thus, 3° is fulfilled.

Next, for $x \in X$, define (see [15])

$$\rho x(t) = \begin{cases} x(t) & \text{if } x(t) \geq 0 \text{ on } [0,T], \\ \frac{1}{2}(x(t) - \min\{x(t) : t \in [0,T]\}) & \text{if } x(\tilde{t}) < 0 \text{ for some } \tilde{t} \in [0,T]. \end{cases}$$

Clearly, ρ is a retraction and maps subsets of $\overline{\Omega}_2$ into bounded subsets of C, so 4° holds.

To verify 5°, it is enough to consider, for $x \in \operatorname{Ker} L \cap \Omega_2$ and $\lambda \in [0,1]$, the mapping

$$H(x,\lambda)(t) = x(t) - \lambda\left(\frac{1}{T}\int_0^T (\rho x)(s)\,ds + \frac{M}{\int_0^T \psi(s)\,ds}\int_0^T \psi(s)f\bigl(s,(\rho x)(s),(\rho x)'(s)\bigr)\,ds\right).$$

Observe that if $x \in \operatorname{Ker} L \cap \Omega_2$, then $x(t) = c$ on $[0,T]$ and $\|x\| < R$. Suppose $H(x,\lambda) = 0$ for $x \in \partial\Omega_2$. Then $c = \pm R$. For $c = R$, we have $(\rho x)(t) = x(t)$ and in view of (H3), we get

$$0 \leq R(1-\lambda) = \lambda\frac{M}{\int_0^T \psi(s)\,ds}\int_0^T \psi(s)f(s,R,0)\,ds < 0,$$

which is a contradiction. If $c = -R$, then $(\rho x)(t) = 0$, hence

$$-R = \lambda\frac{M}{\int_0^T \psi(s)\,ds}\int_0^T \psi(s)f(s,0,0)\,ds,$$

which contradicts (H2). Thus, $H(x,\lambda) \neq 0$ for $x \in \partial\Omega_2$ and $\lambda \in [0,1]$. This implies

$$d_B\bigl(H(x,0), \operatorname{Ker} L \cap \Omega_2, 0\bigr) = d_B\bigl(H(x,1), \operatorname{Ker} L \cap \Omega_2, 0\bigr),$$

and

$$d_B\bigl(\bigl[I - (P+JQN)\rho\bigr]\big|_{\operatorname{Ker} L}, \operatorname{Ker} L \cap \Omega_2, 0\bigr) = d_B\bigl(H(c,1), \operatorname{Ker} L \cap \Omega_2, 0\bigr) \neq 0.$$

We next show that 6° holds. Let $x \in C(u_0) \cap \partial\Omega_1$. Then for $t \in [0,T]$, we have $r \geq x(t) \geq m\|x\|_\infty > 0$, $r \geq |x'(t)| \geq \|x'\|_\infty$, and by (H6) and (H7), we obtain

$$\begin{aligned} \Psi x(\eta) &= \frac{1}{T}\int_0^T x(s)\,ds + \int_0^T K(\eta,s)f\bigl(s,x(s),x'(s)\bigr)\,ds \\ &\geq \int_0^T K(\eta,s)g(s)\bigl[x(s) + \bigl|x'(s)\bigr|\bigr]\,ds \geq m\int_0^T K(\eta,s)g(s)\bigl[\|x\|_\infty + \bigl\|x'\bigr\|_\infty\bigr]\,ds \\ &\geq m\|x\|\int_0^T K(\eta,s)g(s)\,ds \geq \|x\|. \end{aligned}$$

This implies $\|x\| \leq \|\Psi x\|$ for $x \in C(u_0) \cap \partial\Omega_1$, so 6° is satisfied.

Finally, we must check if 7° holds. If $x \in \partial\Omega_2$, then in view of (H2), we get

$$\begin{aligned} (P+JQN)(\rho x)(t) &= \frac{1}{T}\int_0^T (\rho x)(s)\,ds + \frac{M}{\int_0^T \psi(s)\,ds}\int_0^T \psi(s)f\bigl(s,(\rho x)(s),(\rho x)'(s)\bigr)\,ds \\ &\geq \frac{1}{T}\int_0^T (\rho x)(s)\,ds + \frac{M}{\int_0^T \psi(s)\,ds}\int_0^T \psi(s)\bigl[-\alpha(\rho x)(s) + \beta\bigl|(\rho x)'(s)\bigr|\bigr]\,ds \end{aligned}$$

$$\begin{aligned}
&\geq \int_0^T \left[\frac{1}{T} - \frac{\alpha M \psi(s)}{\int_0^T \psi(\tau)\,d\tau}\right](\rho x)(s)\,ds \\
&\geq \int_0^T \left[\frac{1}{T} - \frac{\alpha M}{e(T)(1-e(T))\int_0^T \psi(\tau)\,d\tau}\right](\rho x)(s)\,ds \geq 0.
\end{aligned}$$

Moreover, for $x \in \overline{\Omega}_2 \setminus \Omega_1$, we have from (H2) and (H7)

$$\begin{aligned}
\Psi_\rho x(t) &= \frac{1}{T}\int_0^T (\rho x)(s)\,ds + \int_0^T K(t,s) f\bigl(s,(\rho x)(s),(\rho x)'(s)\bigr)\,ds \\
&\geq \frac{1}{T}\int_0^T (\rho x)(s)\,ds + \int_0^T K(t,s)\bigl[-\alpha(\rho x)(s) + \beta\bigl|(\rho x)'(s)\bigr|\bigr]\,ds \\
&\geq \int_0^T \left[\frac{1}{T} - \alpha K(t,s)\right](\rho x)(s)\,ds \geq 0.
\end{aligned}$$

Thus, 7° is fulfilled and the assertion follows. □

We now give two examples illustrating Theorem 2. Some calculations have been made with *Mathematica*. In the first example, the function h is constant, while in the second $h(t) = 1/(1+t)$ and f is independent of t.

Example 1 Consider the following PBVP:

$$\begin{cases} x''(t) + x'(t) + (t(1-t)+1)(-\frac{2}{9}x(t) + \frac{3}{4}|x'(t)| + 1) = 0, & t \in [0,1], \\ x(0) = x(1), \qquad x'(0) = x'(1). \end{cases} \tag{12}$$

Then $e(t) = e^{-t}$, $\varphi(t) = 1 - e^{-t}$, $\Phi(t) = t + e^{-t} - 1$, $\psi(t) = \frac{e}{e-1}$, and

$$k(t,s) = \begin{cases} -s + \frac{e^{s-t+1} - e^{1-t}}{e-1}, & 0 \leq s \leq t \leq 1, \\ -s + 1 + \frac{e^{s-t} - e^{1-t}}{e-1}, & 0 \leq t \leq s \leq 1. \end{cases}$$

Moreover, (7) with $M = \frac{3}{2}$ reads

$$K(t,s) = \begin{cases} t - s + \frac{e^{s-t+1}}{e-1}, & 0 \leq s \leq t \leq 1, \\ t - s + 1 + \frac{e^{s-t}}{e-1}, & 0 \leq t \leq s \leq 1, \end{cases}$$

and the assumptions (H2)-(H7) are met with $R = 20$, $\alpha = \frac{2}{9}$, $\beta = \frac{3}{4}$, $r = \frac{36}{53}$, $m \in [\frac{12(e-1)}{17+7e}, 1)$, $\eta = 0$ and $g(t) = t(1-t) + 1$. By Theorem 2, problem (12) has a positive solution.

Example 2 Consider the PBVP

$$\begin{cases} x''(t) + \frac{1}{1+t}x'(t) + \frac{1}{10} - \frac{1}{9}x(t) + (x'(t))^{4/5} = 0, & t \in [0, \frac{1}{2}], \\ x(0) = x(\frac{1}{2}), \qquad x'(0) = x'(\frac{1}{2}). \end{cases} \tag{13}$$

In this case, we have $e(t) = \frac{1}{1+t}$, $\varphi(t) = \ln(1+t)$, $\Phi(t) = -t + \ln(1+t) + t\ln(1+t)$ and

$$\psi(t) = (1+t)\left(3 - \frac{\ln(1+t)}{\ln(\frac{3}{2})}\right).$$

The assumptions of Theorem 2 are fulfilled with $M = 1$, $R = 10$, $\alpha = \frac{1}{3}$, $\beta = \frac{1}{2}$, $r = \frac{1}{100}$, $m = 0.9$, $\eta = \frac{1}{4}$ and $g(t) = 3$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

MZ and PD contributed equally to the manuscript and read and approved its final version.

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