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Exponential and polynomial stabilization of the Kirchhoff string by nonlinear boundary control

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Abstract

This paper addresses the stabilization problem of the nonlinear Kirchhoff string using nonlinear boundary control. Nonlinear boundary control is the negative feedback of the transverse velocity of the string at one end, which satisfies a polynomial-type constraint. Employing the multiplier method, we establish explicit exponential and polynomial stability for the Kirchhoff string. The theoretical results are assured by numerical results of the asymptotic behavior for the system.

1 Introduction

Stabilization and vibration controllability of string or beam systems arising from different engineering backgrounds has attracted attention of many researchers [1–4]. In particular, boundary feedback stabilization of string and beam systems has become an important research area [5–7]. This is because, in a practice system, vibration is more easily controlled through a boundary point than using point sensors or actuators away from the boundaries [8, 9].

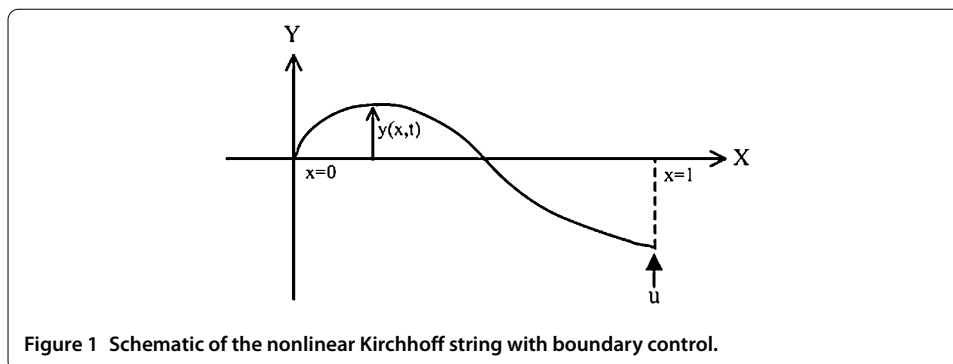
There are several nonlinear mathematical models that describe the transversal vibration of stretched strings. One such model is presented in the following equation:

$$y_{tt}(x, t) = \left[a + b \int_0^1 y_x^2(x, t) dx \right] y_{xx}(x, t) \quad (1)$$

for all $x \in (0, 1)$ and $t \geq 0$, where $a > 0$, $b \geq 0$ are two constants. Obviously, the above equation is a simple prototype of the classical equation

$$\rho h y_{tt}(x, t) = \left[T_0 + \frac{hE}{2l} \int_0^l y_x^2(x, t) dx \right] y_{xx}(x, t),$$

which was proposed by Kirchhoff [10]. Here l is the length of the string; E is Young's modulus of the material; ρ is density; h is the area of the cross section; $y(x, t)$ is the transversal displacement of the point x of the string at time t . This model has been studied by researchers from the physical and mathematical points of view; see, e.g., references [11–13] and the references therein.



In this paper, we consider Kirchhoff string (1) with the following boundary conditions (see Figure 1):

$$y(0, t) = 0, \quad (2)$$

$$T(t)y_x(1, t) = u(t) \quad (3)$$

for all $t \geq 0$. $T(t)$ denotes the tension in the string at time t . The boundary condition in equation (2) implies that the string is fixed at $x = 0$. The boundary condition in equation (3) represents the balance of the transversal component of the tension in the string and the control input u which is applied transversally at $x = 1$. Because the tension in the string represented by equation (1) is not constant and is given by

$$T(t) = a + b \int_0^1 y_x^2(x, t) dx \quad (4)$$

for all $t \geq 0$ (see [14]), the boundary condition in equation (3) can be written as

$$\left[a + b \int_0^1 y_x^2(x, t) dx \right] y_x(1, t) = u(t). \quad (5)$$

Shahrzad and Krishna [13] investigated the stabilization of Kirchhoff string (1) with a linear negative velocity control, which means the boundary control u has a linear negative velocity feedback form

$$u(t) := u(y_t(1, t)) = -Ly_t(1, t) \quad (6)$$

for all $t \geq 0$, where L is a positive constant. They established exponential stability. In [15], the absolute stability of the Kirchhoff string (1) with linear sector boundary control was considered. It is well known that linear strings represented by equation (1), for which $b = 0$, can be stabilized by the control law in equation (6); see *e.g.*, references [16–18]. Moreover, Shahrzad [19], Fung *et al.* [5], and Li and Hou [20] developed linear boundary control laws for axially moving strings. It is worth mentioning that Kobayashi [21] designed a linear parallel compensator based on boundary displacement observer and proved the string (1) can be stabilized by parallel compensator control.

In the literature mentioned above, such as [13, 16] and [20], the exponential stabilization result for various string systems by linear boundary control mainly relies on the Lyapunov

direct method. In this work, we investigate the stabilization of string (1) with a more general and ‘flexible’ boundary control (see hypothesis (H) in Section 2). The feedback function u is not required to satisfy a strict control law such as (6), but just satisfies some appropriate polynomial-type constraint. In this general boundary control case, it seems that the Lyapunov direct method is no more applicable. So, we need to use a more meticulous method to deal with the stabilization problem. Applying the multiplier method, we establish not only exponential stability result but also polynomial stability result for Kirchhoff string (1).

The remainder of this technical paper is arranged as follows. Section 2 describes the model of the Kirchhoff nonlinear string and introduces the control assumption. The problem of exponential and polynomial stability is addressed in Section 3. Finally, a numerical example is demonstrated where the nonlinear distributed parameter infinite-dimensional equation is solved by applying the finite element method in Section 4.

2 Problem formulation

Consider the nonlinear Kirchhoff string model as shown in Figure 1. For the sake of easy reading and later referring, the governing equation, the boundary conditions and the initial functions are put together as

$$\begin{cases} y_{tt}(x, t) = [a + b \int_0^1 y_x^2(x, t) dx] y_{xx}(x, t), & (7a) \\ y(0, t) = 0, & (7b) \\ [a + b \int_0^1 y_x^2(x, t) dx] y_x(1, t) = u(y_t(1, t)), & (7c) \\ y(x, 0) = f(x), \quad y_t(x, 0) = g(x) & (7d) \end{cases}$$

for all $x \in (0, 1)$ and $t \geq 0$. Here $f(x)$ and $g(x)$ in equation (7d) are the initial displacement and velocity of the string, respectively. We assume that $f \in C^1[0, 1]$, $g \in C[0, 1]$ and that at least one of the functions f or g is not identically zero over $[0, 1]$.

To obtain a precise stabilization result, we make the following hypothesis on the continuous control feedback $u : R \rightarrow R$:

(H) There exist constants $L_2 \geq L_1 > 0$ and $r \geq 1$ such that

$$u(w) \in \overline{\Omega}(w) := \begin{cases} -\text{Co}\{L_1 w, L_2 w\}, & |w| > 1, \\ -\text{Co}\{L_1 w^r, L_2 w^{\frac{1}{r}}\}, & |w| \leq 1. \end{cases} \quad (8)$$

Remark 2.1 Obviously, condition (H) is equivalent to, for all $w \in R$,

$$\begin{cases} u(w)w \leq 0, \\ L_1 \min\{|w|, |w|^r\} \leq |u(w)| \leq L_2 \max\{|w|, |w|^{\frac{1}{r}}\}. \end{cases}$$

Obviously, hypothesis (H) is a ‘flexible’ and ‘robust’ condition, which allows the feedback function u to vary in an appropriate geometric region given by a polynomial-type constraint. For example, Figure 2 illustrates a feedback control u satisfying a linear sector constraint (H) with $r = 1$, $L_1 = 1$, $L_2 = 2$, and Figure 3 shows a feedback control u satisfying a nonlinear constraint (H) with $r = 2$, $L_1 = 1$, $L_2 = 2$. Since $u(y_t(1, t))y_t(1, t) \leq 0$, for all $t \geq 0$, the boundary control (7c) is the negative feedback of transversal velocity of the string at $x = 1$.

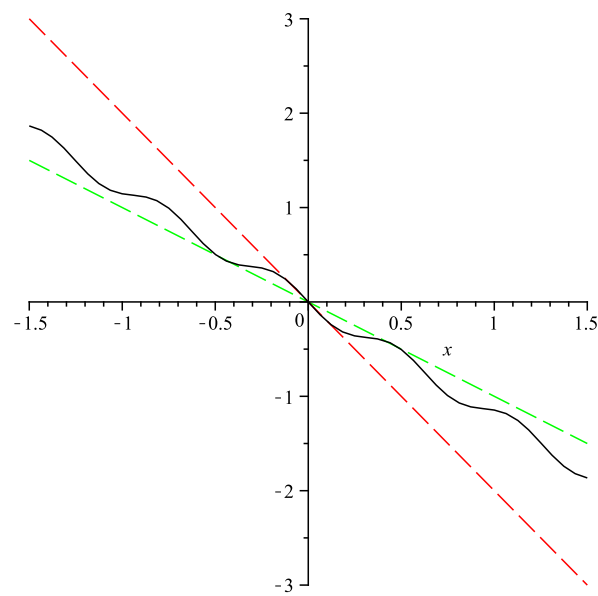


Figure 2 Linear sector constraint.

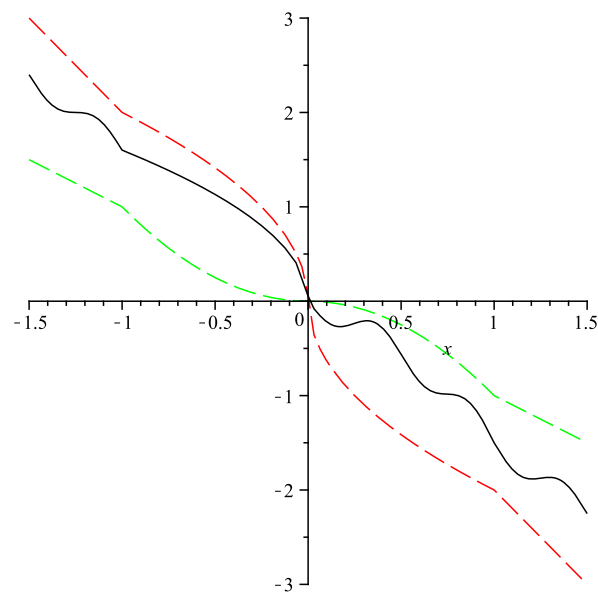


Figure 3 Nonlinear sector constraint.

For the existence and uniqueness of the solution of the general Kirchhoff equation, we refer to [11, 12] and references therein. In this work, we study the stabilization of the string in (7a) by this negative feedback boundary control u , which provides a dissipative effect.

Remark 2.2 According to boundary condition (7b) at $x = 0$, we easily get

$$y_t(0, t) = 0 \quad \text{for all } t \geq 0. \quad (9)$$

We define the natural energy function of time for system (7a)-(7d) and denote it by $t \rightarrow E(t)$. The scalar-valued function E is defined as

$$E(t) := \frac{1}{2} \int_0^1 y_t^2(x, t) dx + \frac{1}{2} \int_0^1 a y_x^2(x, t) dx + \frac{b}{4} \left[\int_0^1 y_x^2(x, t) dx \right]^2. \quad (10)$$

Especially, from the initial displacement and velocity condition (7d), we obtain the initial energy as

$$E(0) = \frac{1}{2} \int_0^1 [g^2(x) + a f_x^2(x)] dx + \frac{b}{4} \left[\int_0^1 f_x^2(x) dx \right]^2.$$

Since at least one of the functions f and g is not identically zero over $(0, 1)$ we have $E(0) > 0$.

3 Stabilization by boundary control

In this section we state and prove our main result. For this purpose we establish several lemmas.

Lemma 3.1 *Let $y(\cdot, \cdot)$ be the solution for system (7a)-(7d). Then*

$$\int_0^1 y_x y_{xt} dx = y_x(1, t) y_t(1, t) - \int_0^1 y_{xx} y_t dx, \quad (11)$$

$$\int_0^1 x y_x y_{xx} dx = \frac{1}{2} y_x^2(1, t) - \frac{1}{2} \int_0^1 y_x^2(x, t) dx, \quad (12)$$

$$\int_0^1 x y_x y_{tt} dx = \int_0^1 \left([x y_x y_t]_t + \frac{1}{2} y_t^2 \right) dx - \frac{1}{2} y_t^2(1, t). \quad (13)$$

Proof See the Appendix. □

Now, we give a property of the energy function E .

Proposition 3.1 *The time-derivative of the energy function E in equation (10), along the solution of system (7a)-(7d) satisfies*

$$E'(t) = u(t) y_t(1, t) \leq 0 \quad (14)$$

for all $t \geq 0$.

Proof Differentiating the energy function (10) with respect to t , we get

$$E'(t) = \int_0^1 y_t y_{tt} dx + \left[a + b \int_0^1 y_x^2 dx \right] \int_0^1 y_x y_{xt} dx. \quad (15)$$

According to equation (11) and boundary control (7c), we get

$$\begin{aligned} & \left[a + b \int_0^1 y_x^2 dx \right] \int_0^1 y_x y_{xt} dx \\ &= u(t) y_t(1, t) - \left[a + b \int_0^1 y_x^2 dx \right] \int_0^1 y_{xx} y_t dx. \end{aligned} \quad (16)$$

Substituting equation (16) into equation (15) and observing (7a), we obtain

$$\begin{aligned} E'(t) &= u(t)y_t(1, t) + \int_0^1 y_{tt}y_t dx - \left[a + b \int_0^1 y_x^2 dx \right] \int_0^1 y_{xx}y_t dx \\ &= u(t)y_t(1, t) \end{aligned} \quad (17)$$

for all $t \geq 0$. We obtain equation (14). \square

Remark 3.1 From Proposition 3.1, we obtain the energy identity for system (7a)-(7d),

$$E(S) - E(T) = \int_T^S u(t)y_t(1, t) dt.$$

Therefore, the energy E is a decreasing function of time.

During the subsequent stability analysis, we utilize the following inequality.

Lemma 3.2 *Let $y(\cdot, \cdot)$ be the solution for system (7a)-(7d). Then*

$$\int_0^1 xy_x(x, t)y_t(x, t) dx \leq \max\left\{1, \frac{1}{a}\right\} E(t) \quad (18)$$

for all $t \geq 0$.

Proof Applying the Cauchy-Schwarz inequality, we get

$$\begin{aligned} \int_0^1 xy_x(x, t)y_t(x, t) dx &\leq \frac{1}{2} \int_0^1 x^2 y_x^2 dx + \frac{1}{2} \int_0^1 y_t^2 dx \\ &\leq \frac{1}{2} \int_0^1 y_x^2 dx + \frac{1}{2} \int_0^1 y_t^2 dx \end{aligned} \quad (19)$$

for all $t \geq 0$. On the other hand, the definition of energy function (10) implies

$$\frac{a}{2} \int_0^1 y_x^2 dx + \frac{1}{2} \int_0^1 y_t^2 dx \leq E(t)$$

for all $t \geq 0$. It follows from the above inequality that

$$\frac{1}{2} \int_0^1 y_x^2 dx + \frac{1}{2} \int_0^1 y_t^2 dx \leq \begin{cases} E(t), & a \geq 1, \\ \frac{1}{a} E(t), & 0 < a < 1. \end{cases} \quad (20)$$

Together with (19) and (20), we get equation (18). Hence we complete the proof of Lemma 3.2. \square

Now, we present a Gronwall-type lemma (see Komornik [22], pp.124), which will play an essential role when establishing the stabilization result.

Lemma 3.3 *Let $G : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a non-increasing function. Assume that there exists a constant $\omega > 0$ such that*

$$\int_T^{+\infty} G^{q+1}(t) dt \leq \frac{1}{\omega} G(T) \quad \text{for all } T \geq 0.$$

Then the following estimation is true, for all $t \geq 0$,

$$\begin{cases} G(t) \leq G(0)e^{1-\omega t} & \text{if } q = 0, \\ G(t) \leq G(0)\left(\frac{1+q}{q\omega t}\right)^{\frac{1}{q}} & \text{if } q > 0. \end{cases}$$

We give *a priori* estimation for the energy function $E(t)$, which was established in [15]. For the sake of completeness, we give the proof here.

Lemma 3.4 *The energy function E in equation (10), along the solution of system (7a)-(7d), satisfies*

$$E(t) \leq - \int_0^1 [xy_x y_t]_t dx + \frac{1}{2} y_t^2(1, t) + \frac{1}{2a} u^2(y_t(1, t)) \quad (21)$$

for all $t \geq 0$.

Proof We multiply equation (7a) by $xy_x(x, t)$ and do integration over $[0, 1]$, with respect to x . We obtain

$$\begin{aligned} 0 &= \int_0^1 xy_x \left(y_{tt} - \left[a + b \int_0^1 y_x^2 dx \right] y_{xx} \right) dx \\ &= \int_0^1 \left([xy_x y_t]_t + \frac{1}{2} y_t^2 \right) dx - \frac{1}{2} y_t^2(1, t) \\ &\quad - \frac{1}{2} \left[a + b \int_0^1 y_x^2 dx \right] \left(y_x^2(1, t) - \int_0^1 y_x^2 dx \right), \end{aligned} \quad (22)$$

using equations (13) and (12) in Lemma 3.1. It follows from (10) that

$$\begin{aligned} &\frac{1}{2} \int_0^1 y_t^2 dx + \frac{1}{2} \left[a + b \int_0^1 y_x^2 dx \right] \int_0^1 y_x^2 dx \\ &= E(t) + \frac{b}{4} \left[\int_0^1 y_x^2 dx \right]^2. \end{aligned} \quad (23)$$

Since $a + b \int_0^1 y_x^2 dx \geq a$, according to boundary control (7c), we have

$$\left[a + b \int_0^1 y_x^2 dx \right] y_x^2(1, t) \leq \frac{1}{a} u^2(y_t(1, t)). \quad (24)$$

Hence, substituting equation (23) into equation (22) and using equation (24), one has

$$\begin{aligned} E(t) &+ \frac{b}{4} \left[\int_0^1 y_x^2 dx \right]^2 \\ &= - \int_0^1 [xy_x y_t]_t dx + \frac{1}{2} y_t^2(1, t) + \frac{1}{2} \left[a + b \int_0^1 y_x^2 dx \right] y_x^2(1, t) \\ &\leq - \int_0^1 [xy_x y_t]_t dx + \frac{1}{2} y_t^2(1, t) + \frac{1}{2a} u^2(y_t(1, t)). \end{aligned}$$

Since $\frac{b}{4} \left[\int_0^1 y_x^2 dx \right]^2 \geq 0$, we complete the proof of Lemma 3.4. \square

Lemma 3.5 For any constant $q \geq 0$, the energy function E along the solution of system (7a)-(7d) satisfies the following estimation, for all $S > T \geq 0$,

$$\begin{aligned} \int_T^S E^{q+1}(t) dt &\leq C_1 E^{q+1}(T) + \frac{1}{2} \int_T^S E^q(t) y_t^2(1, t) dt \\ &\quad + \frac{1}{2a} \int_T^S E^q(t) u^2(y_t(1, t)) dt, \end{aligned} \quad (25)$$

where $C_1 = \frac{3q+2}{q+1} \max\{1, \frac{1}{a}\}$.

Proof According to inequality (21) in Lemma 3.4, we have, for all $0 \leq T < S$,

$$\begin{aligned} \int_T^S E^{q+1}(t) dt &\leq - \int_T^S E^q(t) \int_0^1 [xy_x y_t]_t dx dt \\ &\quad + \frac{1}{2} \int_T^S E^q(t) y_t^2(1, t) dt + \frac{1}{2a} \int_T^S E^q(t) u^2(y_t(1, t)) dt. \end{aligned} \quad (26)$$

Moreover, using integration by parts, we get

$$\begin{aligned} \int_T^S E^q(t) \int_0^1 [xy_x y_t]_t dx dt &= \int_T^S \left[E^q(t) \int_0^1 xy_x y_t dx \right]_t dt \\ &\quad - \int_T^S qE'(t)E^{q-1}(t) \int_0^1 (xy_x y_t) dx dt. \end{aligned}$$

Hence, inequality (26) becomes

$$\begin{aligned} \int_T^S E^{q+1}(t) dt &\leq A_1 + A_2 + \frac{1}{2} \int_T^S E^q(t) y_t^2(1, t) dt \\ &\quad + \frac{1}{2a} \int_T^S E^q(t) u^2(y_t(1, t)) dt, \end{aligned} \quad (27)$$

where

$$\begin{aligned} A_1 &:= - \int_T^S \left[E^q(t) \int_0^1 xy_x y_t dx \right]_t dt, \\ A_2 &:= \int_T^S qE'(t)E^{q-1}(t) \int_0^1 (xy_x y_t) dx dt. \end{aligned}$$

Firstly, we estimate A_1 and A_2 ,

$$\begin{aligned} A_1 &= - \int_T^S \left[E^q(t) \int_0^1 xy_x y_t dx \right]_t dt \\ &\leq \left| E^q(T) \int_0^1 xy_x(x, T) y_t(x, T) dx \right| + \left| E^q(S) \int_0^1 xy_x(x, S) y_t(x, S) dx \right| \quad (\text{by (18)}) \\ &\leq \max\left\{1, \frac{1}{a}\right\} E^{q+1}(T) + \max\left\{1, \frac{1}{a}\right\} E^{q+1}(S) \\ &\leq 2 \max\left\{1, \frac{1}{a}\right\} E^{q+1}(T), \end{aligned} \quad (28)$$

where the last inequality follows from the fact $E(t)$ is a decreasing function. On the other hand,

$$\begin{aligned} A_2 &= \int_T^S qE'(t)E^{q-1}(t) \int_0^1 xy_x y_t dx dt \\ &\leq \int_T^S |qE'(t)E^{q-1}(t)| \left| \int_0^1 xy_x y_t dx \right| dt \\ &\leq \max \left\{ 1, \frac{1}{a} \right\} \left| \int_T^S qE'(t)E^q(t) dt \right| \quad (\text{by (18)}) \\ &= \frac{q}{q+1} \max \left\{ 1, \frac{1}{a} \right\} |E^{q+1}(S) - E^{q+1}(T)| \\ &\leq \frac{q}{q+1} \max \left\{ 1, \frac{1}{a} \right\} E^{q+1}(T). \end{aligned} \quad (29)$$

Finally, inserting the two inequalities, (28) and (29), in (27), we get inequality (25). This completes the proof of Lemma 3.5. \square

We now state the main stabilization result for system (7a)-(7d).

Theorem 3.1 Assume that assumption (H) holds. Then there exist three constants $k_1, k_2, \sigma > 0$ such that, for all $t \geq 0$,

$$\begin{cases} E(t) \leq k_1 e^{-\sigma t} & \text{if } r = 1, \\ E(t) \leq k_2 t^{-\frac{2}{r-1}} & \text{if } r > 1. \end{cases} \quad (30)$$

Remark 3.2 It is worth to mention that Theorem 2.4 in [13] can be viewed as a special cases of Theorem 3.1. Indeed, in the linear control case (6), the exponential stability in Theorem 3.1 coincides with the result in [13].

Proof of Theorem 3.1 We distinguish two cases related to the parameter r to establish the energy decay rate.

Case (I): $r = 1$;

Case (II): $r > 1$.

In Case (I), we choose $q = 0$. According to hypothesis (H), we know that

$$s^2 \leq -\frac{1}{L_1} u(s)s, \quad u^2(s) \leq -L_2 u(s)s \quad \text{for all } s \in R.$$

Hence, from inequality (25) and equation (14), we deduce that, for all $S > T \geq 0$,

$$\begin{aligned} \int_T^S E(t) dt &\leq C_1 E(T) + \frac{1}{2} \int_T^S \left[y_t^2(1, t) + \frac{1}{a} u^2(y_t(1, t)) \right] dt \\ &\leq C_1 E(T) - \frac{a + L_1 L_2}{2aL_1} \int_T^S u(t) y_t(1, t) dt \\ &= C_1 E(T) + \frac{a + L_1 L_2}{2aL_1} \int_T^S -E'(t) dt \\ &\leq C_2 E(T), \end{aligned} \quad (31)$$

where $C_2 = C_1 + \frac{1}{2}(\frac{1}{L_1} + \frac{L_2}{a})$ and C_1 is given in Lemma 3.5.

Now we deal with Case (II). In this case, we choose $q = \frac{r-1}{2} > 0$. We first admit the following fact (the proof is given in the Appendix).

Claim 1 For any $\delta > 0$, we have the following estimates, for all $S > T \geq 0$,

$$\begin{aligned} \int_T^S E^q(t) y_t^2(1, t) dt &\leq \frac{r-1}{r+1} \delta^{\frac{r+1}{r-1}} \int_T^S E^{q\frac{r+1}{r-1}}(t) dt \\ &\quad + \frac{2\delta^{-\frac{r+1}{2}}}{(r+1)L_1} E(T) + \frac{1}{(q+1)L_1} E^{q+1}(T), \end{aligned} \quad (32)$$

$$\begin{aligned} \int_T^S E^q(t) u^2(y_t(1, t)) dt &\leq \frac{r-1}{r+1} \delta^{\frac{r+1}{r-1}} \int_T^S E^{q\frac{r+1}{r-1}}(t) dt \\ &\quad + \frac{2L_2^r \delta^{-\frac{r+1}{2}}}{(r+1)} E(T) + \frac{L_2}{(q+1)} E^{q+1}(T). \end{aligned} \quad (33)$$

Now, inserting inequalities (32) and (33) into (25), we obtain, for all $S > T \geq 0$,

$$\begin{aligned} \int_T^S E^{q+1}(t) dt &\leq \left(\frac{1}{2} + \frac{1}{2a} \right) \frac{r-1}{r+1} \delta^{\frac{r+1}{r-1}} \int_T^S E^{q\frac{r+1}{r-1}}(t) dt \\ &\quad + C_3 E^{q+1}(T) + C_4 E(T), \end{aligned} \quad (34)$$

where $C_3 = C_1 + \frac{a+L_1L_2}{2a(q+1)L_1}$, $C_4 = \frac{1}{(r+1)L_1} \delta^{-\frac{r+1}{2}} + \frac{L_2^r}{(r+1)a} \delta^{-\frac{r+1}{2}}$ and C_1 is given in Lemma 3.5. Now we choose $\delta = [(1 + \frac{1}{a})^{\frac{r-1}{r+1}}]^{-\frac{r-1}{r+1}}$. Then it is obvious that $(1 + \frac{1}{a})^{\frac{r-1}{r+1}} \delta^{\frac{r+1}{r-1}} = 1$. Hence, inequality (34) becomes

$$\int_T^S E^{q+1}(t) dt \leq \frac{1}{2} \int_T^S E^{q\frac{r+1}{r-1}}(t) dt + C_3 E^{q+1}(T) + C_4 E(T).$$

Recalling $q = \frac{r-1}{2}$, we get $q+1 = q\frac{r+1}{r-1} = \frac{r+1}{2}$. Hence, the above inequality is rewritten as

$$\begin{aligned} \int_T^S E^{\frac{r+1}{2}}(t) dt &\leq 2C_3 E^{\frac{r+1}{2}}(T) + 2C_4 E(T) \\ &\leq 2(C_3 E^{\frac{r-1}{2}}(0) + C_4) E(T), \end{aligned} \quad (35)$$

where the last inequality follows from Remark 3.1. Finally, by letting $S \rightarrow +\infty$ in (31), (35) and using Lemma 3.3 with $G(t) = E(t)$, we complete the proof of Theorem 3.1. \square

Remark 3.3 According to the proof of Theorem 3.1, it is easy to see that the constants σ , k_1 and k_2 in Theorem 3.1 can be chosen as, respectively, $\sigma^{-1} = 2 \max\{1, \frac{1}{a}\} + \frac{1}{2}(\frac{1}{L_1} + \frac{L_2}{a})$, $k_1 = eE(0)$, and $k_2 = E(0)[2\frac{r+1}{r-1}(C_3 E^{\frac{r-1}{2}}(0) + C_4)]^{\frac{2}{r-1}}$ with $C_3 = \frac{3r-1}{r+1} \max\{1, \frac{1}{a}\} + \frac{a+L_1L_2}{a(r+1)L_1}$, $C_4 = \frac{1}{r+1}(\frac{1}{L_1} + \frac{L_2^r}{a})[(1 + \frac{1}{a})^{\frac{r-1}{r+1}}]^{\frac{r-1}{2}}$. This means that the coefficients of the exponential or polynomial decay rate are exactly determined only by the initial tension a , the initial energy $E(0)$ and the feedback control u . However, in the polynomial decay case, the order of decay rate is determined only by the feedback control u .

Finally, it is shown that the boundary control u stabilizes the nonlinear Kirchhoff string.

Theorem 3.2 Assume that assumption (H) holds. Then there exist two constants $k'_1, k'_2 > 0$ such that for all $x \in (0, 1)$ and $t \geq 0$,

$$\begin{cases} |y(x, t)| \leq k'_1 e^{-\frac{\sigma t}{2}} & \text{if } r = 1, \\ |y(x, t)| \leq k'_2 t^{-\frac{1}{r-1}} & \text{if } r > 1, \end{cases} \quad (36)$$

where $k'_1 = \sqrt{\frac{2k_1}{a}}$, $k'_2 = \sqrt{\frac{2k_2}{a}}$ and k_1, k_2, σ are given in Theorem 3.1.

Proof According to the fact that $y_x(0, t) = 0$, for all $t \geq 0$, we get

$$\begin{aligned} |y(x, t)| &= \left| \int_0^x y_x(z, t) dz \right| \leq \int_0^1 |y_x(z, t)| dz \\ &\leq \left(\int_0^1 y_x^2(x, t) dx \right)^{\frac{1}{2}} \quad (\text{by the Cauchy inequality}) \\ &\leq \sqrt{\frac{2E(t)}{a}} \quad (\text{by equation (14)}) \end{aligned} \quad (37)$$

for all $x \in (0, 1)$ and $t \geq 0$. By combining (37) with (30) in Theorem 3.1, we complete the proof of Theorem 3.2. \square

4 Numerical results

In this section we consider a computational example for the closed-loop system (7a)-(7d). To illustrate the control performance of the boundary control law satisfying condition (H), numerical simulations by using the finite element method (FEM) are performed. We use Lagrange 'hat' basis with FEM equidistant meshes. The system parameters used in the simulations are $a = 2$, $b = 3$. The initial conditions are $f(x) = 0.6 \sin(5x)$ and $g(x) = 0.5 \sin(3x)$. That is we consider the following Kirchhoff system:

$$\begin{cases} y_{tt}(x, t) = [2 + 3 \int_0^1 y_x^2(x, t) dx] y_{xx}(x, t), \\ y(0, t) = 0, \\ [a + b \int_0^1 y_x^2(x, t) dx] y_x(1, t) = u(y_t(1, t)), \\ y(x, 0) = 0.6 \sin(5x), \quad y_t(x, 0) = 0.5 \sin(3x). \end{cases} \quad (38)$$

The dynamic responses of the controlled Kirchhoff string were simulated under two feedback control laws:

$$u_1(x) = -2x - \sin x, \quad x \in R$$

and

$$u_2(x) = \begin{cases} -\sqrt{|x|}, & |x| < 1, \\ -x, & |x| \geq 1. \end{cases}$$

Obviously, the feedback control function u_1 satisfies the constraint condition (H) with $r = 1$, $L_1 = 1$, $L_2 = 3$, and u_2 satisfies the constraint condition (H) with $r = 2$, $L_1 = L_2 = 1$. Then, according to Theorem 3.2, the asymptotic behavior of the transverse vibration of system (38) under control law u_1 (or control law u_2) possesses exponential decay (or poly-

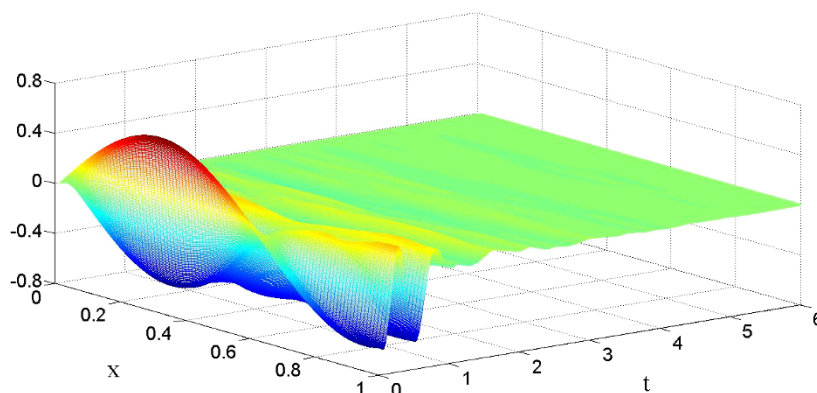


Figure 4 String response of closed-loop system (38) under control law u_1 .

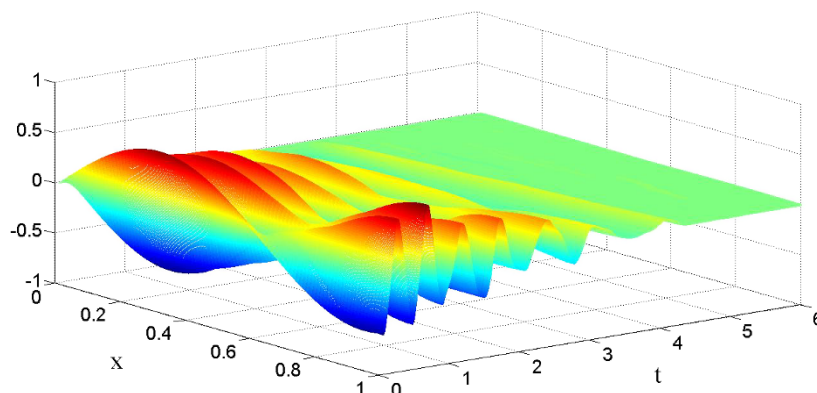


Figure 5 String response of closed-loop system (38) under control law u_2 .

nomial decay with degree -1 , because $r = 2$). The string response $y(x, t)$ of closed-loop system (38) with control law u_1 and control law u_2 are shown in Figure 4 and Figure 5, respectively. The corresponding transversal displacement at $x = 0.5$ is shown in Figures 6 and 7, respectively.

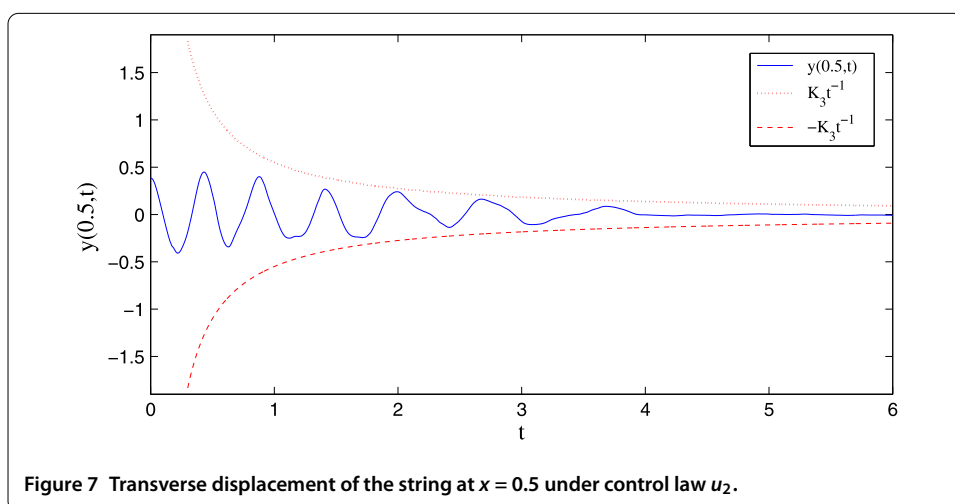
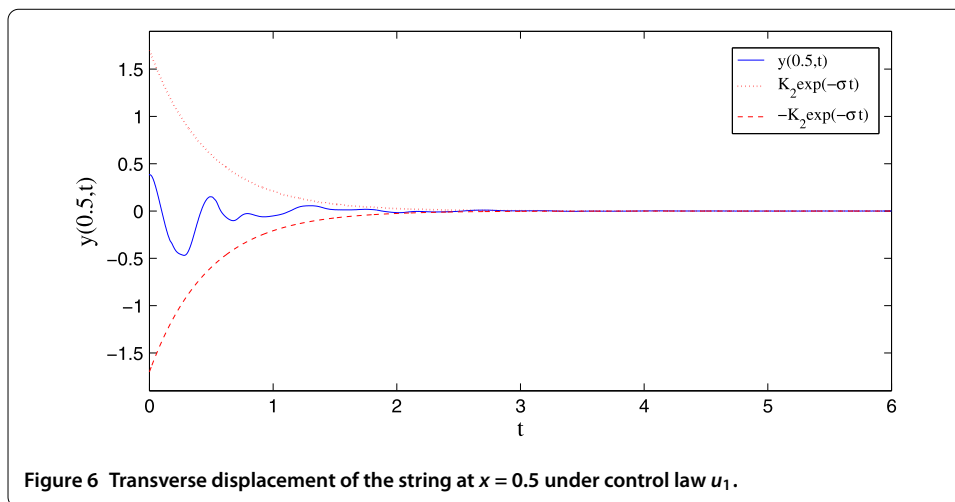
From Figures 4 and 5, it can be seen that, in the case of control law u_2 , the decay of the transverse vibration is relatively slow compared to the case of control law u_1 . Indeed, from Figure 6 and Figure 7, we know that in the case of control law u_1 , the transverse vibration has been suppressed exponentially ($|y(0.5, t)| \leq K_2 e^{-\sigma t}$), whereas in the case of control law u_2 , the transverse vibration has been suppressed polynomially ($|y(0.5, t)| \leq K_3 t^{-1}$). It is coincident with the results of Theorem 3.2.

Appendix

A.1 Proof of Lemma 3.1

Integrating by parts and (9) we get, for all $t \geq 0$,

$$\begin{aligned} \int_0^1 y_x(x, t) y_{xt}(x, t) dx &= \int_0^1 [y_x(x, t) y_t(x, t)]_x dx - \int_0^1 y_{xx}(x, t) y_t(x, t) dx \\ &= y_x(1, t) y_t(1, t) - \int_0^1 y_{xx}(x, t) y_t(x, t) dx. \end{aligned}$$



So we obtain equation (11). Next, integrating by parts we compute

$$\int_0^1 xy_x y_{xx} dx = \frac{1}{2} \int_0^1 [xy_x^2]_x dx - \frac{1}{2} \int_0^1 y_x^2 dx = \frac{1}{2} y_x^2(1, t) - \frac{1}{2} \int_0^1 y_x^2 dx$$

for all $t \geq 0$. That is, equation (12) holds. Finally, we write

$$\int_0^1 xy_x y_{tt} dx = \int_0^1 [xy_x y_t]_t dx - \int_0^1 xy_{tt} y_{xt} dx \quad (39)$$

and

$$\int_0^1 xy_t y_{xt} dx = \frac{1}{2} \int_0^1 [xy_t^2]_x dx - \frac{1}{2} \int_0^1 y_t^2 dx \quad (40)$$

for all $t \geq 0$. Substituting equation (40) into equation (39), we obtain equation (13). Thus the proof of Lemma 3.1 is complete.

A.2 Proof of Claim 1

Since $r > 1$, hypothesis (H) implies

$$L_1|s|^r \leq |u(s)| \leq L_2|s|^{\frac{1}{r}} \quad \text{if } |s| \leq 1,$$

$$L_1|s| \leq |u(s)| \leq L_2|s| \quad \text{if } |s| > 1.$$

Hence, it is true that, for all $s \in R$,

$$s^2 \leq \left(-\frac{1}{L_1}u(s)s\right)^{\frac{2}{r+1}} - \frac{1}{L_1}u(s)s, \quad (41)$$

$$u^2(s) \leq (-L_2^r u(s)s)^{\frac{2}{r+1}} - L_2 u(s)s. \quad (42)$$

Then, from (41) and Proposition 3.1 we have

$$\begin{aligned} & \int_T^S E^q(t) y_t^2(1, t) dt \\ & \leq \int_T^S E^q(t) \left(-\frac{1}{L_1}u(t)y_t(1, t)\right)^{\frac{2}{r+1}} dt + \int_T^S E^q(t) \left(-\frac{1}{L_1}u(t)y_t(1, t)\right) dt \\ & = \int_T^S E^q(t) \left(-\frac{E'(t)}{L_1}\right)^{\frac{2}{r+1}} dt - \frac{1}{L_1} \int_T^S E^q(t) E'(t) dt \\ & \leq \int_T^S E^q(t) \left(-\frac{E'(t)}{L_1}\right)^{\frac{2}{r+1}} dt + \frac{E^{q+1}(T)}{(q+1)L_1}. \end{aligned} \quad (43)$$

From Young's inequality, we have, for any $\alpha, \beta, \delta > 0$,

$$\alpha\beta \leq \delta^\rho \frac{\alpha^\rho}{\rho} + \delta^{-\rho'} \frac{\beta^{\rho'}}{\rho'}, \quad (44)$$

where $\rho, \rho' > 1$ and $\frac{1}{\rho} + \frac{1}{\rho'} = 1$. Applying the above inequality with $\alpha = E^q(t)$, $\beta = \left(-\frac{E'(t)}{L_1}\right)^{\frac{2}{r+1}}$, $\rho = \frac{r+1}{r-1}$ and $\rho' = \frac{r+1}{2}$, we obtain, for any $\delta > 0$,

$$\begin{aligned} & \int_T^S E(t)^q \left(-\frac{E'(t)}{L_1}\right)^{\frac{2}{r+1}} dt \\ & \leq \frac{r-1}{r+1} \delta^{\frac{r+1}{r-1}} \int_T^S E^{q\frac{r+1}{r-1}}(t) dt + \frac{2\delta^{-\frac{r+1}{2}}}{(r+1)L_1} \int_T^S -E'(t) dt \\ & \leq \frac{r-1}{r+1} \delta^{\frac{r+1}{r-1}} \int_T^S E^{q\frac{r+1}{r-1}}(t) dt + \frac{2\delta^{-\frac{r+1}{2}}}{(r+1)L_1} E(T). \end{aligned} \quad (45)$$

By inserting (45) into (43), we deduce inequality (32) in Claim 1. Similarly, from (42) and Proposition 3.1 we have

$$\begin{aligned} & \int_T^S E^q(t) u^2(y_t(1, t)) dt \\ & \leq \int_T^S E^q(t) (-L_2^r u(t)y_t(1, t))^{\frac{2}{r+1}} dt + L_2 \int_T^S E^q(t) (-u(t)y_t(1, t)) dt \end{aligned}$$

$$\begin{aligned}
 &= \int_T^S E^q(t) (-L_2^r E'(t))^{\frac{2}{r+1}} dt - L_2 \int_T^S E^q(t) E'(t) dt \\
 &\leq \int_T^S E^q(t) (-L_2^r E'(t))^{\frac{2}{r+1}} dt + \frac{L_2}{(q+1)} E^{q+1}(T).
 \end{aligned} \tag{46}$$

Applying again inequality (44) with $\alpha = E^q(t)$, $\beta = [-L_2^r E'(t)]^{\frac{2}{r+1}}$, $\rho = \frac{r+1}{r-1}$ and $\rho' = \frac{r+1}{2}$, we obtain, for any $\delta > 0$

$$\begin{aligned}
 &\int_T^S E(t)^q (-L_2^r E'(t))^{\frac{2}{r+1}} dt \\
 &\leq \frac{r-1}{r+1} \delta^{\frac{r+1}{r-1}} \int_T^S E^{q\frac{r+1}{r-1}}(t) dt + \frac{2L_2^r \delta^{-\frac{r+1}{2}}}{(r+1)} \int_T^S -E'(t) dt \\
 &\leq \frac{r-1}{r+1} \delta^{\frac{r+1}{r-1}} \int_T^S E^{q\frac{r+1}{r-1}}(t) dt + \frac{2L_2^r \delta^{-\frac{r+1}{2}}}{(r+1)} E(T).
 \end{aligned} \tag{47}$$

Inserting (47) into (46), we get inequality (33) in Claim 1.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the manuscript and read and approved the final manuscript.

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Acknowledgements

This work is supported by the National Natural Science Foundation of China (Grant 11226128) and the Natural Science Foundation of Heilongjiang Province of China (Grant F201113).

Received: 22 April 2013 Accepted: 6 August 2013 Published: 21 October 2013

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doi:10.1186/1687-2770-2013-215

Cite this article as: Wu and Wang: Exponential and polynomial stabilization of the Kirchhoff string by nonlinear boundary control. *Boundary Value Problems* 2013 **2013**:215.

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