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Multiple solutions for the $p(x)$-Laplacian problem involving critical growth with a parameter

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Abstract

By energy estimates and establishing a local $(PS)_c$ condition, existence of solutions for the $p(x)$-Laplacian problem involving critical growth in a bounded domain is obtained via the variational method under the presence of symmetry.

MSC: 35J20; 35J62

Keywords: $p(x)$-Laplacian problem; critical Sobolev exponents concentration-compactness principle

1 Introduction

In recent years, the study of problems in differential equations involving variable exponents has been a topic of interest. This is due to their applications in image restoration, mathematical biology, dielectric breakdown, electrical resistivity, polycrystal plasticity, the growth of heterogeneous sand piles and fluid dynamics, *etc.* We refer readers to [1–7] for more information. Furthermore, new applications are continuing to appear, see, for example, [8] and the references therein.

With the variational techniques, the $p(x)$-Laplacian problems with subcritical nonlinearities have been investigated, see [9–13] *etc.* However, the existence of solutions for $p(x)$-Laplacian problems with critical growth is relatively new. In 2010, Bonder and Silva [14] extended the concentration-compactness principle of Lions to the variable exponent spaces, and a similar result can be found in [15]. After that, there have been many publications for this case, see [16–19] *etc.*

In this paper, we study the existence and multiplicity of solutions for the quasilinear elliptic problem

$$\begin{cases} -\Delta_{p(x)}u = \lambda|u|^{q(x)-2}u + f(x,u), & x \in \Omega; \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta_{p(x)}u = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$, $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary, $\lambda > 0$ is a real parameter, $p(x)$, $q(x)$ are continuous functions on $\bar{\Omega}$ with

$$1 < p^- := \min_{x\in\bar{\Omega}} p(x) \leq p^+ := \max_{x\in\bar{\Omega}} p(x) < N, \qquad 1 \leq q(x) \leq p^*(x), \quad \forall x \in \bar{\Omega}, \tag{1.2}$$

where

$$p^*(x) = \frac{Np(x)}{N - p(x)}, \quad \forall x \in \bar{\Omega},$$

and

$$\{x \in \bar{\Omega}, q(x) = p^*(x)\} \neq \emptyset. \tag{1.3}$$

Related to f, we assume that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying $\sup\{|f(x,s)|; x \in \Omega, |s| \leq M\} < \infty$ for every $M > 0$, and the subcritical growth condition:

(f_1) $f(x,s) \leq C_1(1 + |s|^{\beta(x)-1})$ for all $(x,s) \in \Omega \times \mathbb{R}$, where $\beta(x)$ is a continuous function in $\bar{\Omega}$ satisfying $\beta(x) < p^*(x)$, $\forall x \in \bar{\Omega}$.

For $F(x,s) = \int_0^s f(x,t)\,dt$, we suppose that f satisfies the following:

(f_2) there are constants $\sigma \in [0, p^-)$ and $a_1, a_2 > 0$ such that for every $s \in \mathbb{R}$, a.e. in Ω,

$$\frac{1}{p^+} f(x,s)s - F(x,s) \geq -a_1 - a_2|s|^\sigma;$$

(f_3) there are constants $b_1, b_2 > 0$ and a continuous function $r(x) < p^*(x)$, $\forall x \in \bar{\Omega}$, with $r^+ > p^-$, such that for every $s \in \mathbb{R}$, a.e. in Ω,

$$F(x,s) \leq b_1|s|^{r(x)} + b_2;$$

(f_4) there are $c_1 > 0$, $h_1 \in L^{p'(x)}(\Omega)$ and $\Omega_0 \subset \Omega$ with $|\Omega_0| > 0$ such that

$$F(x,s) \geq -h_1(x)|s|^{p(x)} - c_1 \quad \text{for every } s \in \mathbb{R}, \text{ a.e. in } \Omega,$$

and

$$\liminf_{|s| \to \infty} \frac{F(x,s)}{|s|^{p^+}} = \infty \quad \text{uniformly a.e. in } \Omega_0.$$

Now we state our result.

Theorem 1.1 *Assume that* (1.2), (1.3) *and* (f_1)-(f_4) *are satisfied with* $p^+ < q^-$, $f(x,s)$ *is odd in* s. *Then, given* $k \in \mathbb{N}$, *there exists* $\lambda_k \in (0, \infty]$ *such that problem* (1.1) *possesses at least* k *pairs of nontrivial solutions for all* $\lambda \in (0, \lambda_k)$.

Our paper is motivated by [17]. In [17], the authors considered the multiple solutions to problem (1.1) under the conditions that f has the form $f(x,t) = a(x)|t|^{p(x)-2}t + g(x,t)$ with $a \in L^\infty(\Omega)$ and g satisfies the following:

(g_1) there is $\alpha > 0$ such that

$$\int_\Omega \frac{1}{p(x)} \left(|\nabla u|^{p(x)} - a(x)|u|^{p(x)}\right) dx \geq \alpha \int_\Omega \frac{1}{p(x)} |u|^{p(x)}\,dx;$$

(g_2) $g \in C(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$, odd with respect to t and

$$g(x,t) = o\big(|t|^{p(x)-1}\big), \quad |t| \to 0 \text{ uniformly in } x,$$
$$g(x,t) = o\big(|t|^{q(x)-1}\big), \quad |t| \to \infty \text{ uniformly in } x;$$

(g_3) $G(x,t) \leq \frac{1}{p^+} g(x,t)t$ for all $t \in \mathbb{R}$ and a.e. in Ω, where $G(x,t) = \int_0^t g(x,s)\,ds$.

Moreover, they assumed that

$$p(x) = p^+, \quad \forall x \in \Gamma = \big\{x \in \Omega : a(x) > 0\big\}, \tag{1.4}$$

and the result is the following theorem.

Theorem 1.2 *Assume that* (1.2), (1.3), (1.4) *and* (g_1)-(g_3) *are satisfied with* $p^+ < q^-$. *Then there exists a sequence* $\{\lambda_k\} \subset (0,\infty)$ *with* $\lambda_k > \lambda_{k+1}$ *such that for* $\lambda \in (\lambda_k, \lambda_{k+1})$, *problem* (1.1) *has at least k pairs of nontrivial solutions*.

Note that (f_2) is a weaker version of (g_3). This condition combined with (f_1) and the concentration-compactness principle in [14] will allow us to verify that the associated functional satisfies the (*PS*) condition [20] below a fixed level for $\lambda > 0$ sufficiently small. Conditions (f_3) and (f_4) provide the geometry required by the symmetric mountain pass theorem [20]. Compared with (g_2), there is no condition imposed on f near zero in Theorem 1.1. Furthermore, we should mention that our Theorem 1.1 improves the main result found in [21]. In that paper, the authors considered only the case where $p(x)$ is constant, while in our present paper, we have showed that the main result found in [21] is still true for a large class of $p(x)$ functions.

The paper is organized as follows. In Section 2, we introduce some necessary preliminary knowledge. Section 3 contains the proof of our main result.

2 Preliminaries

We recall some definitions and basic properties of the generalized Lebesgue-Sobolev spaces $L^{p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$, where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary. And C will denote generic positive constants which may vary from line to line.

Set

$$C_+(\bar{\Omega}) = \big\{p(x) \in C(\bar{\Omega}) : p(x) > 1, \forall x \in \bar{\Omega}\big\}.$$

For any $p(x) \in C_+(\bar{\Omega})$, we define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \left\{u \in M(\Omega) : \int_\Omega \big|u(x)\big|^{p(x)}\,dx < \infty\right\}$$

with the norm

$$|u|_{p(x)} = \inf\left\{\mu > 0 : \int_\Omega \left|\frac{u}{\mu}\right|^{p(x)} dx \leq 1\right\},$$

where $M(\Omega)$ is the set of all measurable real functions defined on Ω.

Define the space

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\}$$

with the norm

$$\|u\|_{1,p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}.$$

By $W_0^{1,p(x)}(\Omega)$, we denote the subspace of $W^{1,p(x)}(\Omega)$ which is the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_{1,p(x)}$. Further, we have

Lemma 2.1 [22, 23] *There is a constant $C > 0$ such that for all $u \in W_0^{1,p(x)}(\Omega)$,*

$$|u|_{p(x)} \leq C|\nabla u|_{p(x)}.$$

So, $|\nabla u|_{p(x)}$ and $\|u\|_{1,p(x)}$ are equivalent norms in $W_0^{1,p(x)}(\Omega)$. Hence we will use the norm $\|u\| = |\nabla u|_{p(x)}$ for all $u \in W_0^{1,p(x)}(\Omega)$.

Lemma 2.2 [22, 23] *Set $\rho(u) = \int_\Omega |u|^{p(x)}\,dx$. For $u, u_n \in L^{p(x)}(\Omega)$, we have*:

(1) $|u|_{p(x)} < 1\ (= 1; > 1) \Leftrightarrow \rho(u) < 1\ (= 1; > 1)$.
(2) *If* $|u|_{p(x)} > 1$, *then* $|u|_{p(x)}^{p^-} \leq \rho(u) \leq |u|_{p(x)}^{p^+}$.
(3) *If* $|u|_{p(x)} < 1$, *then* $|u|_{p(x)}^{p^+} \leq \rho(u) \leq |u|_{p(x)}^{p^-}$.
(4) $\lim_{n\to\infty} u_n = u \Leftrightarrow \lim_{n\to\infty} \rho(u_n - u) = 0$.
(5) $\lim_{n\to\infty} |u_n|_{p(x)} = \infty \Leftrightarrow \lim_{n\to\infty} \rho(u_n) = \infty$.

Lemma 2.3 [23] *If $p_1(x), p_2(x) \in C_+(\bar{\Omega})$ with $p_1(x) \leq p_2(x)$ a.e. in Ω, then there exists the continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$.*

Lemma 2.4 [22] *If $q(x) \in C_+(\bar{\Omega})$ and $q(x) < p^*(x)$ for any $x \in \Omega$, the embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is compact.*

Lemma 2.5 [23] *The conjugate space of $L^{p(x)}(\Omega)$ is $L^{p'(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$,*

$$\int_\Omega |uv|\,dx \leq \left(\frac{1}{p^-} + \frac{1}{p'^-}\right)|u|_{p(x)}|v|_{p'(x)}.$$

The energy functional corresponding to problem (1.1) is defined on $W_0^{1,p(x)}(\Omega)$ as follows:

$$I_\lambda(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx - \lambda \int_\Omega \frac{1}{q(x)}|u|^{q(x)}\,dx - \int_\Omega F(x,u)\,dx. \tag{2.1}$$

Then $I_\lambda \in C^1(W_0^{1,p(x)}(\Omega), \mathbb{R})$ and $\forall u, \phi \in W_0^{1,p(x)}(\Omega)$,

$$\langle I'_\lambda(u), \phi\rangle = \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla\phi\,dx - \lambda \int_\Omega |u|^{q(x)-2}u\phi\,dx - \int_\Omega f(x,u)\phi\,dx.$$

We say that $u \in W_0^{1,p(x)}(\Omega)$ is a weak solution of problem (1.1) in the weak sense if for any $\phi \in W_0^{1,p(x)}(\Omega)$,

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla \phi \, dx - \lambda \int_\Omega |u|^{q(x)-2}u\phi \, dx - \int_\Omega f(x,u)\phi \, dx = 0.$$

So, the weak solution of problem (1.1) coincides with the critical point of I_λ. Next, we need only to consider the existence of critical points of $I_\lambda(u)$.

We say that $I_\lambda(u)$ satisfies the $(PS)_c$ condition if any sequence $\{u_n\} \subseteq W_0^{1,p(x)}(\Omega)$, such that $I_\lambda(u_n) \to c$ and $I'_\lambda(u_n) \to 0$ as $n \to \infty$, possesses a convergent subsequence. In this article, we shall be using the following version of the symmetric mountain pass theorem [20].

Lemma 2.6 [20] *Let $E = V \oplus X$, where E is a real Banach space and V is finite dimensional. Suppose that $I \in C^1(E, \mathbb{R})$ is an even functional satisfying $I(0) = 0$ and*

(i) *there is a constant $\rho > 0$ such that $I_{\partial B_\rho \cap X} \geq 0$;*
(ii) *there is a subspace W of E with $\dim V < \dim W < \infty$ and there is $M > 0$ such that $\max_{u \in W} I(u) < M$;*
(iii) *considering $M > 0$ given by (ii), I satisfies $(PS)_c$ for $0 \leq c \leq M$.*

Then I possesses at least $\dim W - \dim V$ pairs of nontrivial critical points.

Next we would use the concentration-compactness principle for variable exponent spaces. This will be the keystone that enables us to verify that I_λ satisfies the $(PS)_c$ condition.

Lemma 2.7 [14] *Let $q(x)$ and $p(x)$ be two continuous functions such that*

$$1 < \inf_{x \in \bar\Omega} p(x) \leq \sup_{x \in \bar\Omega} p(x) < N \quad \text{and} \quad 1 \leq q(x) \leq p^*(x) \quad \text{in } \Omega.$$

Let $\{u_n\}$ be a weakly convergent sequence in $W_0^{1,p(x)}(\Omega)$ with weak limit u such that:

- *$|\nabla u_n|^{p(x)} \rightharpoonup \mu$ weakly in the sense of measures;*
- *$|u_n|^{q(x)} \rightharpoonup \nu$ weakly in the sense of measures.*

Also assume that $\mathcal{A} = \{x \in \Omega : q(x) = p^*(x)\}$ is nonempty. Then, for some countable index set K, we have:

$$\nu = |u|^{q(x)} + \sum_{i \in K} \nu_i \delta_{x_i}, \quad \nu_i \geq 0, \tag{2.2}$$

$$\mu \geq |\nabla u|^{p(x)} + \sum_{i \in K} \mu_i \delta_{x_i}, \quad \mu_i \geq 0, \tag{2.3}$$

$$S\nu_i^{1/p^*(x_i)} \leq \mu_i^{1/p(x_i)} \quad \forall i \in K, \tag{2.4}$$

where $\{x_i\}_{i \in K} \subset \mathcal{A}$ and S is the best constant in the Gagliardo-Nirenberg-Sobolev inequality for variable exponents, namely

$$S = S_q(\Omega) := \inf_{\phi \in C_0^\infty(\Omega)} \frac{\|\phi\|}{|\phi|_{q(x)}}. \tag{2.5}$$

3 Proof of main results

Lemma 3.1 *Assume that f satisfies* (f_1) *and* (f_2) *with $p^+ < q^-$. Then, given $M > 0$, there exists $\lambda_* > 0$ such that I_λ satisfies the $(PS)_c$ condition for all $c < M$, provided* $0 < \lambda < \lambda_*$.

Proof (1) The boundedness of the $(PS)_c$ sequence.

Let $\{u_n\}$ be a $(PS)_c$ sequence, *i.e.*, $\{u_n\}$ satisfies $I_\lambda(u_n) \to c$, and $I'_\lambda(u_n) \to 0$ as $n \to \infty$. If $\|u_n\| \le 1$, we have done. So we only need to consider the case that $\|u_n\| > 1$ with $|u_n|_{q(x)} > 1$. We know that

$$\begin{aligned} I_\lambda(u_n) &= \int_\Omega \frac{1}{p(x)} |\nabla u_n|^{p(x)}\,dx - \lambda \int_\Omega \frac{1}{q(x)} |u_n|^{q(x)}\,dx - \int_\Omega F(x,u_n)\,dx, \\ \langle I'_\lambda(u_n), u_n\rangle &= \int_\Omega |\nabla u_n|^{p(x)}\,dx - \lambda \int_\Omega |u_n|^{q(x)}\,dx - \int_\Omega f(x,u_n)u_n\,dx. \end{aligned} \tag{3.1}$$

From (f_2), we get

$$\begin{aligned} c + 1 + \|u_n\| &\ge I_\lambda(u_n) - \frac{1}{p^+}\langle I'_\lambda(u_n), u_n\rangle \\ &\ge \left(\frac{1}{p^+} - \frac{1}{q^-}\right)\lambda \int_\Omega |u_n|^{q(x)}\,dx + \int_\Omega \left(\frac{1}{p^+} f(x,u_n)u_n - F(x,u_n)\right)dx \\ &\ge \left(\frac{1}{p^+} - \frac{1}{q^-}\right)\lambda \int_\Omega |u_n|^{q(x)}\,dx - a_1|\Omega| - a_2 \int_\Omega |u_n|^\sigma\,dx. \end{aligned}$$

Notice that $q^- \le q(x)$, $\forall x \in \bar{\Omega}$, then from Lemmas 2.3, 2.4, $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega) \hookrightarrow L^{q^-}(\Omega)$, so $|u|_{q^-} \le C_1|u|_{q(x)} \le C\|u\|$. Let $\alpha = (q^- - \sigma)/q^-$, then $0 < \alpha < 1$, and from the Hölder inequality,

$$\begin{aligned} \int_\Omega |u_n|^\sigma\,dx &\le \left(\int_\Omega |u_n|^{q^-}\,dx\right)^{\frac{\sigma}{q^-}} |\Omega|^{\frac{q^- - \sigma}{q^-}} \\ &= \left(\int_\Omega |u_n|^{q^-}\,dx\right)^{(1-\alpha)} |\Omega|^\alpha \\ &\le |\Omega|^\alpha C^{(1-\alpha)q^-} \|u_n\|^{(1-\alpha)q^-}. \end{aligned}$$

In addition, from Lemma 2.2(2), we can also obtain that

$$\begin{aligned} \int_\Omega |u_n|^\sigma\,dx &\le \left(\int_\Omega |u_n|^{q^-}\,dx\right)^{(1-\alpha)} |\Omega|^\alpha \\ &\le |\Omega|^\alpha \big(C_1 |u_n|_{q(x)}\big)^{(1-\alpha)q^-} \\ &\le |\Omega|^\alpha C_1^{(1-\alpha)q^-} \left(\int_\Omega |u_n|^{q(x)}\,dx\right)^{(1-\alpha)}. \end{aligned}$$

Then

$$\begin{aligned} & I_\lambda(u_n) - \frac{1}{p^+}\langle I'_\lambda(u_n), u_n\rangle \\ &\quad \ge \left(\frac{1}{p^+} - \frac{1}{q^-}\right)\lambda \int_\Omega |u_n|^{q(x)}\,dx - a_1|\Omega| - a_2|\Omega|^\alpha C_1^{(1-\alpha)q^-} \left(\int_\Omega |u_n|^{q(x)}\,dx\right)^{(1-\alpha)}, \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} c+1+\|u_n\| &\geq I_\lambda(u_n)-\frac{1}{p^+}\langle I'_\lambda(u_n),u_n\rangle \\ &\geq \left(\frac{1}{p^+}-\frac{1}{q^-}\right)\lambda\int_\Omega |u_n|^{q(x)}\,dx - a_1|\Omega| - C\|u_n\|^{(1-\alpha)q^-}. \end{aligned}$$

So we have

$$\int_\Omega |u_n|^{q(x)}\,dx \leq C + C\|u_n\| + C\|u_n\|^{(1-\alpha)q^-}. \tag{3.3}$$

From (3.1), (3.3) and (f_1), we have

$$\begin{aligned} \frac{1}{p^+}\|u_n\|^{p^-} &\leq \int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx = I_\lambda(u_n)+\lambda\int_\Omega \frac{1}{q(x)}|u_n|^{q(x)}\,dx + \int_\Omega F(x,u_n)\,dx \\ &\leq C + C\int_\Omega |u_n|^{q(x)}\,dx \\ &\leq C + C\|u_n\| + C\|u_n\|^{(1-\alpha)q^-}. \end{aligned}$$

Noting that $(1-\alpha)q^- = \sigma < p^-$, we have that $\{u_n\}$ is bounded.

(2) Up to a subsequence, $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$.

By Lemma 2.7, we can assume that there exist two measures μ, ν and a function $u \in W_0^{1,p(x)}(\Omega)$ such that

$$\begin{aligned} & u_n \rightharpoonup u \quad \text{weakly in } W_0^{1,p(x)}(\Omega), \\ & |\nabla u_n|^{p(x)} \rightharpoonup \mu \quad \text{weakly in the sense of measures}, \\ & |u_n|^{q(x)} \rightharpoonup \nu \quad \text{weakly in the sense of measures}, \\ & \nu = |u|^{q(x)} + \sum_{j\in K}\nu_j\delta_{x_j}, \\ & \mu \geq |\nabla u|^{p(x)} + \sum_{j\in K}\mu_j\delta_{x_j}, \\ & S\nu_j^{1/p^*(x_j)} \leq \mu_j^{1/p(x_j)}. \end{aligned}$$

Choose a function $\varphi(x) \in C_0^\infty(\mathbb{R}^N)$ such that $0 \leq \varphi(x) \leq 1$, $\varphi(x) \equiv 1$ on $B(0,1)$ and $\varphi(x) \equiv 0$ on $\mathbb{R}^N \setminus B(0,2)$. For any $x \in \mathbb{R}^N$, $\varepsilon > 0$ and $j \in K$, let $\varphi_{j,\varepsilon}(x) = \varphi(\frac{x-x_j}{\varepsilon})$. It is clear that $\{\varphi_{j,\varepsilon}u_n\}$ is bounded in $W_0^{1,p(x)}(\Omega)$. From $I'_\lambda(u_n) \to 0$, we can obtain $\langle I'_\lambda(u_n), \varphi_{j,\varepsilon}u_n\rangle \to 0$, as $n \to \infty$, *i.e.*,

$$\begin{aligned} & \int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n \cdot u_n\nabla\varphi_{j,\varepsilon}\,dx + \int_\Omega |\nabla u_n|^{p(x)}\varphi_{j,\varepsilon}\,dx \\ & \quad - \lambda\int_\Omega |u_n|^{q(x)}\varphi_{j,\varepsilon}\,dx - \int_\Omega f(x,u_n)u_n\varphi_{j,\varepsilon}\,dx \to 0. \end{aligned} \tag{3.4}$$

From (f_1), by Lemma 2.7, we have

$$\begin{aligned}&\lim_{n\to\infty}\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n\cdot u_n\nabla\varphi_{j,\varepsilon}\,dx\\&\quad=\lambda\int_\Omega\varphi_{j,\varepsilon}\,d\nu-\int_\Omega\varphi_{j,\varepsilon}\,d\mu+\int_\Omega f(x,u)u\varphi_{j,\varepsilon}\,dx.\end{aligned}\tag{3.5}$$

By the Hölder inequality, it is easy to check that

$$\lim_{n\to\infty}\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n\cdot u_n\nabla\varphi_{j,\varepsilon}\,dx=0.$$

From (3.5), as $\varepsilon\to 0$, we obtain $\lambda\nu_j=\mu_j$. From Lemma 2.7, we conclude that

$$\nu_j=0\quad\text{or}\quad\nu_j\geq S^N\max\bigl\{\lambda^{-\frac{N}{p^+}},\lambda^{-\frac{N}{p^-}}\bigr\}.\tag{3.6}$$

Given $M>0$, set

$$\begin{aligned}\lambda_*=\min\biggl\{&S^{p^+},S^{p^-},\biggl(\frac{S^N(\frac{1}{p^+}-\frac{1}{q^-})^{\frac{1}{\alpha}}}{(M+a_1|\Omega|+a_2|\Omega|^\alpha C_1^{(1-\alpha)q^-})^{\frac{1}{\alpha}}}\biggr)^{\frac{1}{\frac{N}{p^+}-\frac{1}{\alpha}}},\\&\biggl(\frac{S^N(\frac{1}{p^+}-\frac{1}{q^-})^{\frac{1}{\alpha}}}{(M+a_1|\Omega|+a_2|\Omega|^\alpha C_1^{(1-\alpha)q^-})^{\frac{1}{\alpha}}}\biggr)^{\frac{1}{\frac{N}{p^-}-\frac{1}{\alpha}}}\biggr\},\end{aligned}$$

where S is given by (2.5). Considering $0<\lambda<\lambda_*$, we have

$$1<S^N\lambda^{-\frac{N}{p^+}},\qquad 1<S^N\lambda^{-\frac{N}{p^-}},\tag{3.7}$$

and

$$\biggl(\frac{M+a_1|\Omega|+a_2|\Omega|^\alpha C_1^{(1-\alpha)q^-}}{(\frac{1}{p^+}-\frac{1}{q^-})\lambda}\biggr)^{\frac{1}{\alpha}}<S^N\min\bigl\{\lambda^{-\frac{N}{p^+}},\lambda^{-\frac{N}{p^+}}\bigr\}.\tag{3.8}$$

We claim that $\int_\Omega d\nu<S^N\min\{\lambda^{-\frac{N}{p^+}},\lambda^{-\frac{N}{p^-}}\}$. Indeed, if $\int_\Omega d\nu\leq 1$, this follows by (3.7). Otherwise, taking $n\to\infty$ in (3.2), we obtain

$$\begin{aligned}\biggl(\frac{1}{p^+}-\frac{1}{q^-}\biggr)\lambda\int_\Omega d\nu&\leq a_1|\Omega|+a_2|\Omega|^\alpha C_1^{(1-\alpha)q^-}\biggl(\int_\Omega d\nu\biggr)^{1-\alpha}+c\\&\leq\bigl(M+a_1|\Omega|+a_2|\Omega|^\alpha C_1^{(1-\alpha)q^-}\bigr)\biggl(\int_\Omega d\nu\biggr)^{1-\alpha}.\end{aligned}$$

Therefore, by (3.8), the claim is proved. As a consequence of this fact, we conclude that $\nu_j=0$ for all $j\in K$. Therefore, $u_n\to u$ in $L^{q(x)}(\Omega)$. Then, with the similar step in [17], we can get that $u_n\to u$ in $W_0^{1,p(x)}(\Omega)$. □

Next we prove Theorem 1.1 by verifying that the functional I_λ satisfies the hypotheses of Lemma 2.6. First, we recall that each basis $\{e_i\}_{i\in\mathbb{N}}$ for a real Banach space E is a Schauder basis for E, *i.e.*, given $n \in \mathbb{N}$, the functional $e_n^* : E \to \mathbb{R}$ defined by

$$e_n^*(v) = \alpha_n, \quad v = \sum_{i=1}^{\infty} \alpha_i e_i \in E$$

is a bounded linear functional [24, 25]. Now, fixing a Schauder basis $\{e_i\}_{i \in \mathbb{N}}$ for $W_0^{1,p(x)}(\Omega)$, for $j \in \mathbb{N}$, we set

$$\begin{aligned} V_j &= \{u \in W_0^{1,p(x)}(\Omega) : e_i^*(u) = 0, i > j\}, \\ X_j &= \{u \in W_0^{1,p(x)}(\Omega) : e_i^*(u) = 0, i \leq j\}, \end{aligned} \tag{3.9}$$

then $W_0^{1,p(x)}(\Omega) = V_j \oplus X_j$.

Lemma 3.2 *Given $1 \leq r(x) < p^*(x)$ for all $x \in \Omega$ and $\delta > 0$, there is $j \in \mathbb{N}$ such that for all $u \in X_j$, $|u|_{r(x)} \leq \delta \|u\|$.*

Proof We prove the lemma by contradiction. Suppose that there exist $\delta > 0$ and $u_j \in X_j$ for every $j \in \mathbb{N}$ such that $|u_j|_{r(x)} \geq \delta \|u_j\|$. Taking $v_j = \frac{u_j}{|u_j|_{r(x)}}$, we have $|v_j|_{r(x)} = 1$ for every $j \in \mathbb{N}$ and $\|v_j\| \leq \frac{1}{\delta}$. Hence $\{v_j\} \subset W_0^{1,p(x)}(\Omega)$ is a bounded sequence, and we may suppose, without loss of generality, that $v_j \rightharpoonup v$ in $W_0^{1,p(x)}(\Omega)$. Furthermore, $e_n^*(v) = 0$ for every $n \in \mathbb{N}$ since $e_n^*(v_j) = 0$ for all $j \geq n$. This shows that $v = 0$. On the other hand, by the compactness of the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega)$, we conclude that $|v|_{r(x)} = 1$. This proves the lemma. □

Lemma 3.3 *Suppose that f satisfies* (f_3), *then there exist $j \in \mathbb{N}$ and $\rho, \alpha, \tilde{\lambda} > 0$ such that $I|_{\partial B_\rho \cap X_j} \geq \alpha$ for all $0 < \lambda < \tilde{\lambda}$.*

Proof Now suppose that $\|u\| > 1$, with $|u|_{r(x)} > 1$, $|u|_{q(x)} > 1$. From (f_3), we know that

$$\begin{aligned} I_\lambda(u) &= \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}\, dx - \lambda \int_\Omega \frac{|u|^{q(x)}}{q(x)}\, dx - \int_\Omega F(x,u)\, dx \\ &\geq \frac{1}{p^+} \int_\Omega |\nabla u|^{p(x)}\, dx - \frac{\lambda}{q^-} \int_\Omega |u|^{q(x)}\, dx - b_1 \int_\Omega |u|^{r(x)}\, dx - b_2 |\Omega|. \end{aligned}$$

Consequently, considering $\delta > 0$ to be chosen posteriorly by Lemma 3.2, we have, for all $u \in X_j$ and j sufficiently large,

$$\begin{aligned} I_\lambda(u) &\geq \frac{1}{p^+} \|u\|^{p^-} - \frac{\lambda C}{q^-} \|u\|^{q^+} - b_1 \delta^{r^+} \|u\|^{r^+} - b_2 |\Omega| \\ &\geq \|u\|^{p^-} \left(\frac{1}{p^+} - b_1 \delta^{r^+} \|u\|^{r^+ - p^-} \right) - b_2 |\Omega| - \frac{C\lambda}{q^-} \|u\|^{q^+}. \end{aligned}$$

Now taking $1 < \|u\| = \rho(\delta)$ such that $b_1 \delta^{r^+} \rho^{r^+ - p^-} = \frac{1}{2p^+}$ and noting that $r^+ > p^-$, so $\rho(\delta) \to +\infty$, if $\delta \to 0$. We can choose $\delta > 0$ such that $\frac{\rho^{p^-}}{2p^+} - b_1 |\Omega| > \frac{\rho^{p^-}}{4p^+}$. Next, we take $\tilde{\lambda} > 0$ such that for $0 < \lambda < \tilde{\lambda}$,

$$I_\lambda(u) \geq \frac{\rho^{p^-}}{4p^+} - \frac{C\lambda}{q^-}\rho^{q^+} > 0$$

for every $u \in X_j$, $\|u\| = \rho$, the proof is complete. □

Lemma 3.4 *Suppose that* f *satisfies* (f_4), *then, given* $m \in \mathbb{N}$, *there exist a subspace* W *of* $W_0^{1,p(x)}(\Omega)$ *and a constant* $M_m > 0$ *such that* $\dim W = m$ *and* $\max_{u\in W} I(u) < M_m$.

Proof Let $x_0 \in \Omega_0$ and $r_0 > 0$ be such that $\overline{B(x_0,r_0)} \subset \Omega$, and $0 < |\overline{B(x_0,r_0)} \cap \Omega_0| < \frac{|\Omega_0|}{2}$. First, we take $v_1 \in C_0^\infty(\Omega)$ with $\operatorname{supp}(v_1) = \overline{B(x_0,r_0)}$. Considering $\Omega_1 = \Omega_0 \setminus [\overline{B(x_0,r_0)} \cap \Omega_0] \subset \widehat{\Omega}_0 = \overline{\Omega \setminus B(x_0,r_0)}$, we have $|\Omega_1| \geq \frac{|\Omega_0|}{2} > 0$. Let $x_1 \in \Omega_1$ and $r_1 > 0$ such that $\overline{B(x_1,r_1)} \subset \widehat{\Omega}_0$, and $0 < |\overline{B(x_1,r_1)} \cap \Omega_1| < \frac{|\Omega_1|}{2}$. Next, we take $v_2 \in C_0^\infty(\Omega)$ with $\operatorname{supp}(v_2) = \overline{B(x_1,r_1)}$. After a finite number of steps, we get $v_1, v_2, \ldots, v_m$ such that $\operatorname{supp}(v_i) \cap \operatorname{supp}(v_j) = \emptyset$, $i \neq j$, and $|\operatorname{supp}(v_j) \cap \Omega_0| > 0$ for all $i,j \in \{1,2,\ldots,m\}$. Let $W = \operatorname{span}\{v_1, v_2, \ldots, v_m\}$, by construction, $\dim W = m$, and for every $v \in W \setminus \{0\}$,

$$\int_{\Omega_0} |v|^{p^+}\,dx > 0.$$

Since

$$\max_{u\in W\setminus\{0\}} I_0(u) = \max_{t>0, v\in W\cap\partial B_1(0)} \left(\int_\Omega \frac{(t|\nabla v|)^{p(x)}}{p(x)}\,dx - \int_\Omega F(x,tv)\,dx \right),$$

consider the case that $t > 1$, then $I_0(tv) \leq \frac{t^{p^+}}{p^-} - \int_\Omega F(x,tv)\,dx = t^{p^+}(\frac{1}{p^-} - \frac{1}{t^{p^+}}\int_\Omega F(x,tv)\,dx)$. Now it suffices to verify that

$$\lim_{t\to\infty} \frac{1}{t^{p^+}} \int_\Omega F(x,tv)\,dx > \frac{1}{p^-}.$$

From condition (f_4), given $L > 0$, there is $C > 0$ such that for every $s \in \mathbb{R}$, a.e. x in Ω_0,

$$F(x,s) \geq L|s|^{p^+} - C.$$

Consequently, for $v \in \partial B_1(0) \cap W$ and $t > 1$,

$$\int_\Omega F(x,tv)\,dx \geq Lt^{p^+} \int_{\Omega_0} |v|^{p^+}\,dx - Ct^{p^+} \int_{\Omega\setminus\Omega_0} h_1(x)|v|^{p(x)}\,dx - C_2$$

and

$$\lim_{t\to\infty} \frac{\int_\Omega F(x,tv)\,dx}{t^{p^+}} \geq L \int_{\Omega_0} |v|^{p^+}\,dx - C \int_{\Omega\setminus\Omega_0} h_1(x)|v|^{p(x)}\,dx \geq Lr - CR,$$

where $r = \min\{\int_{\Omega_0} |v|^{p^+}\,dx, v \in \partial B_1(0) \cap W\}$ and $R = \max\{\int_{\Omega\setminus\Omega_0} h_1(x)|v|^{p(x)}\,dx, v \in \partial B_1(0) \cap W\}$. Observing that W is finite dimensional, we have $R < +\infty$, $r > 0$, and the inequality is obtained by taking $L > \frac{1}{r}(\frac{1}{p^-} + CR)$. The proof is complete. □

Proof of Theorem 1.1 First, we recall that $W_0^{1,p(x)}(\Omega) = V_j \oplus X_j$, where V_j and X_j are defined in (3.9). Invoking Lemma 3.3, we find $j \in \mathbb{N}$, and I_λ satisfies (i) with $X = X_j$. Now, by Lemma 3.4, there is a subspace W of $W_0^{1,p(x)}(\Omega)$ with $\dim W = k + j = k + \dim V_j$ such that I_λ satisfies (ii). By Lemma 3.1, I_λ satisfies (iii). Since $I_\lambda(0) = 0$ and I_λ is even, we may apply Lemma 2.6 to conclude that I_λ possesses at least k pairs of nontrivial critical points. The proof is complete. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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