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Multiple solutions to nonlinear Schrödinger equations with critical growth

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Abstract
In 2000, Cingolani and Lazzo (J. Differ. Equ. 160:118-138, 2000) studied nonlinear Schrödinger equations with competing potential functions and considered only the subcritical growth. They related the number of solutions with the topology of the global minima set of a suitable ground energy function. In the present paper, we establish these results in the critical case. In particular, we remove the condition $c_0 < c_\infty$, which is a key condition in their paper. In the proofs we apply variational methods and Ljusternik-Schnirelmann theory.

MSC: 35J60; 35Q55

Keywords: nonlinear Schrödinger equations; critical growth; variational methods

1 Introduction and main result

We investigate the following nonlinear Schrödinger equation:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + W(x)\psi - g(x,\psi), \tag{1.1}$$

which arises in quantum mechanics and provides a description of the dynamics of the particle in a non-relativistic setting. $\hbar$ is the Planck's constant, $m > 0$ denotes the mass of the particle, $W : \mathbb{R}^N \to \mathbb{R}$ is the electric potential, g is the nonlinear coupling, and ψ is the wave function representing the state of the particle. A standing wave solution of equation (1.1) is a solution of the form $\psi(x,t) = u(x)e^{-i\frac{Et}{\hbar}}$. It is clear that $\psi(x,t)$ solves (1.1) if and only if $u(x)$ solves the following stationary equation:

$$-\frac{\hbar^2}{2m}\Delta u + \big(W(x) - E\big)u = g(x,u). \tag{1.2}$$

For simplicity and without loss of generality, we set $\varepsilon = \hbar$, $V(x) = 2m(W(x) - E)$ and $\tilde{g} = 2mg$, then equation (1.2) is equivalent to

$$-\varepsilon^2\Delta u + V(x)u = \tilde{g}(x,u). \tag{1.3}$$

A considerable amount of work has been devoted to investigating solutions of (1.3). The existence, multiplicity and qualitative property of such solutions have been extensively studied. For single interior spikes solutions in the whole space $\mathbb{R}^N$, please see [1–9] *etc.* For multiple interior spikes, please see [10, 11] *etc.* For single boundary spike solutions with

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Neumann boundary condition, please see [6, 12–15] *etc.* For multiple boundary spikes, please see [16–18] *etc.* In particular, Wang and Zeng [9] studied the existence and concentration behavior of solutions for NLS with competing potential functions. Cingolani and Lazzo in [19] obtained the multiple solutions for the similar equation. In those papers only the subcritical growth was considered. In the present paper, we complete these studies by considering a class of nonlinearities with the critical growth. In particular, we remove the condition $c_0 < c_\infty$, which is a key condition in [19].

In the sequel, we restrict ourselves to the critical case in which $\tilde{g}(x,u) = |u|^{2^*-2}u + f(x,u)$. More specifically, we study the following problem:

$$\begin{cases} -\varepsilon^2 \Delta u + V(x)u = |u|^{2^*-2}u + f(x,u) & \text{in } \mathbb{R}^N, \\ u > 0 \quad \text{in } \mathbb{R}^N, \\ \lim_{|x|\to\infty} u(x) = 0, \end{cases} \tag{1.4}$$

where $2^* := 2N/(N-2)$ if $N \geq 3$, and $2^* := \infty$ if $N = 1, 2$. $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ satisfies

(f_1) $f(x,t) = 0$ for each $t \leq 0$;
(f_2) $\lim_{t\to 0^+} \frac{f(x,t)}{t} = 0$;
(f_3) there exists $q \in (2, 2^*)$ such that

$$\limsup_{t\to\infty} \frac{f(x,t)}{t^{q-1}} < \infty;$$

(f_4) there exists $2 < \theta < 2^*$ such that

$$0 < \theta F(x,t) \leq f(x,t)t, \quad \forall t > 0,$$

where $F(t) := \int_0^t f(\tau)\,d\tau$;
(f_5) the function $\frac{f(x,t)}{t}$ is strictly increasing in $t \geq 0$ for any $x \in \Omega$.

Our main results are the following theorem.

Theorem 1.1 *Let* $N \geq 4$. *Suppose that* f *satisfies* (f_1)-(f_5), V *is a continuous function in* $\mathbb{R}^N$ *and satisfies* $\inf_{x\in\mathbb{R}^N} V(x) > 0$. *Then when* ε *is sufficiently small, the problem* (1.4) *has at least* $\operatorname{cat}(\Sigma, \Sigma_\delta)$ *distinct nontrivial solutions.*

Here $\operatorname{cat}(\Sigma, \Sigma_\delta)$ denotes the Ljusternik-Schnirelmann category of Σ in Σ_δ. By definition (*e.g.*, [20]), the category of A with respect to M, denoted by $\operatorname{cat}(A, M)$, is the least integer k such that $A \subset A_1 \cup \cdots \cup A_k$, with A_i $(i = 1, \ldots, k)$ closed and contractible in M. We set $\operatorname{cat}(\emptyset, M) = 0$ and $\operatorname{cat}(A, M) = +\infty$ if there are no integers with the above property. We will use the notation $\operatorname{cat}(M)$ for $\operatorname{cat}(M, M)$.

To prove Theorem 1.1, we mainly use the idea of [15, 19, 21]. More precisely, we can show that the $(PS)_c$-condition holds in the subset $\tilde{\mathcal{N}}_\varepsilon$ (see (4.6)). Hence the standard Ljusternik-Schnirelmann category theory can be applied in $\tilde{\mathcal{N}}_\varepsilon$ to yield the existence of at least $\operatorname{cat}(\tilde{\mathcal{N}}_\varepsilon)$ critical points of I_ε. And then we construct two continuous mappings

$$\phi_\varepsilon : \Sigma \to \tilde{\mathcal{N}}_\varepsilon \tag{1.5}$$

and

$$\beta : \tilde{\mathcal{N}_\varepsilon} \to \Sigma_\delta, \tag{1.6}$$

where

$$\Sigma_\delta = \{x \in \mathbb{R}^N : \operatorname{dist}(x, \Sigma) \leq \delta\}, \quad \forall \delta > 0. \tag{1.7}$$

Then a topological argument asserts that

$$\operatorname{cat}(\tilde{\mathcal{N}_\varepsilon}) \geq 2\operatorname{cat}(\Sigma, \Sigma_\delta).$$

We will also prove that if u is a critical point of I_ε satisfying $I_\varepsilon(u) \leq \varepsilon^N(c_0 + h(\varepsilon))$, then u cannot change sign. Hence we obtain at least $\operatorname{cat}(\Sigma, \Sigma_\delta)$ nontrivial critical points of I_ε.

The paper is organized as follows. In Section 2, we collect some notations and preliminaries. A compactness result is given in Section 3, which is a key step in our proof. Finally, in Section 4, we prove Theorem 1.1.

2 Notations and preliminaries

$H^1(\mathbb{R}^N)$ is the usual Sobolev space of real-valued functions defined by

$$H^1(\mathbb{R}^N) := \{u : \nabla u \in L^2(\mathbb{R}^N) \text{ and } u \in L^2(\mathbb{R}^N)\}$$

with the normal

$$\|u\|^2 := \int_{\mathbb{R}^N} (|\nabla u|^2 + V(a)u^2)\,dx.$$

Let H_ε be the subspace of a Hilbert space $H^1(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_\varepsilon^2 := \int_{\mathbb{R}^N} (\varepsilon^2|\nabla u|^2 + V(x)u^2)\,dx < \infty.$$

We denote by S the Sobolev constant for the embedding $\mathcal{D}^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$, namely

$$S = \inf_{0 \neq u \in \mathcal{D}^{1,2}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2\,dx}{(\int_{\mathbb{R}^N} |u|^{2^*}\,dx)^{2/2^*}}, \tag{2.1}$$

where $\mathcal{D}^{1,2}(\mathbb{R}^N)$ is the usual Sobolev space of real-valued functions defined by

$$\mathcal{D}^{1,2}(\mathbb{R}^N) := \{u : \nabla u \in L^2(\mathbb{R}^N) \text{ and } u \in L^{2^*}(\mathbb{R}^N)\}.$$

We say that a function $u \in H_\varepsilon$ is a weak solution of the problem (1.4) if

$$\int_{\mathbb{R}^N} (\varepsilon^2 \nabla u \nabla v + V(x)uv - |u|^{2^*-2}uv - f(x,u)v)\,dx = 0, \quad \forall v \in H_\varepsilon.$$

In view of (f_2) and (f_3), we have that the associated functional $I_\varepsilon : H_\varepsilon \to \mathbb{R}$ given by

$$I_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^N} (\varepsilon^2|\nabla u|^2 + V(x)|u|^2)\,dx - \frac{1}{2^*}\int_{\mathbb{R}^N} |u|^{2^*}\,dx - \int_{\mathbb{R}^N} F(x,u)\,dx$$

is well defined. Moreover, $I_\varepsilon \in C^1(H_\varepsilon)$ with the following derivative:

$$\langle I'_\varepsilon(u), v\rangle = \int_{\mathbb{R}^N} \left(\varepsilon^2 \nabla u \nabla v + V(x)uv - |u|^{2^*-2}uv - f(x,u)v\right) dx.$$

Hence, the weak solutions of (1.4) are exactly the critical points of I_ε.

Let us recall some known facts about the limiting problem, namely the problem

$$-\Delta u + V(a)u = |u|^{2^*-2}u + f(a,u) \quad \text{in } \mathbb{R}^N, \tag{2.2}$$

here $a \in \mathbb{R}^N$ acts as a parameter instead of an independent variable. Solutions of (2.2) will be sought in the Sobolev space $H^1(\mathbb{R}^N)$ as critical points of the functional

$$J_a(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(a)|u|^2\right) dx - \frac{1}{2^*}\int_{\mathbb{R}^N} |u|^{2^*} dx - \int_{\mathbb{R}^N} F(a,u)\, dx.$$

The least positive critical value $G(a)$ can be characterized as

$$G(a) := \inf_{u \in \mathcal{M}_a} J_a(u),$$

where

$$\mathcal{M}_a := \left\{v \in H^1\left(\mathbb{R}^N\right) \setminus \{0\} : \langle J'_a(u), u\rangle = 0\right\}. \tag{2.3}$$

An associated critical point w actually solves equation (2.2) and is called a ground state solution or the least energy solution, *i.e.*, w satisfies

$$J_a(w) = \inf_{u \in \mathcal{M}_a} J_a(u).$$

Moreover, there exist $C > 0$ and $\delta > 0$ such that

$$w(x) \le e^{-\delta|x|}, \qquad \nabla w(x) \le e^{-\delta|x|} \quad \text{for all } x \in \mathbb{R}^N. \tag{2.4}$$

For more details, please see [22, 23].

Set

$$c_0 := \inf_{a \in \mathbb{R}^N} G(a), \qquad \Sigma := \left\{a \in \mathbb{R}^N : G(a) = c_0\right\}.$$

For any $\delta > 0$, we denote $\Sigma_\delta = \{x \in \mathbb{R}^N : \operatorname{dist}(x, \Sigma) \le \delta\}$. We need to estimate the super bound of c_0. In order to do this, we estimate $G(a)$. We shall use a family of radial function defined by

$$U_\varepsilon(x) = \left(N(N-2)\right)^{\frac{N-2}{4}} \left(\frac{\varepsilon}{\varepsilon^2 + |x|^2}\right)^{\frac{N-2}{2}}.$$

It is known [20] that

$$-\Delta U_\varepsilon = U_\varepsilon^{(N+2)/(N-2)} \quad \text{in } \mathbb{R}^N.$$

Moreover, we have

$$\int_{\mathbb{R}^N} |\nabla U_\varepsilon|^2\,dx = \int_{\mathbb{R}^N} |U_\varepsilon|^{2^*}\,dx = S^{N/2}.$$

Set $u_\varepsilon(x) = \phi(x)U_\varepsilon(x)$, where $\phi \in C^1$ is a cut-off function satisfying $\phi(x) \equiv 1$ if $|x| \leq \delta/2$, $\phi(x) \equiv 0$ if $|x| \geq \delta$ and $0 \leq \phi(x) \leq 1$. After a detailed calculation, we have the following estimates:

$$\int_{\mathbb{R}^N} \bigl|\nabla(\phi U_\varepsilon)\bigr|^2\,dx = S^{N/2} + O\bigl(\varepsilon^{N-2}\bigr), \tag{2.5}$$

$$\int_{\mathbb{R}^N} \bigl|\phi U_\varepsilon\bigr|^{2^*}\,dx = S^{N/2} + O\bigl(\varepsilon^{N}\bigr), \tag{2.6}$$

$$\int_{\mathbb{R}^N} \bigl|\phi U_\varepsilon\bigr|^{2}\,dx = \alpha(\varepsilon) := \begin{cases} C\varepsilon^2|\log\varepsilon| + O(\varepsilon^2) & \text{if } N = 4, \\ C\varepsilon^2 + O(\varepsilon^{N-2}) & \text{if } N \geq 5. \end{cases} \tag{2.7}$$

Since $F(t) \geq 0$, from (2.5)-(2.7), we conclude

$$\begin{aligned} G(a) &\leq \max_{t>0} J_a(tu_\varepsilon) \\ &\leq \max_{t>0}\left(\frac{t^2}{2}\int_{\mathbb{R}^N}\bigl(|\nabla u_\varepsilon|^2 + V(a)|u_\varepsilon|^2\bigr)\,dx - \frac{t^{2^*}}{2^*}\int_{\mathbb{R}^N}|u_\varepsilon|^{2^*}\,dx\right), \end{aligned} \tag{2.8}$$

then the maximum value of the right-hand side is achieved at

$$\tau = \left(\frac{\int_{\mathbb{R}^N}(|\nabla u_\varepsilon|^2 + V(a)|u_\varepsilon|^2)\,dx}{\int_{\mathbb{R}^N}|u_\varepsilon|^{2^*}\,dx}\right)^{\frac{N-2}{4}} \tag{2.9}$$

and

$$\begin{aligned} \max_{t>0} J_a(tu_\varepsilon) &\leq \frac{1}{N}\frac{(\int_{\mathbb{R}^N}(|\nabla u_\varepsilon|^2 + V(a)|u_\varepsilon|^2)\,dx)^{N/2}}{(\int_{\mathbb{R}^N}|u_\varepsilon|^{2^*}\,dx)^{(N-2)/2}} \\ &= \frac{1}{N}\left(\frac{\int_{\mathbb{R}^N}(|\nabla u_\varepsilon|^2 + V(a)|u_\varepsilon|^2)\,dx}{(\int_{\mathbb{R}^N}|u_\varepsilon|^{2^*}\,dx)^{2/2^*}}\right)^{N/2} \\ &\leq \frac{1}{N}\left(\frac{S^{N/2} + V(a)\alpha(\varepsilon) + O(\varepsilon^{N-2})}{S^{(N-2)/2} + O(\varepsilon^{N-2})}\right)^{N/2} \\ &< \frac{1}{N}S^{N/2}. \end{aligned} \tag{2.10}$$

Hence we have

$$c_0 < \frac{1}{N}S^{N/2}. \tag{2.11}$$

We denote the Nehari manifold of I_ε by

$$\mathcal{N}_\varepsilon = \bigl\{u \in H_{\varepsilon,A} \setminus \{0\} : \bigl\langle I'_\varepsilon(u), u\bigr\rangle = 0\bigr\}.$$

3 Compactness result

Proposition 3.1 *Let $\varepsilon_n \to 0$ as $n \to \infty$. Assume that $(u_n) \subset \mathcal{N}_{\varepsilon_n}$ satisfies $\varepsilon_n^{-N} I_{\varepsilon_n}(u_n) \to c_0$ as $n \to \infty$. Then uniformly in $a \in \Sigma$, there exist a subsequence of $v_n(y) := u_n(\varepsilon_n y + a)$ (still denoted by v_n), and $t_n > 0$ such that $w_n := t_n v_n \in \mathcal{M}_a$. Furthermore, w_n converges strongly in $H^1(\mathbb{R}^N)$ to w, the positive ground state solution of equation* (2.2).

Proof Let $(u_n) \subset \mathcal{N}_{\varepsilon_n}$ be such that $\varepsilon_n^{-N} I_{\varepsilon_n}(u_n) \to c_0$. Then, by a change of variable $x = \varepsilon_n y + a$, we have

$$\begin{aligned} c_0 + 1 &\geq \varepsilon_n^{-N}\left(I_{\varepsilon_n}(u_n) - \frac{1}{\theta}\langle I_{\varepsilon_n}(u_n), u_n\rangle\right) \\ &\geq \varepsilon_n^{-N}\left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_n\|_{\varepsilon_n}^2 \\ &= \left(\frac{1}{2} - \frac{1}{\theta}\right)\int_{\mathbb{R}^N}\left(\left|\nabla u_n(\varepsilon_n y + a)\right|^2 + V(\varepsilon_n y + a)\left|u_n(\varepsilon_n y + a)\right|^2\right) dy. \end{aligned} \tag{3.1}$$

This implies that (u_n) is bounded in $H^1(\mathbb{R}^N)$. Noting that

$$\begin{aligned} c_0 + o(1) = {}& \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla u_n + V(\varepsilon_n y + a)|u_n|^2\right) dy - \frac{1}{2^*}\int_{\mathbb{R}^N} |u_n|^{2^*}\, dy \\ & - \int_{\mathbb{R}^N} F(x, u_n)\, dy, \end{aligned} \tag{3.2}$$

hence

$$m := \lim_{n\to\infty} \int_{\mathbb{R}^N}\left(|\nabla u_n + V(\varepsilon_n y + a)|u_n|^2\right) dy > 0, \tag{3.3}$$

since $c_0 > 0$. Now we prove that there exists a sequence $(z_n) \subset \mathbb{R}^N$ and constants $R, \gamma > 0$ such that

$$\liminf_{n\to\infty} \int_{B_R(z_n)} |u_n|^2\, dy \geq \gamma > 0. \tag{3.4}$$

Indeed, if this is not true, then the boundedness of (u_n) in $H^1(\mathbb{R}^N)$ and a lemma due to Lions [24, Lemma I.1] imply that $u_n \to 0$ in $L^s(\mathbb{R}^N)$ for all $2 < s < 2^*$. Given $\delta > 0$, we can use (f_2), (f_3) and $u_n \in \mathcal{N}_{\varepsilon_n}$ to get

$$\int_{\mathbb{R}^N} f(x, u_n) u_n\, dy \leq \delta \int_{\mathbb{R}^N} |u_n|^2\, dy + C_\delta \int_{\mathbb{R}^N} |u_n|^q\, dy.$$

Moreover,

$$\int_{\mathbb{R}^N}\left(|\nabla u_n + V(\varepsilon_n y + a)|u_n|^2\right) dy = \int_{\mathbb{R}^N} |u_n|^{2^*}\, dy$$

as $n \to \infty$. Therefore

$$m = \lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{2^*}\, dy > 0, \tag{3.5}$$

and consequently (3.2) yields

$$c_0 = \frac{m}{2} - \frac{m}{2^*} = \frac{m}{N},$$

i.e.,

$$m = Nc_0 < S^{N/2} \quad \text{(see (2.11))}. \tag{3.6}$$

However, recall the definition of S in (2.1),

$$m = \lim_{n\to\infty} \int_{\mathbb{R}^N} \left(|\nabla u_n + V(\varepsilon_n y + a)|u_n|^2 \right) dy \geq S \lim_{n\to\infty} \left(\int_{\mathbb{R}^N} |u_n|^{2^*} dy \right)^{2/2^*} = Sm^{2/2^*},$$

equivalent to $m \geq S^{N/2}$, contradicting (3.6). Thus, (3.4) holds. Using the idea of [21, 25], along a subsequence as $n \to \infty$, we may assume that

$$v_n(y) := u_n(\varepsilon_n y + a) \rightharpoonup v \not\equiv 0 \quad \text{weakly in } H^1(\mathbb{R}^N).$$

We now consider $t_n > 0$ such that $w_n := t_n v_n \in \mathcal{M}_a$ (see (2.3)). By a change of variable $x = \varepsilon_n y + a$, it follows that

$$\begin{aligned} c_0 &\leq J_a(w_n) = J_a(t_n v_n) \leq \sup_{t>0} J_a(t v_n) \leq \sup_{t>0} \varepsilon_n^{-N} I_{\varepsilon_n}(t u_n) + o(1) = \varepsilon_n^{-N} I_{\varepsilon_n}(u_n) + o(1) \\ &= c_0 + o(1). \end{aligned} \tag{3.7}$$

Hence $J_a(w_n) \to c_0$, from which it follows that $w_n \nrightarrow 0$ in $H^1(\mathbb{R}^N)$.

Since (v_n) and (w_n) are bounded in $H^1(\mathbb{R}^N)$ and $v_n \nrightarrow 0$ in $H^1(\mathbb{R}^N)$, the sequence (t_n) is bounded. Thus, up to a subsequence, $t_n \to t_0 \geq 0$. If $t_0 = 0$, then $\|w_n\|_{H^1(\mathbb{R}^N)} \to 0$, which does not occur. Hence $t_0 > 0$, and therefore the sequence (w_n) satisfies

$$J_a(w_n) \to c_0, w_n \rightharpoonup w := t_0 v \not\equiv 0 \quad \text{weakly in } H^1(\mathbb{R}^N). \tag{3.8}$$

For fixed $v \in H^1(\mathbb{R}^N)$, define

$$b(w) = \int_{\mathbb{R}^N} \nabla w \nabla v \, dy \quad \text{for all } w \in H^1(\mathbb{R}^N).$$

By the Hölder inequality,

$$\left| b(w) \right| \leq |\nabla w|_2 |\nabla v|_2 \leq \|w\|_{H^1(\mathbb{R}^N)} \|v\|_{H^1(\mathbb{R}^N)}.$$

Hence $b \in H^{-1}$, the dual space of $H^1(\mathbb{R}^N)$. Consequently, as $n \to \infty$, $w_n \rightharpoonup w$ in $H^1(\mathbb{R}^N)$ implies $b(w_n) \to b(w)$, *i.e.*,

$$\int_{\mathbb{R}^N} \nabla w_n \nabla v \, dx = \int_{\mathbb{R}^N} \nabla w \nabla v \, dx + o(1). \tag{3.9}$$

Since w_n converges weakly to w in $H^1(\mathbb{R}^N)$, w_n is bounded in $L^{2^*}(\mathbb{R}^N)$. Thus $|w_n|^{2^*-2} w_n$ is bounded in $L^{(2^*)'}(\mathbb{R}^N)$. It then follows that there is a subsequence of (w_n), still denoted by

(w_n), such that $|w_n|^{2^*-2}w_n$ converges weakly to some $\tilde{w}$ in $L^{(2^*)'}(\mathbb{R}^N)$. Next we will show $\tilde{w} = |w|^{2^*-2}w$. Choose a sequence $(K_m)_{m\geq 1}$ of open relatively compact subsets, with regular boundaries, of $\mathbb{R}^N$ covering $\mathbb{R}^N$, *i.e.*, $\mathbb{R}^N = \bigcup_{m\geq 1} K_m$. It is easy to see that, by compact embedding, $w_n \to w$ in $L^q(K_m)$ for any $q < 2^*$. Hence $w_n \to w$ a.e. on K_m. Hence $|w_n|^{2^*-2}w_n \to |w|^{2^*-2}w$ a.e. on K_m. By the Brezis and Lieb lemma [26], we conclude that $|w_n|^{2^*-2}w_n \to |w|^{2^*-2}w$ strongly in $L^{(2^*)'}(K_m)$. Thus $\tilde{w} = |w|^{2^*-2}w$ a.e. on each K_m, and then the diagonal rule implies a.e. on $\mathbb{R}^N$. Hence

$$\int_{\mathbb{R}^N} |w_n|^{2^*-2}w_n v\,dx = \int_{\mathbb{R}^N} |w|^{2^*-2}wv\,dx + o(1). \tag{3.10}$$

Similarly, we have

$$\int_{\mathbb{R}^N} V(a)w_n v\,dx = \int_{\mathbb{R}^N} V(a)wv\,dx + o(1). \tag{3.11}$$

By (f_2) and (f_3),

$$\int_{\mathbb{R}^N} \bigl|f(x,w_n)v\bigr|\,dx \leq \delta \int_{\mathbb{R}^N} |w_n||v|\,dx + C_\delta \int_{\mathbb{R}^N} |w_n|^{q-1}|v|\,dx \leq \delta|w_n|_2|v|_2 + C_\delta|w_n|_q^{q-1}|v|_q.$$

Hence when R is large enough, we get

$$\int_{B_R^c(z_n)\cap\mathbb{R}^N} \bigl|f(a,w_n)v\bigr|\,dx = o(1).$$

Noting that $w_n \to w$ in $L^q(B_R(z_n))$, $2 \leq q < 2^*$. Therefore we have

$$\int_{B_R(z_n)\cap\mathbb{R}^N} f(a,w_n)v\,dx = \int_{B_R(z_n)\cap\mathbb{R}^N} f(a,w)v\,dx + o(1).$$

Hence

$$\int_{\mathbb{R}^N} f(a,w_n)v\,dx = \int_{\mathbb{R}^N} f(a,w)v\,dx + o(1). \tag{3.12}$$

By (3.9)-(3.12), we derive that

$$-\Delta w + V(a)w = |w|^{2^*-2}w + f(a,w) \quad \text{in } \mathbb{R}^N, \tag{3.13}$$

i.e., $J_a'(w) = 0$.

For any $n \in \mathbb{N}$ let us consider the measure sequence μ_n defined by

$$\int_{\mathbb{R}^N} \mu_n(dy) = \int_{\mathbb{R}^N} \bigl(|w_n|^{2^*} + |w_n|^2 + |w_n|^q\bigr)\,dy.$$

We assume

$$\int_{\mathbb{R}^N} \mu_n(dy) \to l.$$

By the concentration-compactness lemma [24], there exists a subsequence of (μ_n) (denoted in the same way) satisfying one of the three following possibilities.

Compactness: There exists a sequence $z_n \in \mathbb{R}^N$ such that for any $\delta > 0$ there is a radius $R > 0$ with the property that

$$\lim_{n\to\infty} \int_{B_R(z_n)} \mu_n(dy) \geq l - \delta.$$

Vanishing: For all $R > 0$,

$$\lim_{n\to\infty} \left(\sup_{z\in\mathbb{R}^N} \int_{B_R(z)} \mu_n(dy) \right) = 0.$$

Dichotomy: There exists a number $\tilde{a}$, $0 < \tilde{a} < l$, such that for any $\delta > 0$ there is a number $R > 0$ and a sequence (z_n) with the following property: Given $R' > R$ there are non-negative measures μ_n^1, μ_n^2 such that

(i) $0 \leq \mu_n^1 + \mu_n^2 \leq \mu_n$,
(ii) $\operatorname{supp}(\mu_n^1) \subset B_R(z_n)$, $\operatorname{supp}(\mu_n^2) \subset \mathbb{R}^N \setminus B_{R'}(z_n)$,
(iii) $\limsup_{n\to\infty}(|\tilde{a} - \int_{\mathbb{R}^N} \mu_n^1(dy)| + |(l-\tilde{a}) - \int_{\mathbb{R}^N} \mu_n^2(dy)|) \leq \delta$.

We are going to rule out the last two possibilities so that compactness holds. Our first goal is to show that vanishing cannot occur. Otherwise,

$$\begin{aligned} \|w_n\|_{H^1(\mathbb{R}^N)} &= \int_{\mathbb{R}^N} |w_n|^{2^*}\,dy + \int_{\mathbb{R}^N} f(x, w_n) w_n\,dy \\ &\leq \int_{\mathbb{R}^N} |w_n|^{2^*}\,dy + \delta \int_{\mathbb{R}^N} |w_n|^2\,dy + C_\delta \int_{\mathbb{R}^N} |w_n|^q\,dy \to 0. \end{aligned}$$

Hence $J(w_n) \to 0$, contradicting $c_0 > 0$.

Now for the harder part. Let η be a smooth nonincreasing cut-off function, defined in $[0,\infty)$, such that $\eta = 1$ if $0 \leq t \leq 1$; $\eta = 0$ if $t \geq 2$; $0 \leq \eta \leq 1$ and $|\eta'(t)| \leq 2$. Also, let $\eta_r(\cdot) = \eta(\frac{\cdot}{r})$. We define

$$\xi(t) = 1 - \eta(t),$$

a nondecreasing function on $[0,\infty)$. Denote by $\xi_r(\cdot) = \xi(\frac{\cdot}{r})$. We show now that dichotomy does not occur. Otherwise there exists $\tilde{a} \in (0,l)$ such that for some $R' > R \to \infty$ and $z_n \in \mathbb{R}^N$ the function μ_n splits into μ_n^1 and μ_n^2 with the following properties:

$$\int_{\mathbb{R}^N} \mu_n^1(dy) \to \tilde{a}, \qquad \int_{\mathbb{R}^N} \mu_n^2(dy) \to l - \tilde{a}. \tag{3.14}$$

If we denote

$$w_n^1 = \eta_R(x - z_n) w_n(x), \qquad w_n^2 = \xi_{R'}(x - z_n) w_n(x),$$

(3.14) becomes

$$\begin{aligned} &\int_{\mathbb{R}^N} \left(\left|w_n^1\right|^{2^*} + \left|w_n^1\right|^2 + \left|w_n^1\right|^q\right) dy \to \tilde{a}, \\ &\int_{\mathbb{R}^N} \left(\left|w_n^2\right|^{2^*} + \left|w_n^2\right|^2 + \left|w_n^2\right|^q\right) dy \to l - \tilde{a}. \end{aligned} \tag{3.15}$$

Denote by $\Omega' := B_{R'}(z_n) \setminus B_R(z_n)$, then

$$0 = \langle J_a'(w_n), \chi_{\Omega'} w_n \rangle = \int_{\Omega'} \left(|\nabla w_n|^2 + V(a)|w_n|^2 \right) dy - \int_{\Omega'} |w_n|^{2^*} dy - \int_{\Omega'} f(a, w_n) w_n \, dy.$$

Using Dichotomy (iii), we get

$$\begin{aligned} \int_{\Omega'} \mu_n(dy) &= \int_{\mathbb{R}^N} \mu_n(dy) - \int_{B_R(z_n)} \mu_n(dy) - \int_{B^c_{R'}(z_n)} \mu_n(dy) \\ &\le \int_{\mathbb{R}^N} \mu_n(dy) - \int_{B_R(z_n)} \mu_n^1(dy) - \int_{B^c_{R'}(z_n)} \mu_n^2(dy) \to 0, \end{aligned}$$

which implies

$$\int_{\Omega'} |w_n|^{2^*} dy \to 0, \qquad \int_{\Omega'} |w_n|^2 \, dy \to 0, \qquad \int_{\Omega'} |w_n|^q \, dy \to 0.$$

Hence

$$\begin{aligned} \int_{\Omega'} \left(|\nabla w_n|^2 + V(a)|w_n|^2 \right) dy &= \int_{\Omega'} |w_n|^{2^*} dy + \int_{\Omega'} f(a, w_n) w_n \, dy \\ &\le \int_{\Omega'} |w_n|^{2^*} dy + \delta \int_{\Omega'} |w_n|^2 \, dy + C_\delta \int_{\Omega'} |w_n|^q \, dy \to 0. \end{aligned}$$

Now we observe that supp $w_n^1 \cap$ supp $w_n^2 = \emptyset$, therefore

$$\begin{aligned} &\int_{\mathbb{R}^N} \left(|\nabla w_n|^2 + V(a)|w_n|^2 \right) dy \\ &\quad = \int_{\mathbb{R}^N} \left(\left|\nabla w_n^1\right|^2 + V(a)\left|w_n^1\right|^2 \right) dy + \int_{\mathbb{R}^N} \left(\left|\nabla w_n^2\right|^2 + V(a)\left|w_n^2\right|^2 \right) dy + o(1), \end{aligned} \tag{3.16}$$

$$\int_{\mathbb{R}^N} |w_n|^{2^*} dy = \int_{\mathbb{R}^N} \left|w_n^1\right|^{2^*} dy + \int_{\mathbb{R}^N} \left|w_n^2\right|^{2^*} dy + o(1), \tag{3.17}$$

$$\int_{\mathbb{R}^N} F(a, w_n) \, dx = \int_{\mathbb{R}^N} F\left(a, w_n^1\right) dy + \int_{\mathbb{R}^N} F\left(a, w_n^2\right) dy + o(1), \tag{3.18}$$

and

$$\int_{\mathbb{R}^N} f(a, w_n) w_n \, dy = \int_{\mathbb{R}^N} f\left(a, w_n^1\right) w_n^1 \, dy + \int_{\mathbb{R}^N} f\left(a, w_n^2\right) w_n^2 \, dy + o(1), \tag{3.19}$$

where $o(1) \to 0$ as $n \to \infty$.

Recall that $(w_n) \subset \mathcal{M}_a$ (see (2.3)), which implies

$$\begin{aligned} \langle J_a'(w_n), w_n \rangle &= \int_{\mathbb{R}^N} \left(|\nabla w_n|^2 + V(a)|w_n|^2 \right) dy - \int_{\mathbb{R}^N} |w_n|^{2^*} dy - \int_{\mathbb{R}^N} f(a, w_n) w_n \, dy \\ &= o(1). \end{aligned} \tag{3.20}$$

Then using w_n^1 and w_n^2 in place of w_n, respectively, we get

$$\begin{aligned}\langle J_a'(w_n), w_n^1\rangle &= \int_{\mathbb{R}^N}\left(\left|\nabla w_n^1\right|^2 + V(a)\left|w_n^1\right|^2\right)dy - \int_{\mathbb{R}^N}\left|w_n^1\right|^{2^*}dy - \int_{\mathbb{R}^N} f\left(a, w_n^1\right)w_n^1\,dy \\ &= o(1), \\ \langle J_a'(w_n), w_n^2\rangle &= \int_{\mathbb{R}^N}\left(\left|\nabla w_n^2\right|^2 + V(a)\left|w_n^2\right|^2\right)dy - \int_{\mathbb{R}^N}\left|w_n^2\right|^{2^*}dy - \int_{\mathbb{R}^N} f\left(a, w_n^2\right)w_n^2\,dy \\ &= o(1).\end{aligned} \tag{3.21}$$

There exists $t_n^1 > 0$ such that $t_n^1 w_n^1 \in \mathcal{M}_0$, *i.e.*,

$$\begin{aligned}&\left(t_n^1\right)^2\int_{\mathbb{R}^N}\left(\left|\nabla w_n^1\right|^2 + V(a)\left|w_n^1\right|^2\right)dy - \int_{\mathbb{R}^N}\left|t_n^1 w_n^1\right|^{2^*}dy \\ &\quad - \int_{\mathbb{R}^N} f\left(x, t_n^1 w_n^1\right)t_n^1 w_n^1\,dy = 0.\end{aligned} \tag{3.22}$$

By (f_2) and (f_3), $f(a,w_n)w_n \le \delta|w_n|^2 + C_\delta|w_n|^q$, we see t_n^1 cannot go zero, that is, $t_n^1 \ge t_0^1 > 0$. If $t_n^1 \to \infty$, by (3.21), (3.22) and (f_4), we get

$$\begin{aligned}\int_{\mathbb{R}^N}\left|w_n^1\right|^{2^*}dy + \int_{\mathbb{R}^N} f\left(x, w_n^1\right)w_n^1\,dy &= \int_{\mathbb{R}^N}\left(\left|\nabla w_n^1\right|^2 + V(a)\left|w_n^1\right|^2\right)dy \\ &= \int_{\mathbb{R}^N}\frac{|t_n^1 w_n^1|^{2^*} + f(x, t_n^1 w_n^1)t_n^1 w_n^1}{(t_n^1)^2}\,dy \to \infty,\end{aligned} \tag{3.23}$$

since (f_5). By (3.15),

$$\begin{aligned}&\int_{\mathbb{R}^N}\left|w_n^1\right|^{2^*}dy + \int_{\mathbb{R}^N} f\left(x, w_n^1\right)w_n^1\,dy \\ &\quad\le \int_{\mathbb{R}^N}\left|w_n^1\right|^{2^*}dy + \delta\int_{\mathbb{R}^N}\left|w_n^1\right|^2 dy + C_\delta\int_{\mathbb{R}^N}\left|w_n^1\right|^q dy \\ &\quad\le C\int_{\mathbb{R}^N}\left(\left|w_n^1\right|^{2^*} + \left|w_n^1\right|^2 + \left|w_n^1\right|^q\right)dy \le C\left(\tilde{a} + o(1)\right),\end{aligned} \tag{3.24}$$

a contradiction. Thus $0 < t_0^1 \le t_n^1 \le C$. Assume that $t_n^1 \to t^1$, we will show $t^1 = 1$. By (3.21) and (3.22), we have

$$\begin{aligned}o(1) &= \left(t_n^1\right)^{2^*-2}\int_{\mathbb{R}^N}\left|w_n^1\right|^{2^*}dy + \int_{\mathbb{R}^N}\frac{f(x, t_n^1 w_n^1)w_n^1}{t_n^1}\,dy - \int_{\mathbb{R}^N}\left|w_n^1\right|^{2^*}dy - \int_{\mathbb{R}^N} f\left(x, w_n^1\right)w_n^1\,dy \\ &= \left(\left(t_n^1\right)^{2^*-2} - 1\right)\int_{\mathbb{R}^N}\left|w_n^1\right|^{2^*}dy + \int_{\mathbb{R}^N}\left(\frac{f(x, t_n^1 w_n^1)}{t_n^1} - f\left(x, w_n^1\right)\right)w_n^1\,dy.\end{aligned}$$

Hence by the Lebesgue dominated convergence theorem, we get

$$\left(\left(t^1\right)^{2^*-2} - 1\right)\int_{\mathbb{R}^N}\left|w^1\right|^{2^*}dy + \int_{\mathbb{R}^N}\left(\frac{f(x, t^1 w^1)}{t^1} - f\left(x, w^1\right)\right)w^1\,dy = 0.$$

By (f_5), we have $t^1 = 1$. Similarly, $t_n^2 \to t^2 = 1$. Using this together with (3.16), (3.17), (3.18) and (3.19), we obtain

$$c_0 + o(1) = J_a(w_n) = J_a\left(w_n^1\right) + J_a\left(w_n^2\right) + o(1) = J_a\left(t_n^1 w_n^1\right) + J_a\left(t_n^2 w_n^2\right) + o(1) \geq 2c_0 + o(1).$$

Contradiction! Thus dichotomy does not occur.

With vanishing and dichotomy ruled out, we obtain the compactness of a sequence μ_n, *i.e.*, there exist $z_n \in \mathbb{R}^N$ and for each $\delta > 0$, there exists $R > 0$ such that

$$\int_{B_R^c(z_n)} \left(|w_n|^{2^*} + |w_n|^2 + |w_n|^q\right) dy \leq \delta. \tag{3.25}$$

Then (z_n) must be bounded, for otherwise (3.25) would imply, in the limit $n \to \infty$,

$$\int_{\mathbb{R}^N} \left(|w|^{2^*} + |w|^2 + |w|^q\right) dy \leq C_1\delta \tag{3.26}$$

for some positive constants C_1, independent of δ, which implies $w \equiv 0$, contrary to (3.8).

From the foregoing, it follows that there exist bounded nonnegative measures $\tilde{\mu}$, $\tilde{\nu}$ on $\mathbb{R}^N$ such that $|\nabla w_n|^2 \rightharpoonup \tilde{\mu}$ weakly and $|w_n|^{2^*} \rightharpoonup \tilde{\nu}$ tightly as $n \to \infty$. Lemma I.1 in [27] declares that there exist sequences $(x_j) \subset \mathbb{R}_+^N$, $(\tilde{\mu}_j), (\tilde{\nu}_j) \subset (0, \infty)$ such that

$$\begin{aligned} &(1) \quad \tilde{\mu} \geq |\nabla w|^2 + \sum_{j \in \tilde{J}} \tilde{\mu}_j \delta_{x_j}, \\ &(2) \quad \tilde{\nu} = |w|^{2^*} + \sum_{j \in \tilde{J}} \tilde{\nu}_j \delta_{x_j}, \\ &(3) \quad \tilde{\mu}_j \geq S\tilde{\nu}_j^{2/2^*}, \end{aligned} \tag{3.27}$$

where δ_{x_j} denotes a Dirac measure, $j \in \tilde{J}$. Take $x_j \in \mathbb{R}_+^N$ in the support of the singular part of $\tilde{\mu}$, $\tilde{\nu}$. We consider $\phi \in C_c^\infty(\mathbb{R}^N)$ such that

$$\phi = 1 \quad \text{on } B(x_j, \varepsilon), \qquad \phi = 0 \quad \text{on } B(x_j, 2\varepsilon)^c, \qquad |\nabla\phi| \leq 2/\varepsilon. \tag{3.28}$$

Choosing the test function ϕw_n, from $\langle I_\varepsilon'(w_n), \phi w_n\rangle \to 0$, we have

$$\begin{aligned} &\int_{\mathbb{R}^N} \phi\tilde{\mu}(dx) + \int_{\mathbb{R}^N} \phi V(a)|w|^2\,dx - \int_{\mathbb{R}^N} \phi\tilde{\nu}(dx) - \int_{\mathbb{R}^N} \phi f(x,w)w\,dx \\ &\quad = \lim_{n\to\infty} \int_{\mathbb{R}^N} \nabla w_n w_n \nabla\phi\,dx \leq C_1 \lim_{n\to\infty} \left(\int_{B(x_j,2\varepsilon)} |w_n|^2\,dx\right)^{1/2} \xrightarrow[\varepsilon\to 0]{} 0. \end{aligned} \tag{3.29}$$

This reduces to

$$\tilde{\mu}_j = \tilde{\nu}_j,$$

hence (3.27)(3) states

$$S\tilde{\nu}_j^{2/2^*} \leq \tilde{\nu}_j,$$

i.e.,

$$\tilde{\nu}_j = 0 \quad \text{or} \quad \tilde{\nu}_j \geq S^{N/2}.$$

Consequently,

$$\tilde{\nu} \geq |w|^{2^*} + S^{N/2} \sum_{j \in J} \delta_{x_j},$$

and hence

$$\int_{\mathbb{R}^N} |w_n|^{2^*}\, dx \to \int_{\mathbb{R}^N} \tilde{\nu}(dx) \geq \int_{\mathbb{R}^N} |w|^{2^*}\, dx + S^{N/2} \operatorname{Card} J, \tag{3.30}$$

which implies that the set J is at most finite. Here $\operatorname{Card} J$ is the cardinal numbers of set J. Hence

$$\begin{aligned} J_a(w_n) - \frac{1}{2}\langle J_a'(w_n), w_n\rangle &= \left(\frac{1}{2} - \frac{1}{2^*}\right) \int_{\mathbb{R}^N} |w_n|^{2^*}\, dy + \int_{\mathbb{R}^N} \big(f(a, w_n)w_n - F(a, w_n)\big)\, dy \\ &\geq \frac{1}{N} \int_{\mathbb{R}^N} |w_n|^{2^*}\, dy, \end{aligned} \tag{3.31}$$

since

$$f(x,t)t \geq \theta F(t) \geq F(t) > 0.$$

When n is large enough, recall $c_0 < \frac{1}{N} S^{N/2}$ (see (2.11)), together with (3.30) and (3.31), we obtain

$$\frac{1}{N} S^{N/2} > c_0 \geq \frac{1}{N} \int_{\Omega} |w|^{2^*}\, dx + \frac{1}{N} \sum_{j \in J} \nu_j \geq \frac{1}{N} S^{N/2},$$

a contradiction. Therefore J is empty, that is, $|w_n|_{2^*}^{2^*} \to |w|_{2^*}^{2^*}$ as $n \to \infty$. By the Brezis and Lieb lemma [26] again, we get

$$|w_n - w|_{2^*}^{2^*} \to 0. \tag{3.32}$$

Equation (3.25) and compact embedding theorem imply

$$\int_{\mathbb{R}^N} f(a, w_n) w_n\, dx = \int_{\mathbb{R}^N} f(a, w) w\, dx + o(1). \tag{3.33}$$

This together with (3.13), (3.20) and (3.32) allows us to deduce easily

$$\|w_n\|_{H^1(\mathbb{R}^N)} = \|w\|_{H^1(\mathbb{R}^N)} + o(1).$$

Since $H^1(\mathbb{R}^N_+)$ is a uniformly convex Banach space, hence

$$\|w_n - w\|_{H^1(\mathbb{R}^N)} \to 0. \tag{3.34}$$

From (3.32), (3.33) and (3.34), we can obtain

$$c_0 = \lim_{n\to\infty} J_a(w_n) = J_a(w), \tag{3.35}$$

i.e., w is the ground state solution of (2.2) in view of (3.13). The proof of Proposition 3.1 is complete. □

4 Proof of Theorem 1.1

Proposition 4.1 *Suppose f satisfies* (f_2)-(f_4). *Then I_ε satisfies the $(PS)_c$-condition for all $c < \varepsilon^N S^{N/2}/N$, that is, every sequence (u_n) in H_ε such that $I_\varepsilon(u_n) \to c$, $I'_\varepsilon(u_n) \to 0$, as $n \to \infty$, possesses a convergent subsequence.*

Proof Suppose that (u_n) is a sequence in H_ε such that $I_\varepsilon(u_n) \to c < \varepsilon^N S^{N/2}/N$, $I'_\varepsilon(u_n) \to 0$, as $n \to \infty$. Using (f_4), by a change of variable $x = \varepsilon y$, we obtain that

$$\begin{aligned} c + o(1)\|u_n\|_\varepsilon &\ge I_\varepsilon(u_n) - \frac{1}{\theta}\langle I_\varepsilon(u_n), u_n\rangle \\ &= \left(\frac{1}{2} - \frac{1}{\theta}\right)\int_{\mathbb{R}^N} \left(\varepsilon^2|\nabla u_n|^2 + V(x)|u_n|^2\right)dx + \left(\frac{1}{\theta} - \frac{1}{2^*}\right)\int_{\mathbb{R}^N} |u_n|^{2^*}\,dx \\ &\quad + \frac{1}{\theta}\int_{\mathbb{R}^N} \left(f(x,u_n)u_n - \theta F(x,u_n)\right)dx \\ &\ge \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_n\|_\varepsilon^2 = \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_n\|^2. \end{aligned} \tag{4.1}$$

This implies that (u_n) is bounded in H^1. Therefore we may assume $u_n \rightharpoonup u$ in H^1 and $u_n \to u$ a.e. Let $u_n = v_n + u$. Then

$$\begin{aligned} &\int_{\mathbb{R}^N} \left(|\nabla u_n|^2 + V(\varepsilon y)|u_n|^2\right)dy \\ &\quad = \int_{\mathbb{R}^N} \left(|\nabla v_n|^2 + V(\varepsilon y)|v_n|^2\right)dy + \int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(\varepsilon y)|u|^2\right)dy + o(1), \\ &\int_{\mathbb{R}^N} f(x,u_n)u_n\,dy = \int_{\mathbb{R}^N} f(x,v_n)v_n\,dy + \int_{\mathbb{R}^N} f(x,u)u\,dy + o(1), \\ &\int_{\mathbb{R}^N} F(x,u_n)\,dy = \int_{\mathbb{R}^N} F(x,v_n)\,dy + \int_{\mathbb{R}^N} F(x,u)\,dy + o(1), \end{aligned}$$

and by the Brezis-Lieb lemma [26],

$$\int_{\mathbb{R}^N} |u_n|^{2^*}\,dy = \int_{\mathbb{R}^N} |v_n|^{2^*}\,dy + \int_{\mathbb{R}^N} |u|^{2^*}\,dy + o(1).$$

For convenience, we denote by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(\varepsilon y)|u|^2\right)dy - \frac{1}{2^*}\int_{\mathbb{R}^N} |u|^{2^*}\,dy - \int_{\mathbb{R}^N} F(x,u)\,dy \tag{4.2}$$

and

$$\langle I'(u), v\rangle = \int_{\mathbb{R}^N} \left(\nabla u \nabla v + V(\varepsilon y)uv - |u|^{2^*-2}uv - f(x,u)v\right)dy. \tag{4.3}$$

It is clear that

$$I_\varepsilon(u) = \varepsilon^N I(u), \qquad \langle I'_\varepsilon(u), v\rangle = \varepsilon^N \langle I'(u), v\rangle. \tag{4.4}$$

It is easy to verify that $\langle I'(u), u\rangle = 0$. Hence we have

$$o(1) = \langle I'(u_n), u_n\rangle = \langle I'(v_n), v_n\rangle + \langle I'(u), u\rangle + o(1) = \langle I'(v_n), v_n\rangle + o(1),$$

and thus

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \left(|\nabla v_n|^2 + V(x)|v_n|^2\right) dy = \lim_{n\to\infty} \int_{\mathbb{R}^N} |v_n|^{2^*} dy =: \ell,$$

since $\int_{\mathbb{R}^N} f(|v_n|^2)|v_n|^2\, dy \to 0$ by (f_2) and (f_3). If $\ell = 0$, then $\|v_n\|^2 \to 0$, hence $\|v_n\|_\varepsilon^2 = \varepsilon^N \|v_n\|^2 \to 0$ as $n \to \infty$, and we can obtain the desired conclusion. Hence it remains to show that $\ell = 0$. By a change of variable, from

$$I(v_n) = I(v_n) - \frac{1}{2}\langle I'(v_n), v_n\rangle \geq \frac{1}{N} \int_{\mathbb{R}^N} |v_n|^{2^*} dy \geq 0$$

and

$$c = I(u_n) + o(1) = I(v_n) + I(u) + o(1) \geq I(v_n) + o(1),$$

we get

$$\frac{\varepsilon^N \ell}{N} \leq c < \frac{\varepsilon^N S^{N/2}}{N},$$

i.e.,

$$\ell < S^{N/2}. \tag{4.5}$$

By the Sobolev inequalities,

$$\left(\int_{\mathbb{R}^N} |v_n|^{2^*} dx\right)^{2/2^*} \leq S^{-1} \int_{\mathbb{R}^N} \left(|\nabla v_n|^2 + V(y)|v_n|^2\right) dy.$$

Letting $n \to \infty$, we get $\ell^{2/2^*} \leq S^{-1}\ell$, so either $\ell \geq S^{N/2}$ which contradicts (4.5) or $\ell = 0$. □

Let $\iota > 0$ be fixed. Let η be a smooth nonincreasing cut-off function, defined in $[0, \infty)$, such that $\eta = 1$ if $0 \leq t \leq \iota$; $\eta = 0$ if $t \geq 2\iota$; $0 \leq \eta \leq 1$ and $|\eta'(t)| \leq C$ for some $C > 0$. For any $a \in \Sigma$, let

$$\psi_\varepsilon(a)(x) = \eta\left(|x - a|\right) w\left(\frac{x - a}{\varepsilon}\right),$$

where w is the positive ground state of (2.2). We may assume that $t_\varepsilon > 0$ is the unique positive number such that

$$\max_{t\geq 0} I_\varepsilon\big(t\psi_\varepsilon(\xi)(x)\big) = I_\varepsilon\big(t_\varepsilon\psi_\varepsilon(\xi)(x)\big).$$

Let $h(\varepsilon)$ be any positive function tending to 0 as $\varepsilon \to 0$, we define the sublevel

$$\tilde{\mathcal{N}_\varepsilon} := \big\{u \in \mathcal{N}_\varepsilon : I_\varepsilon(u) \leq \varepsilon^N\big(c_0 + h(\varepsilon)\big)\big\}. \tag{4.6}$$

By Lemma 4.2 below, $\tilde{\mathcal{N}_\varepsilon}$ is not empty for ε sufficiently small. By noticing that $t_\varepsilon\psi_\varepsilon(\xi) \in \mathcal{N}_\varepsilon$, we can define $\phi_\varepsilon : \partial\Omega \to \tilde{\mathcal{N}_\varepsilon}$ as

$$\phi_\varepsilon := t_\varepsilon\psi_\varepsilon(\xi).$$

Lemma 4.2 *Uniformly in $a \in \Sigma$, we have*

$$\lim_{\varepsilon\to 0} \varepsilon^{-N} I_\varepsilon\big(\phi_\varepsilon(a)\big) = c_0. \tag{4.7}$$

Proof Let $a \in \Sigma$. Computing directly, we have

$$\begin{aligned}
&\big\|\psi_\varepsilon(a)(x)\big\|_\varepsilon^2 \\
&\quad= \int_{\mathbb{R}^N} \big(\varepsilon^2\big|\nabla\psi_\varepsilon(a)(x)\big|^2 + V(x)\big|\psi_\varepsilon(a)(x)\big|^2\big)\,dx \\
&\quad= \int_{\mathbb{R}^N} \left(\varepsilon^2\left|w\left(\frac{x-a}{\varepsilon}\right)\nabla\eta\big(|x-a|\big) + \frac{1}{\varepsilon}\eta\big(|x-a|\big)\nabla w\left(\frac{x-a}{\varepsilon}\right)\right|^2\right. \\
&\qquad\left. + V(x)\left|\eta\big(|x-a|\big)w\left(\frac{x-a}{\varepsilon}\right)\right|^2\right)dx \\
&\quad= \int_{\mathbb{R}^N} \left(\eta^2\big(|x-a|\big)\left(\left|\nabla w\left(\frac{x-a}{\varepsilon}\right)\right|^2 + V(x)\left|w\left(\frac{x-a}{\varepsilon}\right)\right|^2\right)\right)dx \\
&\qquad+ \int_{\mathbb{R}^N} \left(\varepsilon^2\big|\nabla\eta\big(|x-a|\big)\big|^2\left|w\left(\frac{x-a}{\varepsilon}\right)\right|^2\right. \\
&\qquad\left. + 2\varepsilon\eta\big(|x-a|\big)w\left(\frac{x-a}{\varepsilon}\right)\nabla\eta\big(|x-a|\big)\nabla w\left(\frac{x-a}{\varepsilon}\right)\right)dx \\
&\quad=: I_1 + I_2.
\end{aligned} \tag{4.8}$$

By a change of variable $y = \frac{x-a}{\varepsilon}$, we obtain

$$\begin{aligned}
I_1 &= \varepsilon^N \int_{\mathbb{R}^N} \eta^2\left(\frac{\varepsilon y}{\rho}\right)\big(\big|\nabla w(y)\big|^2 + V(\varepsilon y + a)\big|w(y)\big|^2\big)\,dy \\
&= \varepsilon^N\left(\int_{\mathbb{R}^N} \big(|\nabla w|^2 + V(a)w^2\big)\,dy + o(1)\right) \quad \text{as } \varepsilon \to 0,
\end{aligned} \tag{4.9}$$

uniformly for $a \in \Sigma$.

$$\begin{aligned} I_2 &= \varepsilon^N \int_{\mathbb{R}^N} \left(\varepsilon^2 \left| \nabla \eta\left(\frac{|\varepsilon y|}{\rho}\right)\right|^2 \omega^2(y) + 2\varepsilon \eta\left(\frac{|\varepsilon y|}{\rho}\right) \omega(y) \nabla \eta\left(\frac{|\varepsilon y|}{\rho}\right) \nabla \omega(y)\right) dy \\ &\le \varepsilon^{N+2} \int_{\{y: \rho/\varepsilon \le |y| \le 2\rho/\varepsilon\}} C \left|\omega(y)\right|^2 dy \\ &\quad + 2\varepsilon^{N+1} \int_{\{y: \rho/\varepsilon \le |y| \le 2\rho/\varepsilon\}} C_1 \left|\omega(y)\right| \left|\nabla \omega(y)\right| dy. \end{aligned} \tag{4.10}$$

By the exponential decay of ω, we get

$$I_2 = \varepsilon^N \big(o(1)\big) \quad \text{as } \varepsilon \to 0 \tag{4.11}$$

uniformly for $a \in \Sigma$. Therefore, in the limit that ε is very small, thanks to (4.8) (4.9) and (4.11), we find

$$\left\| \psi_\varepsilon(a)(x) \right\|_\varepsilon^2 = \varepsilon^N \left(\int_{\mathbb{R}^N} \left(|\nabla w|^2 + V(a) w^2 \right) dy + o(1) \right). \tag{4.12}$$

On the other hand, following the idea of [21, 25], from $\langle I'_\varepsilon(t_\varepsilon \psi_\varepsilon(\xi)(x)), t_\varepsilon \psi_\varepsilon(\xi)(x)\rangle = 0$, by the change of variables $y := (x - \xi)/\varepsilon$, we get

$$\begin{aligned} \left\| t_\varepsilon \psi_\varepsilon(\xi)(x) \right\|_\varepsilon^2 &= \int_{\mathbb{R}^N} \left| t_\varepsilon \psi_\varepsilon(\xi)(x) \right|^{2^*} dx + \int_{\mathbb{R}^N} f\big(x, t_\varepsilon \psi_\varepsilon(\xi)(x)\big) t_\varepsilon \psi_\varepsilon(\xi)(x)\, dx \\ &= \varepsilon^N \left(\int_{\mathbb{R}^N} \left| t_\varepsilon \eta_\rho(\varepsilon y) w(y) \right|^{2^*} dy \right. \\ &\quad \left. + \int_{\mathbb{R}^N} f\big(x, t_\varepsilon \eta_\rho(\varepsilon y) w(y)\big) t_\varepsilon \eta_\rho(\varepsilon y) w(y)\, dy \right), \end{aligned} \tag{4.13}$$

$$\begin{aligned} \varepsilon^{-N} \left\| \psi_\varepsilon(\xi)(x) \right\|_\varepsilon^2 &= |t_\varepsilon|^{2^*-2} \int_{\mathbb{R}^N} \left| \eta_\rho(\varepsilon y) w(y) \right|^{2^*} dy \\ &\quad + \int_{\mathbb{R}^N} \frac{f(x, t_\varepsilon \eta_\rho(\varepsilon y) w(y))}{t_\varepsilon} \eta_\rho(\varepsilon y) w(y)\, dy \\ &\to \infty \quad \text{as } t_\varepsilon \to \infty, \end{aligned} \tag{4.14}$$

which contradicts (4.12). Thus, up to a subsequence, $t_\varepsilon \to t_0 \ge 0$.

Since f has subcritical growth and $t_\varepsilon \psi_\varepsilon(\xi) \in \mathcal{N}_\varepsilon$, it follows that $t_0 > 0$. Thus, we can take the limit in (4.13) to obtain

$$\int_{\mathbb{R}^N} \left| \nabla (t_0 w) \right|^2 + |t_0 w|^2\, dy = \int_{\mathbb{R}^N} |t_0 w|^{2^*} dy + \int_{\mathbb{R}^N} f(a, t_0 w) t_0 w\, dy, \tag{4.15}$$

from which it follows that $t_0 w \in \mathcal{M}_a$. Since w also belongs to $\mathcal{M}_a$, we conclude that $t_0 = 1$. This and Lebesgue's theorem imply that

$$\int_{\mathbb{R}^N} \left| t_\varepsilon \psi_\varepsilon(\xi) \right|^{2^*} dx = \varepsilon^N \left(\int_{\mathbb{R}^N} |w|^{2^*} dy + o(1) \right) \tag{4.16}$$

and

$$\int_\Omega F\bigl(x, t_\varepsilon \psi_\varepsilon(a)\bigr)\,dx = \varepsilon^N \biggl(\int_{\mathbb{R}^N} F(a,w)\,dy + o(1)\biggr) \tag{4.17}$$

uniformly for $a \in \Sigma$. Noting $t_\varepsilon \to 1$, from (4.12), (4.16) and (4.17), we have

$$\begin{aligned}\varepsilon^{-N} I_\varepsilon\bigl(\phi_\varepsilon(\xi)\bigr) &= \frac{1}{2}\int_{\mathbb{R}^N} \bigl(|\nabla w|^2 + V(a)w^2\bigr)\,dy + \frac{1}{2^*}\int_{\mathbb{R}^N} |w|^{2^*}\,dy + \int_{\mathbb{R}^N} F(a,w)\,dy + o(1)\\ &= c_0 + o(1).\end{aligned}$$

Thus (4.7) is proved. □

Let $\beta(u)$ be the center of mass of $u \in \mathcal{N}_\varepsilon$ in terms of the L^2 norm:

$$\beta(u) := \frac{\int_{\mathbb{R}^N} x|u|^2\,dx}{\int_{\mathbb{R}^N} |u|^2\,dx}, \quad \forall u \in \mathcal{N}_\varepsilon.$$

Lemma 4.3 *Let $\varepsilon_n \to 0$ as $n \to \infty$. Then for $u_n \in \tilde{\mathcal{N}}_{\varepsilon_n}$,*

$$\beta(u_n) \to a, \quad \text{as } n \to \infty,$$

uniformly for $a \in \Sigma$.

Proof By change of variable $x = \varepsilon_n y + a$, we have

$$\begin{aligned}\beta(u_n) &= \frac{\int_{\mathbb{R}^N} x|u_n|^2\,dx}{\int_{\mathbb{R}^N} |u_n|^2\,dx} = \frac{\int_{\mathbb{R}^N} (\varepsilon_n y + a)|v_n|^2\,dy}{\int_{\mathbb{R}^N} |v_n|^2\,dy} = a + \varepsilon_n \frac{\int_{\mathbb{R}^N} y|v_n|^2\,dy}{\int_{\mathbb{R}^N} |v_n|^2\,dy}\\ &= a + \varepsilon_n \frac{\int_{\mathbb{R}^N} y|t_n v_n|^2\,dy}{\int_{\mathbb{R}^N} |t_n v_n|^2\,dy}.\end{aligned}$$

By Proposition 3.1, $w_n = t_n v_n$ converges strongly in $H^1(\mathbb{R}^N)$ to w, which is a positive ground state solution of equation (2.2). Thanks to the exponential decay of w (see (2.4)), we obtain $\beta(u_n) \to a \in \Sigma$ as $n \to \infty$. This completes the proof of Lemma 4.3. □

Proof of Theorem 1.1 By Proposition 4.1, I_ε satisfies the $(PS)_c$-condition for all $c < \varepsilon^N S^{N/2}/N$. Now let us choose a function $h(\varepsilon) > 0$ such that $h(\varepsilon) \to 0$ as $\varepsilon \to 0$ and such that $\varepsilon^N(c_0 + h(\varepsilon)) < \varepsilon^N S^{N/2}/N$ is not a critical level for I_ε. For such $h(\varepsilon)$, let us introduce the set $\tilde{\mathcal{N}}_\varepsilon \subset \mathcal{N}_\varepsilon$ as in (4.6). Then the standard Ljusternik-Schnirelmann theory implies that I_ε has at least $\operatorname{cat}(\tilde{\mathcal{N}}_\varepsilon)$ critical points on $\tilde{\mathcal{N}}_\varepsilon$ (also see [21]).

By Lemma 4.3, we can assume that for any $\delta > 0$, there exists $\varepsilon_\delta > 0$ such that $\beta(\tilde{\mathcal{N}}_\varepsilon) \subset \Sigma_\delta$ for any $\varepsilon < \varepsilon_\delta$. For such ε, by Lemma 4.2, we have $I_\varepsilon(\phi_\varepsilon(a)) \le \varepsilon^N(c_0 + h(\varepsilon))$ uniformly for $a \in \Sigma$, thus $\phi_\varepsilon(\Sigma) \subset \tilde{\mathcal{N}}_\varepsilon$. Recall $t_\varepsilon \to 1$, calculating directly, we get

$$\begin{aligned}\beta\bigl(\phi_\varepsilon(a)\bigr) &= \frac{\int_{\mathbb{R}^N} x|t_\varepsilon \psi_\varepsilon(a)(x)|^2\,dx}{\int_{\mathbb{R}^N} |t_\varepsilon \psi_\varepsilon(a)(x)|^2\,dx}\\ &= \frac{\int_{\mathbb{R}^N} x|\eta(|x-a|)w(\frac{x-a}{\varepsilon})|^2\,dx}{\int_{\mathbb{R}^N} |\eta(|x-a|)w(\frac{x-a}{\varepsilon})|^2\,dx} + o(1)\end{aligned}$$

$$= \frac{\int_{\mathbb{R}^N} (\varepsilon y + a)|\eta(|\varepsilon y|)w(y)|^2\,dx}{\int_{\mathbb{R}^N} |\eta(|\varepsilon y|)w(y)|^2\,dx} + o(1)$$
$$= a + \frac{\int_{\mathbb{R}^N} \varepsilon y|\eta(|\varepsilon y|)w(y)|^2\,dx}{\int_{\mathbb{R}^N} |\eta(|\varepsilon y|)w(y)|^2\,dx} = a + o(1),$$

as $\varepsilon \to 0$ uniformly for $a \in \Sigma$. Hence the map $\beta \circ \phi_\varepsilon$ is homotopical equivalence to the inclusion $i : \Sigma \to \Sigma_\delta$ for ε small enough. We denote $\tilde{\mathcal{N}}_\varepsilon^+ = \tilde{\mathcal{N}}_\varepsilon \cap \{u \in \mathcal{N}_\varepsilon : u \geq 0 \text{ in } \mathbb{R}^N\}$. It is easy to verify that $\operatorname{cat}(\tilde{\mathcal{N}}_\varepsilon^+, \tilde{\mathcal{N}}_\varepsilon) \geq \operatorname{cat}(\Sigma, \Sigma_\delta)$ and $\operatorname{cat}(\tilde{\mathcal{N}}_\varepsilon^-, \tilde{\mathcal{N}}_\varepsilon) \geq \operatorname{cat}(\Sigma, \Sigma_\delta)$ (*cf.* [19, Lemma 2.2]). Hence we have

$$\operatorname{cat}(\tilde{\mathcal{N}}_\varepsilon) = \operatorname{cat}\bigl(\tilde{\mathcal{N}}_\varepsilon^+, \tilde{\mathcal{N}}_\varepsilon\bigr) + \operatorname{cat}\bigl(\tilde{\mathcal{N}}_\varepsilon^-, \tilde{\mathcal{N}}_\varepsilon\bigr) \geq 2\operatorname{cat}(\Sigma, \Sigma_\delta).$$

Next we show that if u is a critical point of I_ε satisfying $I_\varepsilon(u) \leq \varepsilon^N(c_0 + h(\varepsilon))$, then u cannot change sign. Indeed, if $u = u^+ + u^-$ with $u^+ \neq 0$ and $u^- \neq 0$, then from $\langle I_\varepsilon'(u), u\rangle = 0$, we have

$$\bigl\langle I_\varepsilon'\bigl(u^+\bigr), u^+\bigr\rangle = 0, \qquad \bigl\langle I_\varepsilon'\bigl(u^-\bigr), u^-\bigr\rangle = 0.$$

By change of variable $x = \varepsilon_n y + a$, we get

$$\bigl\langle J_a'\bigl(u^+\bigr), u^+\bigr\rangle = 0, \qquad \bigl\langle J_a'\bigl(u^-\bigr), u^-\bigr\rangle = 0, \quad \text{as } \varepsilon_n \to 0,$$

i.e.,

$$u^+, u^- \in \mathcal{M}_a.$$

Also, noting

$$I_\varepsilon(u) = \varepsilon^N J_a(u), \quad \text{as } \varepsilon_n \to 0.$$

Hence

$$c_0 + h(\varepsilon) \geq J_a(u) = J_a\bigl(u^+\bigr) + J_a\bigl(u^-\bigr) \geq 2G(a) \geq 2c_0,$$

which is a contradiction. Therefore there exist at least $\operatorname{cat}(\Sigma, \Sigma_\delta)$ nonzero critical points of I_ε and thus $\operatorname{cat}(\Sigma, \Sigma_\delta)$ solutions of equation (1.4). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
WL carried out the genetic studies, participated in the sequence alignment and drafted the manuscript. PZ checked the references.

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