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Inverse eigenvalue problems for a discontinuous Sturm-Liouville operator with two discontinuities

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Abstract

In this paper, we consider a discontinuous Sturm-Liouville operator with parameter-dependent boundary conditions and two interior discontinuities. We obtain eigenvalues and eigenfunctions together with their asymptotic approximate formulas. Then, we give some uniqueness theorems by using Weyl function and spectral data, which are called eigenvalues and normalizing constants for solution of inverse problem.

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Keywords: Sturm-Liouville problem; eigenvalues; eigenfunctions; transmission conditions; Weyl function

1 Introduction

It is well known that the theory of Sturm-Liouville problems is one of the most actual and extensively developing fields of theoretical and applied mathematics, since it is an important tool in solving many problems in mathematical physics (see [1–4]). In recent years, there has been increasing interest in spectral analysis of discontinuous Sturm-Liouville problems with eigenvalue-linearly and nonlinearly dependent boundary conditions [1, 5–12]. Various physics applications of such problems can be found in [1, 3, 4, 13–19] and corresponding bibliography cited therein.

Some boundary value problems with discontinuity conditions arise in heat and mass transfer problems, mechanics, electronics, geophysics and other natural sciences (see [3] also [20–29]). For instance, discontinuous inverse problems appear in electronics for building parameters of heterogeneous electronic lines with attractive technical characteristics [20, 30, 31]. Such discontinuity problems also appear in geophysical forms for oscillations of the earth [32, 33]. Furthermore, discontinuous inverse problems appear in mathematics for exploring spectral properties of some classes of differential and integral operators.

Inverse problems of spectral analysis form recovering operators by their spectral data. The inverse problem for the classical Sturm-Liouville operator was studied first by Ambarsumian in 1929 [34] and then by Borg in 1945 [35]. After that, direct and inverse problems for Sturm-Liouville operator have been extended to so many different areas.

We consider a discontinuous Sturm-Liouville problem L with function $\rho(x)$

$$ly := -\rho(x)y''(x) + q(x)y(x) = \lambda y(x), \quad x \in [a, \delta_1] \cup (\delta_1, \delta_2) \cup (\delta_2, b] = \Omega, \tag{1}$$

where

$$\rho(x) = \begin{cases} \rho_1^{-2}, & a \leq x < \delta_1, \\ \rho_2^{-2}, & \delta_1 < x < \delta_2, \\ \rho_3^{-2}, & \delta_2 < x \leq b, \end{cases}$$

and $\rho_1, \rho_2,$ and ρ_3 are given positive real numbers; $q(x) \in L_2[\Omega, \mathbb{R}]$; $\lambda \in \mathbb{C}$ is a complex spectral parameter; boundary conditions at the endpoints

$$l_1y := \lambda(\theta'_1y(a) - \theta'_2y'(a)) - (\theta_1y(a) - \theta_2y'(a)) = 0, \tag{2}$$

$$l_2y := \lambda(\gamma'_1y(b) - \gamma'_2y'(b)) + (\gamma_1y(b) - \gamma_2y'(b)) = 0 \tag{3}$$

with discontinuity conditions at two points $x = \delta_1, x = \delta_2$

$$l_3y := y(\delta_1 + 0) - \theta_3y(\delta_1 - 0) - \gamma_3y'(\delta_1 - 0) = 0, \tag{4}$$

$$l_4y := y'(\delta_1 + 0) - \theta_4y(\delta_1 - 0) - \gamma_4y'(\delta_1 - 0) = 0, \tag{5}$$

$$l_5y := y(\delta_2 + 0) - \theta_5y(\delta_2 - 0) - \gamma_5y'(\delta_2 - 0) = 0, \tag{6}$$

$$l_6y := y'(\delta_2 + 0) - \theta_6y(\delta_2 - 0) - \gamma_6y'(\delta_2 - 0) = 0, \tag{7}$$

where θ_i, γ_i and θ'_j, γ'_j ($i = 1, 6, j = 1, 2$) are real numbers and

$$\alpha_1 = \begin{vmatrix} \theta_3 & \gamma_3 \\ \theta_4 & \gamma_4 \end{vmatrix} > 0, \quad \alpha_2 = \begin{vmatrix} \theta_5 & \gamma_5 \\ \theta_6 & \gamma_6 \end{vmatrix} > 0, \quad \beta_1 = \begin{vmatrix} \theta'_1 & \theta_1 \\ \theta'_2 & \theta_2 \end{vmatrix} > 0 \quad \text{and}$$

$$\beta_2 = \begin{vmatrix} \gamma'_1 & \gamma_1 \\ \gamma'_2 & \gamma_2 \end{vmatrix} > 0.$$

In the present paper, we construct a linear operator T in a suitable Hilbert space such that problem (1)-(7) and the eigenvalue problem for operator T coincide. We investigate eigenvalues and eigenfunctions together with their asymptotic behaviors of operator T . Besides, we study some uniqueness theorems according to Weyl function and spectral data, which are called eigenvalues and normalizing constants.

2 Operator formulation and spectral properties

We make known the inner product in the Hilbert space $H := H_1 \oplus \mathbb{C}^2$, where $H_1 = (L_2(\Omega), \langle \cdot, \cdot \rangle_1)$, \mathbb{C} denotes the Hilbert space of complex numbers and a self-adjoint operator T defined on H such that (1)-(7) can be dealt with as the eigenvalue problem of operator T . We define an inner product in H by

$$\begin{aligned} \langle F, G \rangle := & \alpha_1\alpha_2\rho_1^2 \int_a^{\delta_1} f(x)\bar{g}(x) dx + \alpha_2\rho_1^2 \int_{\delta_1}^{\delta_2} f(x)\bar{g}(x) dx + \rho_3^2 \int_{\delta_2}^b f(x)\bar{g}(x) dx \\ & + \frac{\alpha_1\alpha_2}{\beta_1}f_1\bar{g}_1 + \frac{1}{\beta_2}f_2\bar{g}_2 \end{aligned} \tag{8}$$

for

$$F = \begin{pmatrix} f(x) \\ f_1 \\ f_2 \end{pmatrix} \in H, \quad G = \begin{pmatrix} g(x) \\ g_1 \\ g_2 \end{pmatrix} \in H.$$

Consider the operator T defined by the domain

$$D(T) = \{F \in H : f(x), f'(x) \in AC_{loc}(\Omega), lf \in H_1, l_3 = l_4 = l_5 = l_6 = 0, f_1 = \theta_1'f(a) - \theta_2'f'(a) - \gamma_1'f(b) - \gamma_2'f'(b)\}$$

such that $TF := (lf, \theta_1f(a) - \theta_2f'(a), -(\gamma_1f(b) - \gamma_2f'(b)))$ for $F = (f, \theta_1'f(a) - \theta_2'f'(a), \gamma_1'f(b) - \gamma_2'f'(b)) \in D(T)$ and also l_3 - l_6 are satisfied for f .

Thus, we can rewrite the considered problem (1)-(7) in the operator form as $TF = \lambda F$.

Theorem 1 *The operator T is symmetric in H .*

Proof Let $F, G \in D(T)$. By two partial integrations, we get

$$\begin{aligned} \langle TF, G \rangle &= \langle F, TG \rangle + \alpha_1\alpha_2(W(f, \bar{g}, \delta_1 - 0) - W(f, \bar{g}, a)) \\ &\quad + \alpha_2(W(f, \bar{g}, \delta_2 - 0) - W(f, \bar{g}, \delta_1 + 0)) + W(f, \bar{g}, b) - W(f, \bar{g}, \delta_2 + 0) \\ &\quad + \frac{\alpha_1\alpha_2}{\beta_1}(\theta_1f(a) - \theta_2f'(a))(\theta_1'\bar{g}(a) - \theta_2'\bar{g}'(a)) \\ &\quad - \frac{1}{\beta_2}(\gamma_1f(b) - \gamma_2f'(b))(\gamma_1'\bar{g}(b) - \gamma_2'\bar{g}'(b)) \\ &\quad - \frac{\alpha_1\alpha_2}{\beta_1}(\theta_1\bar{g}(a) - \theta_2\bar{g}'(a))(\theta_1'f(a) - \theta_2'f'(a)) \\ &\quad + \frac{1}{\beta_2}(\gamma_1\bar{g}(b) - \gamma_2\bar{g}'(b))(\gamma_1'f(b) - \gamma_2'f'(b)), \end{aligned}$$

where by $W(f, g; x)$, we denote the Wronskian of the functions f and g as

$$f(x)g'(x) - f'(x)g(x).$$

Since f and g satisfy the boundary conditions (2)-(3) and transmission conditions (4)-(7), we obtain

$$\begin{aligned} &\frac{\alpha_1\alpha_2}{\beta_1}[(\theta_1f(a) - \theta_2f'(a))(\theta_1'\bar{g}(a) - \theta_2'\bar{g}'(a)) - (\theta_1\bar{g}(a) - \theta_2\bar{g}'(a))(\theta_1'f(a) - \theta_2'f'(a))] \\ &= \alpha_1\alpha_2 W(f, \bar{g}, a) \\ &\quad \times \frac{1}{\beta_2}[(\gamma_1\bar{g}(b) - \gamma_2\bar{g}'(b))(\gamma_1'f(b) - \gamma_2'f'(b)) - (\gamma_1f(b) - \gamma_2f'(b))(\gamma_1'\bar{g}(b) - \gamma_2'\bar{g}'(b))] \\ &= -W(f, \bar{g}, b)W(f, \bar{g}, \delta_i + 0) = \alpha_i W(f, \bar{g}, \delta_i - 0). \end{aligned}$$

Thus, we have $\langle TF, G \rangle = \langle F, TG \rangle$, i.e., T is symmetric. □

Lemma 1 *Problem (1)-(7) can be considered as the eigenvalue problem of the symmetric operator T .*

Corollary 1 *All eigenvalues and eigenfunctions of problem (1)-(7) are real, and two eigenfunctions $\varphi(x, \lambda_1)$ and $\varphi(x, \lambda_2)$, corresponding to different eigenvalues λ_1 and λ_2 , are orthogonal in the sense of*

$$\begin{aligned} & \alpha_1 \alpha_2 \rho_1^2 \int_a^{\delta_1} \varphi(x, \lambda_1) \varphi(x, \lambda_2) dx + \alpha_2 \rho_2^2 \int_{\delta_1}^{\delta_2} \varphi(x, \lambda_1) \varphi(x, \lambda_2) dx \\ & + \rho_3^2 \int_{\delta_2}^b \varphi(x, \lambda_1) \varphi(x, \lambda_2) dx \\ & + \frac{\alpha_1 \alpha_2}{\beta_1} (\theta_1' \varphi(a, \lambda_1) - \theta_2' \varphi'(a, \lambda_1)) (\theta_1' \varphi(a, \lambda_2) - \theta_2' \varphi'(a, \lambda_2)) \\ & + \frac{1}{\beta_2} (\gamma_1' \varphi(b, \lambda_1) - \gamma_2' \varphi'(b, \lambda_1)) (\gamma_1' \varphi(b, \lambda_2) - \gamma_2' \varphi'(b, \lambda_2)) = 0. \end{aligned}$$

We define the solutions

$$\varphi(x, \lambda) = \begin{cases} \varphi_1(x, \lambda), & x \in [a, \delta_1), \\ \varphi_2(x, \lambda), & x \in (\delta_1, \delta_2), \\ \varphi_3(x, \lambda), & x \in (\delta_2, b], \end{cases} \quad \psi(x, \lambda) = \begin{cases} \psi_1(x, \lambda), & x \in [a, \delta_1), \\ \psi_2(x, \lambda), & x \in (\delta_1, \delta_2), \\ \psi_3(x, \lambda), & x \in (\delta_2, b] \end{cases}$$

of equation (1) by the initial conditions

$$\begin{aligned} \varphi_1(a, \lambda) &= \lambda \theta_2' - \theta_2, & \varphi_1'(a, \lambda) &= \lambda \theta_1' - \theta_1, \\ \varphi_2(\delta_1, \lambda) &= \theta_3 \varphi_1(\delta_1, \lambda) + \gamma_3 \varphi_1'(\delta_1, \lambda), & \varphi_2'(\delta_1, \lambda) &= \theta_4 \varphi_1(\delta_1, \lambda) + \gamma_4 \varphi_1'(\delta_1, \lambda), \\ \varphi_3(\delta_2, \lambda) &= \theta_5 \varphi_2(\delta_2, \lambda) + \gamma_5 \varphi_2'(\delta_2, \lambda), & \varphi_3'(\delta_2, \lambda) &= \theta_6 \varphi_2(\delta_2, \lambda) + \gamma_6 \varphi_2'(\delta_2, \lambda), \end{aligned} \tag{9}$$

and similarly,

$$\begin{aligned} \psi_3(b, \lambda) &= \lambda \gamma_2' + \gamma_2, & \psi_3'(b, \lambda) &= \lambda \gamma_1' + \gamma_1, \\ \psi_2(\delta_2, \lambda) &= \frac{\gamma_6 \psi_3(\delta_2, \lambda) - \gamma_5 \psi_3'(\delta_2, \lambda)}{\alpha_2}, & \psi_2'(\delta_2, \lambda) &= \frac{\theta_6 \psi_3(\delta_2, \lambda) - \theta_5 \psi_3'(\delta_2, \lambda)}{-\alpha_2}, \\ \psi_1(\delta_1, \lambda) &= \frac{\gamma_4 \psi_2(\delta_1, \lambda) - \gamma_3 \psi_2'(\delta_1, \lambda)}{\alpha_1}, & \psi_1'(\delta_1, \lambda) &= \frac{\theta_4 \psi_2(\delta_1, \lambda) - \theta_3 \psi_2'(\delta_1, \lambda)}{-\alpha_1}, \end{aligned} \tag{10}$$

respectively.

These solutions are entire functions of λ for each fixed $x \in [a, b]$ and satisfy the relation

$$\psi(x, \lambda_n) = \kappa_n \varphi(x, \lambda_n)$$

for each eigenvalue λ_n , where

$$\kappa_n = \frac{\theta_2' \psi'(a, \lambda_n) - \theta_1' \psi(a, \lambda_n)}{\beta_1}.$$

Lemma 2 Let $\lambda = k^2$, $k = \sigma + i\omega$.

Then the following integral equations and also asymptotic behaviors hold for $\nu = 0, 1$:

$$\begin{aligned} \frac{d^\nu}{dx^\nu} \varphi_1(x, \lambda) &= (\lambda\theta'_2 - \theta_2) \frac{d^\nu}{dx^\nu} \cos k\rho_1(x - a) \\ &\quad + \frac{1}{k\rho_1} (\lambda\theta'_1 - \theta_1) \frac{d^\nu}{dx^\nu} \sin k\rho_1(x - a) \\ &\quad + \frac{\rho_1}{k} \int_a^x \frac{d^\nu}{dx^\nu} \sin k\rho_1(x - t) q(t) \varphi_1(t, \lambda) dt \\ &= (\lambda\theta'_2 - \theta_2) \frac{d^\nu}{dx^\nu} \cos k\rho_1(x - a) \\ &\quad + \frac{1}{k\rho_1} (\lambda\theta'_1 - \theta_1) \frac{d^\nu}{dx^\nu} \sin k\rho_1(x - a) + O(|k|^{\nu+1} e^{|\operatorname{Im}k|(x-a)\rho_1}), \\ \frac{d^\nu}{dx^\nu} \varphi_2(x, \lambda) &= (\theta_3\varphi_1(\delta_1, \lambda) + \gamma_3\varphi'_1(\delta_1, \lambda)) \frac{d^\nu}{dx^\nu} \cos k\rho_2(x - \delta_1) \\ &\quad + \frac{1}{k\rho_2} (\theta_4\varphi_1(\delta_1, \lambda) + \gamma_4\varphi'_1(\delta_1, \lambda)) \frac{d^\nu}{dx^\nu} \sin k\rho_2(x - \delta_1) \\ &\quad + \frac{\rho_2}{k} \int_{\delta_1}^x \frac{d^\nu}{dx^\nu} \sin k\rho_2(x - t) q(t) \varphi_2(t, \lambda) dt \\ &= -(\lambda\theta'_2 - \theta_2) \gamma_3 k \rho_1 \sin k\rho_1(\delta_1 - a) \frac{d^\nu}{dx^\nu} \cos k\rho_2(x - \delta_1) \\ &\quad + O(|k|^{\nu+2} e^{|\operatorname{Im}k|((\delta_1-a)\rho_1 + (x-\delta_1)\rho_2)}), \\ \frac{d^\nu}{dx^\nu} \varphi_3(x, \lambda) &= (\theta_5\varphi_2(\delta_2, \lambda) + \gamma_5\varphi'_2(\delta_2, \lambda)) \frac{d^\nu}{dx^\nu} \cos k\rho_3(x - \delta_2) \\ &\quad + \frac{1}{k\rho_3} (\theta_5\varphi_2(\delta_2, \lambda) + \gamma_6\varphi'_2(\delta_2, \lambda)) \frac{d^\nu}{dx^\nu} \sin k\rho_3(x - \delta_2) \\ &\quad + \frac{\rho_3}{k} \int_{\delta_2}^x \frac{d^\nu}{dx^\nu} \sin k\rho_3(x - t) q(t) \varphi_3(t, \lambda) dt \\ &= (\lambda\theta'_2 - \theta_2) \gamma_3 \gamma_5 k^2 \rho_1 \rho_2 \sin k\rho_1(\delta_1 - a) \sin k\rho_2(\delta_2 - \delta_1) \frac{d^\nu}{dx^\nu} \cos k\rho_3(x - \delta_2) \\ &\quad + O(|k|^{\nu+3} e^{|\operatorname{Im}k|((\delta_1-a)\rho_1 + (\delta_2-\delta_1)\rho_2 + (x-\delta_2)\rho_3)}). \end{aligned}$$

Lemma 3 Let $\lambda = k^2$, $k = \sigma + i\omega$.

Then the following integral equations and also asymptotic behaviors hold for $\nu = 0, 1$:

$$\begin{aligned} \frac{d^\nu}{dx^\nu} \psi_3(x, \lambda) &= (\lambda\gamma'_2 + \gamma_2) \frac{d^\nu}{dx^\nu} \cos k\rho_3(x - b) \\ &\quad + \frac{1}{k\rho_3} (\lambda\gamma'_1 + \gamma_1) \frac{d^\nu}{dx^\nu} \sin k\rho_3(x - b) \\ &\quad - \frac{\rho_3}{k} \int_x^b \frac{d^\nu}{dx^\nu} \sin k\rho_3(x - t) q(t) \psi_3(t, \lambda) dt \\ &= (\lambda\gamma'_2 + \gamma_2) \frac{d^\nu}{dx^\nu} \cos k\rho_3(x - b) \\ &\quad + \frac{1}{k\rho_3} (\lambda\gamma'_1 + \gamma_1) \frac{d^\nu}{dx^\nu} \sin k\rho_3(x - b) + O(|k|^{\nu+1} e^{|\operatorname{Im}k|(x-b)\rho_3}), \end{aligned}$$

$$\begin{aligned} \frac{d^v}{dx^v} \psi_2(x, \lambda) &= \frac{\gamma_6 \psi_3(\delta_2, \lambda) - \gamma_5 \psi_3'(\delta_2, \lambda)}{\alpha_2} \frac{d^v}{dx^v} \cos k \rho_2(x - \delta_2) \\ &\quad + \frac{1}{k \rho_2} \frac{\theta_6 \psi_3(\delta_2, \lambda) - \theta_5 \psi_3'(\delta_2, \lambda)}{-\alpha_2} \frac{d^v}{dx^v} \sin k \rho_2(x - \delta_2) \\ &\quad - \frac{\rho_2}{k} \int_x^{\delta_2} \frac{d^v}{dx^v} \sin k \rho_2(x - t) q(t) \psi_2(t, \lambda) dt \\ &= \left(\frac{\lambda \gamma_2' + \gamma_2}{\alpha_2} \right) \left(k \rho_3 \gamma_5 \sin k \rho_3(\delta_2 - b) \frac{d^v}{dx^v} \cos k \rho_2(x - \delta_2) \right) \\ &\quad + O(|k|^{v+2} e^{|\operatorname{Im} k|((\delta_2 - b) \rho_3 + (x - \delta_2) \rho_2)}), \\ \frac{d^v}{dx^v} \psi_1(x, \lambda) &= \frac{\gamma_4 \psi_2(\delta_1, \lambda) - \gamma_3 \psi_2'(\delta_1, \lambda)}{\alpha_1} \frac{d^v}{dx^v} \cos k \rho_1(x - \delta_1) \\ &\quad + \frac{1}{k \rho_1} \frac{\theta_4 \psi_2(\delta_1, \lambda) - \theta_3 \psi_2'(\delta_1, \lambda)}{-\alpha_1} \frac{d^v}{dx^v} \sin k \rho_1(x - \delta_1) \\ &\quad - \frac{\rho_1}{k} \int_x^{\delta_1} \frac{d^v}{dx^v} \sin k \rho_1(x - t) q(t) \psi_1(t, \lambda) dt \\ &= \left(\frac{\lambda \gamma_2' + \gamma_2}{\alpha_2 \alpha_1} \right) \\ &\quad \times \left(k^2 \rho_2 \rho_3 \gamma_3 \gamma_5 \sin k \rho_3(\delta_2 - b) \sin k \rho_2(\delta_1 - \delta_2) \frac{d^v}{dx^v} \cos k \rho_1(x - \delta_1) \right) \\ &\quad + O(|k|^{v+3} e^{|\operatorname{Im} k|((\delta_2 - b) \rho_3 + (\delta_1 - \delta_2) \rho_2 + (x - \delta_1) \rho_1)}). \end{aligned}$$

The function $\Delta(\lambda)$ is called the characteristic function, and numbers $\{\mu_n\}_{n \geq 1}$ are called the normalizing constants of problem (1)-(7) such that

$$\Delta(\lambda) = \lambda(\gamma_1' \varphi(b, \lambda) - \gamma_2' \varphi'(b, \lambda)) + (\gamma_1 \varphi(b, \lambda) - \gamma_2 \varphi'(b, \lambda)), \tag{11}$$

$$\begin{aligned} \mu_n := & \alpha_1 \alpha_2 \rho_1 \int_a^{\delta_1} \varphi^2(x, \lambda_n) dx + \alpha_2 \rho_2 \int_{\delta_1}^{\delta_2} \varphi^2(x, \lambda_n) dx + \rho_3 \int_{\delta_2}^b \varphi^2(x, \lambda_n) dx \\ & + \frac{\alpha_1 \alpha_2}{\beta_1} (\theta_1' \varphi(a, \lambda_n) - \theta_2' \varphi'(a, \lambda_n))^2 + \frac{1}{\beta_2} (\gamma_1' \varphi(b, \lambda_n) - \gamma_2' \varphi'(b, \lambda_n))^2. \end{aligned} \tag{12}$$

Lemma 4 The following equality holds for each eigenvalue λ_n

$$\alpha_1 \alpha_2 \dot{\Delta}(\lambda_n) = \kappa_n \mu_n.$$

Proof Since

$$-\rho(x) \psi''(x, \lambda) + q(x) \psi(x, \lambda) = \lambda \psi(x, \lambda), \quad -\rho(x) \varphi''(x, \lambda_n) + q(x) \varphi(x, \lambda_n) = \lambda \varphi(x, \lambda_n),$$

we get

$$\begin{aligned} & \varphi'(x, \lambda_n) \psi(x, \lambda) - \psi'(x, \lambda) \varphi(x, \lambda_n) (|a|^{\delta_1} + |\delta_1|^{\delta_2} + |b|^{\delta_2}) \\ &= (\lambda - \lambda_n) \rho_1^2 \int_a^{\delta_1} \psi(x, \lambda) \varphi(x, \lambda_n) dx \\ &\quad + (\lambda - \lambda_n) \rho_2^2 \int_{\delta_1}^{\delta_2} \psi(x, \lambda) \varphi(x, \lambda_n) dx + (\lambda - \lambda_n) \rho_3^2 \int_{\delta_2}^b \psi(x, \lambda) \varphi(x, \lambda_n) dx. \end{aligned}$$

After that, add and subtract $\Delta(\lambda)$ on the left-hand side of the last equality, and by using conditions (2)-(7), we obtain

$$\begin{aligned} & \Delta(\lambda) + (\lambda - \lambda_n)(\theta'_1 \psi(a, \lambda) - \theta'_2 \psi'(a, \lambda)) - (\lambda - \lambda_n)(\gamma'_1 \varphi(b, \lambda_n) - \gamma'_2 \varphi'(b, \lambda_n)) \\ & + (1 - \alpha_1)(\psi(\delta_1^-, \lambda) \varphi'(\delta_1^-, \lambda_n) - \varphi(\delta_1^-, \lambda_n) \psi'(\delta_1^-, \lambda)) \\ & + (1 - \alpha_2)(\psi(\delta_2^-, \lambda) \varphi'(\delta_2^-, \lambda_n) - \varphi(\delta_2^-, \lambda_n) \psi'(\delta_2^-, \lambda)) \\ & = (\lambda - \lambda_n) \rho_1^2 \int_a^{\delta_1} \psi(x, \lambda) \varphi(x, \lambda_n) dx + (\lambda - \lambda_n) \rho_2^2 \int_{\delta_1}^{\delta_2} \psi(x, \lambda) \varphi(x, \lambda_n) dx \\ & + (\lambda - \lambda_n) \rho_3^2 \int_{\delta_2}^b \psi(x, \lambda) \varphi(x, \lambda_n) dx, \end{aligned}$$

or

$$\begin{aligned} \frac{\alpha_1 \alpha_2 \Delta(\lambda)}{\lambda - \lambda_n} & = \alpha_1 \alpha_2 \rho_1^2 \int_a^{\delta_1} \psi(x, \lambda) \varphi(x, \lambda_n) dx + \alpha_2 \rho_2^2 \int_{\delta_1}^{\delta_2} \psi(x, \lambda) \varphi(x, \lambda_n) dx \\ & + \rho_3^2 \int_{\delta_2}^b \psi(x, \lambda) \varphi(x, \lambda_n) dx \\ & + \frac{\alpha_1 \alpha_2}{\beta_1} (\theta'_1 \psi(a, \lambda) - \theta'_2 \psi'(a, \lambda)) (\theta'_1 \varphi(a, \lambda_n) - \theta'_2 \varphi'(a, \lambda_n)) \\ & + \frac{1}{\beta_2} (\gamma'_1 \varphi(b, \lambda_n) - \gamma'_2 \varphi'(b, \lambda_n)) (\gamma'_1 \psi(b, \lambda) - \gamma'_2 \psi'(b, \lambda)). \end{aligned}$$

For $\lambda \rightarrow \lambda_n$, $\alpha_1 \alpha_2 \dot{\Delta}(\lambda_n) = \kappa_n \mu_n$ is obtained by using the equality

$$\psi(x, \lambda_n) = \kappa_n \varphi(x, \lambda_n)$$

and (12). □

Corollary 2 *The eigenvalues of problem L are simple.*

Lemma 5 [36] *Let $\{\alpha_i\}_{i=1}^p$ be the set of real numbers satisfying the inequalities $\alpha_0 > \alpha_0 > \dots > \alpha_{p-1} > 0$, and let $\{a_i\}_{i=1}^p$ be the set of complex numbers. If $a_p \neq 0$, then the roots of the equation*

$$e^{\alpha_0 \lambda} + a_1 e^{\alpha_1 \lambda} + \dots + a_{p-1} e^{\alpha_{p-1} \lambda} + a_p = 0$$

have the form

$$\lambda_n = \frac{2\pi ni}{\alpha_0} + \Psi(n) \quad (n = 0, \pm 1, \dots),$$

where $\Psi(n)$ is a bounded sequence.

Now, from Lemma 2 and (11), we can write

$$\Delta(\lambda) - \Delta_0(\lambda) = O(k^6 e^{|\operatorname{Im} k|((\delta_1 - a)\rho_1 + (\delta_2 - \delta_1)\rho_2 + (b - \delta_2)\rho_3)}),$$

where $\Delta_0(\lambda) = k^7 \theta'_2 \gamma'_3 \gamma'_5 \rho_1 \rho_2 \rho_3 \sin k \rho_1 (\delta_1 - a) \sin k \rho_2 (\delta_2 - \delta_1) \sin k \rho_3 (b - \delta_2)$.

We can see that non-zero roots, namely λ_n^0 of the equation $\Delta_0(\lambda) = 0$, are real and analytically simple.

Furthermore, it can be proved by using Lemma 5 that

$$\sqrt{\lambda_n^0} = \frac{n\pi}{(\delta_1 - a)\rho_1 + (\delta_2 - \delta_1)\rho_2 + (b - \delta_2)\rho_3} + \Psi_n, \quad \sup_n |\Psi_n| < \infty. \tag{13}$$

Theorem 2 *The eigenvalues $\{\lambda_n\}_{n \geq 0}$ have the following asymptotic behavior for sufficiently large n :*

$$\sqrt{\lambda_n} = \sqrt{\lambda_{n-4}^0} + o(1). \tag{14}$$

Proof Denote

$$G_n := \{ \lambda : k^2 = \lambda, |k| \leq |k_n^0| + \delta \},$$

where $k_n^0 = \sqrt{\lambda_n^0}$ and δ is a sufficiently small number. The relations

$$|\Delta_0(\lambda)| \geq C_\delta |k|^7 e^{|\operatorname{Im} k|((\delta_1 - a)\rho_1 + (\delta_2 - \delta_1)\rho_2 + (b - \delta_2)\rho_3)}$$

and

$$\Delta(\lambda) - \Delta_0(\lambda) = O(k^6 e^{|\operatorname{Im} k|((\delta_1 - a)\rho_1 + (\delta_2 - \delta_1)\rho_2 + (b - \delta_2)\rho_3)})$$

are valid for $\lambda \in \partial G_n$.

Then, by Rouché's theorem that the number of zeros of $\Delta_0(\lambda)$ coincides with the number of zeros of $\Delta(\lambda)$ in G_n , namely $n + 4$ zeros, $\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_{n+3}$. In the annulus, between G_n and G_{n+1} , $\Delta(\lambda)$ has accurately one positive zero, namely $k_n^2 : k_n = \sqrt{\lambda_n^0} + \delta_n$, for $n \geq 1$. So, it follows that $\lambda_{n+4} = k_n^2$. Applying to Rouché's theorem in $\eta_\varepsilon = \{k : |k - k_n^0| < \varepsilon\}$ for sufficiently small ε and sufficiently large n , we get $\delta_n = o(1)$. Finally, we obtain the asymptotic formula

$$\sqrt{\lambda_n} = \sqrt{\lambda_{n-4}^0} + o(1).$$

Denote

$$\Delta_i(\lambda) := W(\varphi_i, \psi_i, x) := \varphi_i \psi_i' - \varphi_i' \psi_i, \quad x \in \Omega_i \ (i = \overline{1, 3}),$$

which are independent of $x \in \Omega_i$ and are entire functions such that $\Omega_1 = [a, \delta_1)$, $\Omega_2 = (\delta_1, \delta_2)$, $\Omega_3 = (\delta_2, b]$.

It can be easily seen that

$$\Delta(\lambda) := \Delta_3(\lambda) = \alpha_2 \Delta_2(\lambda) = \alpha_1 \alpha_2 \Delta_1(\lambda). \tag{□}$$

Example Let $q = 0, a = 0, b = \pi, \delta_1 = \frac{\pi}{4}, \delta_2 = \frac{\pi}{2}, \theta_3 = \gamma_4 = 0, \gamma_3 = 1, \theta_4 = -1, \theta_5 = \gamma_6 = 0, \gamma_5 = 1, \theta_6 = -1, \gamma_2' = 1, \gamma_6 = -1, \gamma_1' = \gamma_2 = 0, \theta_2' = 1, \theta_1 = -1, \theta_1' = \theta_2 = 0$.

Since

$$\Delta(\lambda) = \rho_1 \rho_2 \rho_3 k^7 \sin k \rho_1 \delta_1 \sin k \rho_2 (\delta_2 - \delta_1) \sin k \rho_3 (\pi - \delta_2) + O(k^6 e^{|\operatorname{Im} k|(\delta_1 \rho_1 + (\delta_2 - \delta_1) \rho_2 + (\pi - \delta_2) \rho_3)}),$$

the eigenvalues of the boundary value problem (1)-(7) satisfy the following asymptotic formulae:

$$\begin{aligned} \sqrt{\lambda_{n_1}} = k_{n_1} &= \frac{4(n-4)}{\rho_1} + \varepsilon_n, & \sqrt{\lambda_{n_2}} = k_{n_2} &= \frac{4(n-4)}{\rho_2} + \varepsilon_n, \\ \sqrt{\lambda_{n_3}} = k_{n_3} &= \frac{2(n-4)}{\rho_3} + \varepsilon_n, \end{aligned}$$

where $\varepsilon_n = O(n^{-1})$.

3 Inverse problems

In this section, we study the inverse problems for the reconstruction of the boundary value problem (1)-(7) by Weyl function and spectral data.

We consider the boundary value problem \tilde{L} with the same form of L but with different coefficients $\tilde{q}(x), \tilde{\theta}_i, \tilde{\gamma}_i, \tilde{\delta}_j, \tilde{\theta}'_j, \tilde{\gamma}'_j, i = 1-6, j = 1, 2$.

If a certain symbol α denotes an object related to L , then the symbol $\tilde{\alpha}$ denotes the corresponding object related to \tilde{L} .

The Weyl function Let $\Phi(x, \lambda)$ be a solution of equation (1), which satisfies the condition $(\lambda \theta'_1 - \theta_1)\Phi(a, \lambda) - (\lambda \theta'_2 - \theta_2)\Phi(a, \lambda) = 1$ and transmissions (4)-(7).

Assume that the function $\chi(x, \lambda)$ is the solution of equation (1) that satisfies the conditions $\chi(a, \lambda) = \beta_1^{-1} \theta'_2, \chi'(a, \lambda) = \beta_1^{-1} \theta'_1$ and the transmission conditions (4)-(7).

Since $W[\chi, \varphi] = 1$, the functions χ and φ are linearly independent. Therefore, the function $\psi(x, \lambda)$ can be represented by

$$\psi(x, \lambda) = \frac{\theta'_2 \psi'(a, \lambda) - \theta'_1 \psi(a, \lambda)}{\beta_1} \varphi(x, \lambda) + \Delta(\lambda) \chi(x, \lambda)$$

or

$$\Phi(x, \lambda) = \frac{\psi(x, \lambda)}{\Delta(\lambda)} = \chi(x, \lambda) + \frac{\theta'_2 \psi'(a, \lambda) - \theta'_1 \psi(a, \lambda)}{\beta_1 \Delta(\lambda)} \varphi(x, \lambda) \tag{15}$$

that is called the Weyl solution, and

$$\frac{\theta'_2 \psi'(a, \lambda) - \theta'_1 \psi(a, \lambda)}{\beta_1 \Delta(\lambda)} = M(\lambda) \tag{16}$$

is called the Weyl function.

Theorem 3 *If $M(\lambda) = \tilde{M}(\lambda)$, then $L = \tilde{L}$, i.e., $q(x) = \tilde{q}(x)$, a.e. and $\theta_i = \tilde{\theta}_i, \gamma_i = \tilde{\gamma}_i, i = 1-6, \delta_j = \tilde{\delta}_j, \theta'_j = \tilde{\theta}'_j, \gamma'_j = \tilde{\gamma}'_j, j = 1, 2$.*

Proof We introduce a matrix $P(x, \lambda) = [P_{kj}(x, \lambda)]_{k,j=1,2}$ by the formula

$$P(x, \lambda) \begin{pmatrix} \tilde{\varphi} & \tilde{\Phi} \\ \tilde{\varphi}' & \tilde{\Phi}' \end{pmatrix} = \begin{pmatrix} \varphi & \Phi \\ \varphi' & \Phi' \end{pmatrix}$$

or

$$\begin{pmatrix} P_{11}(x, \lambda) & P_{12}(x, \lambda) \\ P_{21}(x, \lambda) & P_{22}(x, \lambda) \end{pmatrix} = \begin{pmatrix} -\varphi\tilde{\Phi}' + \Phi\tilde{\varphi}' & \varphi\tilde{\Phi} - \Phi\tilde{\varphi} \\ -\varphi'\tilde{\Phi}' + \tilde{\varphi}'\Phi' & \varphi'\tilde{\Phi} - \tilde{\varphi}\Phi' \end{pmatrix}, \tag{17}$$

where $\Phi(x, \lambda) = \frac{\psi(x, \lambda)}{\Delta(\lambda)}$.

$$P_{11}(x, \lambda) = \chi(x, \lambda)\tilde{\varphi}'(x, \lambda) - \varphi(x, \lambda)\tilde{\chi}'(x, \lambda) + (M(\lambda) - \tilde{M}(\lambda))\varphi(x, \lambda)\tilde{\varphi}'(x, \lambda),$$

$$P_{12}(x, \lambda) = \varphi(x, \lambda)\tilde{\chi}(x, \lambda) - \chi(x, \lambda)\tilde{\varphi}(x, \lambda) + (\tilde{M}(\lambda) - M(\lambda))\varphi(x, \lambda)\tilde{\varphi}(x, \lambda).$$

Thus, if $M(\lambda) \equiv \tilde{M}(\lambda)$, then the functions $P_{11}(x, \lambda)$ and $P_{12}(x, \lambda)$ are entire in λ for each fixed x .

Denote $G_w = \{\lambda : \lambda = k^2, |k - k_\delta| > w, \delta = 1, 2, \dots\}$ and $\tilde{G}_w = \{\lambda : \lambda = k^2, |k - \tilde{k}_\delta| > w, \delta = 1, 2, \dots\}$, where w is sufficiently small number, k_δ and \tilde{k}_δ are square roots of the eigenvalues of the problem L and \tilde{L} , respectively. It is easily shown that

$$\begin{aligned} \Phi_i^{(\nu)}(x, \lambda) &\leq C_w |k|^{v-(2+i)} e^{-|\operatorname{Im} \sqrt{\lambda}|((\delta_1-a)\rho_1 + (\delta_2-\delta_1)\rho_2 + (b-\delta_2)\rho_3)}, \\ x \in \Omega_i \ (i = 1, 2, 3), \nu = 0, 1 \end{aligned} \tag{18}$$

are valid for sufficiently large $|\lambda|$, where $\Omega_1 = [a, \delta_1)$, $\Omega_2 = (\delta_1, \delta_2)$ and $\Omega_3 = (\delta_2, b]$. Hence, Lemma 2 and (18) yield that

$$|P_{11}(x, \lambda)| \leq C_w, \quad |P_{12}(x, \lambda)| \leq C_w |k|^{-1} \quad \text{for } \lambda \in \Omega \text{ and for } \lambda \in G_w \cap \tilde{G}_w. \tag{19}$$

According to (19), and Liouville's theorem, $P_{11}(x, \lambda) = C(x)$ and $P_{12}(x, \lambda) \equiv 0$ for $x \in [a, b] \setminus \{\delta_1, \delta_2, \tilde{\delta}_1, \tilde{\delta}_2\}$. By virtue of (17), we get

$$\varphi(x, \lambda) = C(x)\tilde{\varphi}(x, \lambda), \quad \Phi(x, \lambda) = C(x)\tilde{\Phi}(x, \lambda). \tag{20}$$

It is obvious that

$$\begin{aligned} W[\Phi(x, \lambda), \varphi(x, \lambda)] &= \Phi(a, \lambda)(\theta_1' \lambda - \theta_1) - \Phi'(a, \lambda)(\theta_2' \lambda - \theta_2) \\ &= \frac{\psi(a, \lambda)(\theta_1' \lambda - \theta_1) - \psi'(a, \lambda)(\theta_2' \lambda - \theta_2)}{\Delta(\lambda)} \equiv 1, \end{aligned}$$

and similarly, $W[\tilde{\Phi}(x, \lambda), \tilde{\varphi}(x, \lambda)] \equiv 1$. Thus, we have $C^2(x) \equiv 1$.

Otherwise, the following asymptotic expressions hold

$$\begin{aligned} \varphi_1(x, \lambda) &= \tilde{\varphi}_1(x, \lambda) = \frac{\lambda}{2} e^{-ik(x-a)\rho_1} (1 + o(1)) \quad \text{for } x < \delta_1 \text{ and } x < \tilde{\delta}_1, \\ \varphi_2(x, \lambda) &= \tilde{\varphi}_2(x, \lambda) = -\frac{\lambda^{3/2}}{2} e^{-ik((\delta_1-a)\rho_1+(x-\delta_1)\rho_2)} (1 + o(1)) \\ &\quad \text{for } \delta_1 < x < \delta_2 \text{ and } \tilde{\delta}_1 < x < \tilde{\delta}_2, \\ \varphi_3(x, \lambda) &= \tilde{\varphi}_3(x, \lambda) = \frac{\lambda^2}{2} e^{-ik((\delta_1-a)\rho_1+(\delta_2-\delta_1)\rho_2+(x-\delta_2)\rho_3)} (1 + o(1)) \\ &\quad \text{for } \delta_2 < x \text{ and } \tilde{\delta}_2 < x. \end{aligned} \tag{21}$$

Without loss of generality, we assume that $\delta_1 < \tilde{\delta}_1$ and $\delta_2 < \tilde{\delta}_2$. From (20)-(21), we get $C(x) \equiv 1$ for $x \in [a, \delta_1) \cup (\tilde{\delta}_1, \delta_2) \cup (\delta_2, b]$ and also

$$\begin{aligned} \frac{1}{\lambda^{1/2}} (1 + o(1)) C(x) &= -e^{(x-\delta_1)\rho_2} (1 + o(1)) \quad \text{for } x \in (\delta_1, \tilde{\delta}_1), \\ \frac{1}{\lambda^{1/2}} e^{(x-\delta_1)\rho_2} (1 + o(1)) C(x) &= -e^{(\delta_2-\delta_1)\rho_2+(x-\delta_2)\rho_3} (1 + o(1)) \quad \text{for } x \in (\delta_2, \tilde{\delta}_2). \end{aligned} \tag{22}$$

As $|\lambda| \rightarrow \infty$ in (22), we contradict $C^2(x) \equiv 1$. Therefore, $\delta_1 = \tilde{\delta}_1$, $\delta_2 = \tilde{\delta}_2$. Thus, $\varphi(x, \lambda) \equiv \tilde{\varphi}(x, \lambda)$, $\Phi(x, \lambda) \equiv \tilde{\Phi}(x, \lambda)$ and $\frac{\psi'(x, \lambda)}{\psi(x, \lambda)} \equiv \frac{\tilde{\psi}'(x, \lambda)}{\tilde{\psi}(x, \lambda)}$. Hence, from equation (1) and transmission conditions (4)-(7), $q(x) = \tilde{q}(x)$, a.e., $\theta_i = \tilde{\theta}_i$, $\gamma_i = \tilde{\gamma}_i$, $i = 3 - 6$, and from (9) and (10), $\theta_i = \tilde{\theta}_i$, $\theta'_j = \tilde{\theta}'_j$, $\gamma'_j = \tilde{\gamma}'_j$, $i, j = 1, 2$. \square

Lemma 6 *The following representation holds*

$$M(\lambda) = \sum_{n=0}^{\infty} \frac{\alpha_1 \alpha_2}{\mu_n (\lambda_n - \lambda)}.$$

Proof Weyl function $M(\lambda)$ is a meromorphic function with respect to λ , which has simple poles at λ_n . Therefore, we calculate

$$\operatorname{Res}_{\lambda=\lambda_n} M(\lambda) = \frac{\theta'_2 \psi'(a, \lambda_n) - \theta'_1 \psi(a, \lambda_n)}{\beta_1 \dot{\Delta}(\lambda_n)}.$$

Since $\kappa_n = \frac{\theta'_2 \psi'(a, \lambda_n) - \theta'_1 \psi(a, \lambda_n)}{\beta_1}$ and $\dot{\Delta}(\lambda_n) = \frac{\kappa_n \mu_n}{\alpha_1 \alpha_2}$,

$$\operatorname{Res}_{\lambda=\lambda_n} M(\lambda) = \frac{\alpha_1 \alpha_2}{\mu_n}. \tag{23}$$

Let $\Gamma_N = \{\lambda : \lambda = k^2, |k| = \sqrt{\lambda_N} + \varepsilon\}$, where ε is a sufficiently small number. Consider the contour integral $I_N(\lambda) = \frac{1}{2\pi i} \int_{\Gamma_N} \frac{M(\mu)}{\mu - \lambda} d\mu$, $\lambda \in \operatorname{int} \Gamma_N$. For $\lambda \in G_w$,

$$\Delta(\lambda) \geq |\lambda|^{7/2} C_w e^{|\operatorname{Im} k|((\delta_1-a)\rho_1+(\delta_2-\delta_1)\rho_2+(b-\delta_2)\rho_3)}$$

satisfies. Using this equality and (16), we get

$$|M(\lambda)| \leq \frac{C_w}{|\lambda|} \quad \text{for } \lambda \in G_w.$$

Thus, $\lim_{N \rightarrow \infty} I_N(\lambda) = 0$. As a result, the residue theorem and (23) yield

$$M(\lambda) = \sum_{n=0}^{\infty} \frac{\alpha_1 \alpha_2}{\mu_n(\lambda_n - \lambda)}. \quad \square$$

Theorem 4 *If $\lambda_n = \tilde{\lambda}_n$ and $\mu_n = \tilde{\mu}_n$ for all n , then $L \equiv \tilde{L}$, i.e., $q(x) = \tilde{q}(x)$, a.e., $\theta_i = \tilde{\theta}_i$, $\gamma_i = \tilde{\gamma}_i$, $i = 1 - 6$, $\delta_j = \tilde{\delta}_j$, $\theta'_j = \tilde{\theta}'_j$, $\gamma'_j = \tilde{\gamma}'_j$, $j = 1, 2$. Hence, problem (1)-(7) is uniquely determined by spectral data $\{\lambda_n, \mu_n\}$.*

Proof If $\lambda_n = \tilde{\lambda}_n$ and $\mu_n = \tilde{\mu}_n$ for all n , then $M(\lambda) = \tilde{M}(\lambda)$ by Lemma 6. Therefore, we get $L = \tilde{L}$ by Theorem 3.

Let us consider the boundary value problem L_1 that we get the condition $\theta'_2 y'(a, \lambda) - \theta'_1 y(a, \lambda) = 0$ instead of condition (2) in L . Let $\{\tau_n\}_{n \geq 0}$ be the eigenvalues of the problem L_1 . It is clear that τ_n are zeros of

$$\Delta_1(\tau) := \theta'_2 \psi'(a, \tau) - \theta'_1 \psi(a, \tau). \quad \square$$

Theorem 5 *If $\lambda_n = \tilde{\lambda}_n$ and $\tau_n = \tilde{\tau}_n$ for all n , then $L(q, \delta_j, \theta_k, \gamma_i, \gamma'_j) = L(\tilde{q}, \tilde{\delta}_j, \tilde{\theta}_k, \tilde{\gamma}_i, \tilde{\gamma}'_j)$, $i = 1 - 6$, $k = 3 - 6$, $j = 1, 2$.*

Hence, the problem L is uniquely determined by the sequences $\{\lambda_n\}$ and $\{\tau_n\}$, except coefficients θ_j and θ'_j .

Proof Since the characteristic functions $\Delta(\lambda)$ and $\Delta_1(\tau)$ are entire of order $\frac{1}{2}$, functions $\Delta(\lambda)$ and $\Delta_1(\tau)$ are uniquely determined up to multiplicative constant with their zeros by Hadamard's factorization theorem [37]

$$\Delta(\lambda) = C \prod_{n=0}^{\infty} \left(1 - \frac{\lambda}{\lambda_n}\right),$$

$$\Delta_1(\tau) = C_1 \prod_{n=0}^{\infty} \left(1 - \frac{\tau}{\tau_n}\right),$$

where C and C_1 are constants dependent on $\{\lambda_n\}$ and $\{\tau_n\}$, respectively. Therefore, when $\lambda_n = \tilde{\lambda}_n$ and $\tau_n = \tilde{\tau}_n$ for all n , $\Delta(\lambda) \equiv \tilde{\Delta}(\lambda)$ and $\Delta_1(\tau) \equiv \tilde{\Delta}_1(\tau)$. Hence, $\theta'_2 \psi'(a, \tau) - \theta'_1 \psi(a, \tau) = \tilde{\theta}'_2 \tilde{\psi}'(a, \tau) - \tilde{\theta}'_1 \tilde{\psi}(a, \tau)$. As a result, we get $M(\lambda) = \tilde{M}(\lambda)$ by (16). So, the proof is completed by Theorem 3. \square

Competing interests

The author declares that he has no competing interests.

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