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Regularity of Hölder continuous solutions of the supercritical porous media equation

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Abstract
In this paper, we present a regularity result for weak solutions of the *N*-dimensional ($N = 2$ or 3) porous media equation with supercritical ($\alpha < 1$) dissipation Λ^α. If a Leray-Hopf weak solution is Hölder continuous $\theta \in C^\delta(R^N)$ with $\delta > 1 - \alpha$ on the time interval $[0, t]$, then it is actually a classical solution on $(0, t]$.

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1 Introduction

We use Darcy's law to describe the flow velocity, which reads

$$v = -k(\nabla p + g\gamma\theta),$$

where $v \in \mathbb{R}^N$ is the liquid discharge, p is the scalar pressure, θ is the liquid temperature, k is the matrix position-independent medium permeabilities in the different directions, respectively, divided by the viscosity, g is the acceleration due to gravity and $\gamma \in \mathbb{R}^N$ is the last canonical vector e_N. For brevity, we only consider $k = g = 1$.

In this article, we study the system of heat transfer with a fractional diffusion in an incompressible N (2 or 3)-dimensional flow [1]

$$(DPM)_\alpha \qquad \begin{cases} \partial_t\theta + v\cdot\nabla\theta + \nu\Lambda^\alpha\theta = 0, \\ v = -(\nabla p + \gamma\theta), \qquad \operatorname{div} v = 0, \\ \theta(0,x) = \theta^0(x), \end{cases} \tag{1}$$

where $\nu > 0$ is the dissipative coefficient, and the differential operator Λ^α is given by $\Lambda^\alpha := (-\Delta)^{\frac{\alpha}{2}}$. Considering the scaling transform $\theta(t,x) \to \theta_\lambda(t,x) := \lambda^{\alpha-1}\theta(\lambda^\alpha t, \lambda x)$ for $\lambda > 0$, the system will be divided into three cases: the case $\alpha = 1$ is called the critical case, the case $\alpha > 1$ is subcritical and the case $\alpha < 1$ is supercritical.

Next, by rewriting Darcy's law, we obtain the expression of velocity v only in terms of temperature θ [2, 3]. In the 2D case, thanks to the incompressibility, taking the curl operator first and the $\nabla^\perp := (-\partial_{x_2}, \partial_{x_1})$ operator second on both sides of Darcy's law, we have

$$-\Delta v = \nabla^\perp(\partial_{x_1}\theta) = \left(-\partial_{x_1}\partial_{x_2}\theta, \partial^2_{x_1}\theta\right),$$

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thus the velocity v can be recovered as

$$v(t,x) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \ln|x-y|\left(-\frac{\partial^2\theta}{\partial y_2\partial y_1}(t,y), \frac{\partial^2\theta}{\partial^2 y_1}(t,y)\right)dy, \quad x\in\mathbb{R}^2.$$

Through integration by parts, we finally get

$$v(t,x) = -\frac{1}{2}\big(0,\theta(t,x)\big) + \frac{1}{2\pi}PV\int_{\mathbb{R}^2} H(x-y)\theta(t,y)\,dy, \quad x\in\mathbb{R}^2, \tag{2}$$

where the kernel $H(\cdot)$ is defined by

$$H(x) = \left(\frac{2x_1x_2}{|x|^4}, \frac{x_2^2-x_1^2}{|x|^4}\right).$$

Similarly, in the 3D case, applying the curl operator twice to Darcy's law, we get

$$-\Delta v = \big(-\partial_1\partial_3\theta, -\partial_2\partial_3\theta, \partial_1^2\theta + \partial_2^2\theta\big),$$

where $\partial_i := \frac{\partial}{\partial x_i}$, thus

$$v(t,x) = -\frac{2}{3}\big(0,0,\theta(t,x)\big) + \frac{1}{4\pi}PV\int_{\mathbb{R}^3} K(x-y)\theta(t,y)\,dy, \quad x\in\mathbb{R}^3, \tag{3}$$

where

$$K(x) = \left(\frac{3x_1x_3}{|x|^5}, \frac{3x_2x_3}{|x|^5}, \frac{2x_3^2-x_1^2-x_2^2}{|x|^5}\right).$$

We observe that, in general, each coefficient of $v(\cdot,t)$ (with t as parameter) is only the linear combination of the Calderón-Zygmund singular integral (for the definition, see the sequel) of θ and θ itself. We write the general version as

$$v := \mathcal{T}(\theta) = \mathcal{C}(\theta) + \mathcal{S}(\theta), \tag{4}$$

where $\mathcal{T} = (\mathcal{T}_k)$, $\mathcal{C} = (\mathcal{C}_k)$, $\mathcal{S} = (\mathcal{S}_k)$, $1 \le k \le N$, are all operators mapping scalar functions to vector-valued functions and $\mathcal{C}_k$ equals a constant multiplication operator, whereas $\mathcal{S}_k$ means a Calderón-Zygmund singular integral operator. Especially the corresponding specific forms in 2D or 3D are shown as (2) or (3).

We observe that the system (DPM_α) is not more than a dissipative transport diffusion equation with non-local divergence-free velocity field (the specific relationship between velocity and temperature as (4) shows). It shares many similarities with another flow model - 2D dissipative quasi-geostrophic (QG) equation, which has been intensively studied by many authors [4–11]. From a mathematical point of view, the system (DPM_α) is somewhat a generalization of (QG) equation. Very recently, the system (DPM_α) was introduced and investigated by Córdoba and his group. In [2], the authors obtained some results on strong solutions, weak solutions and attractors for the dissipative system (DPM_α). For finite energy, they obtained global existence and uniqueness in the subcritical and critical cases. In the supercritical case, they obtained local results in H^s, $s > (N-\alpha)/2+1$ and extended to be global under a small condition $\|\theta^0\|_{H^s} \leq c\nu$, for $s > N/2 + 1$, where c is a small fixed constant. In [3], they treated the nondissipative ($\nu = 0$) 2D case and obtained the local existence and uniqueness in the Hölder space C^δ for $0 < \delta < 1$ by the particle-trajectory method and gave some blowup criteria of smooth solutions.

In this paper we present a regularity result of weak solutions of the porous media equation with $\alpha < 1$ (the supercritical case). The result asserts that if a Leray-Hopf weak solution θ of (1) is in the Hölder class C^δ with $\delta > 1 - \alpha$ on the time interval $[0, t]$, then it is actually a classical solution on $(0, t]$. The proof involves representing the functions in the Hölder space in terms of the Littlewood-Paley decomposition and using Besov space techniques. When θ is in C^δ, it also belongs to the Besov space $\dot{B}^{\delta(1-2/p)}_{p,\infty}$ for any $p \geq 2$. By taking p sufficiently large, we have $\theta \in C^{\delta_1} \cap \dot{B}^{\delta_1}_{p,\infty}$ for $\delta_1 > 1 - \alpha$. The idea is to show that $\theta \in C^{\delta_2} \cap \dot{B}^{\delta_2}_{p,\infty}$ with $\delta_2 > \delta_1$. Through iteration, we establish that $\theta \in C^\gamma$ with $\gamma > 1$. Then θ becomes a classical solution.

The rest of this paper is divided into two sections. Section 2 provides the definition of Besov spaces and necessary tools. Section 3 states and proves the main result.

Notation Throughout the paper, C denotes various 'harmless' large finite constants, and c denotes various 'harmless' small constants. We shall sometimes use $X \lesssim Y$ to denote the estimate $X \leq CY$ for some C.

2 Besov spaces and related tools

In this preparatory section, we give the definition of Besov spaces based on the Littlewood-Paley decomposition, introduce the Calderón-Zygmund singular integral, and finally we review some important results that will be used in the following.

Let us recall the Littlewood-Paley decomposition. Let $\mathcal{S}(\mathbb{R}^N)$ be the Schwartz class of rapidly decreasing functions. Given $f \in \mathcal{S}(\mathbb{R}^N)$, its Fourier transform $\mathcal{F}f = \hat{f}$ is defined by

$$\hat{f}(\xi) = (2\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} e^{-ix\cdot\xi} f(x)\,dx.$$

Choose two nonnegative radial functions $\chi, \varphi \in \mathcal{S}(\mathbb{R}^N)$, supported respectively in $\mathcal{B} = \{\xi \in \mathbb{R}^N, |\xi| \leq \frac{4}{3}\}$ and $\mathcal{C} = \{\xi \in \mathbb{R}^N, \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$, such that

$$\chi(\xi) + \sum_{j\geq 0} \varphi(2^{-j}\xi) = 1, \quad \xi \in \mathbb{R}^N,$$

$$\sum_{j\in Z} \varphi(2^{-j}\xi) = 1, \quad \xi \in \mathbb{R}^N \backslash \{0\}.$$

Setting $\varphi_j(\xi) = \varphi(2^{-j}\xi)$. Let $h = \mathcal{F}^{-1}\varphi$ and $\tilde{h} = \mathcal{F}^{-1}\chi$, we define the frequency localization operator as follows:

$$\Delta_j f = \varphi(2^{-j}D)f = 2^{jN} \int_{R^N} h(2^j y) f(x-y)\,dy,$$

$$S_j f = \sum_{k\leq j-1} \Delta_k f = \chi(2^{-j}D)f = 2^{jN} \int_{R^N} \tilde{h}(2^j y) f(x-y)\,dy.$$

Informally, $\Delta_j = S_j - S_{j-1}$ is a frequency projection to the annulus $\{|\xi| \approx 2^j\}$, while S_j is a frequency projection to the ball $\{|\xi| \lesssim 2^j\}$. One easily verifies that with our choice of φ,

$$\Delta_j \Delta_k f \equiv 0 \quad \text{if } |j-k| \geq 2 \quad \text{and} \quad \Delta_j(S_{k-1} f \Delta_k f) \equiv 0 \quad \text{if } |j-k| \geq 5.$$

Now we give the definitions of Besov spaces.

Definition 2.1 Let $s \in \mathbb{R}$, $1 \leq p,q \leq \infty$, the homogeneous Besov space $\dot{B}^s_{p,q}$ is defined by

$$\dot{B}^s_{p,q} = \{ f \in \mathcal{Z}'(\mathbb{R}^N); \|f\|_{\dot{B}^s_{p,q}} < \infty \}.$$

Here,

$$\|f\|_{\dot{B}^s_{p,q}} = \begin{cases} (\sum_{j\in\mathbb{Z}} 2^{jsq} \|\Delta_j f\|_p^q)^{\frac{1}{q}} & \text{for } q < \infty \\ \sup_{j\in\mathbb{Z}} \|\Delta_j f\|_p & \text{for } q = \infty, \end{cases}$$

and $\mathcal{Z}'(\mathbb{R}^N)$ denotes the dual space of $\mathcal{Z}'(\mathbb{R}^N) = \{f \in \mathcal{S}(\mathbb{R}^N); \partial^\gamma \hat{f}(0) = 0; \forall \gamma \in \mathbb{N}^N$ multi-index$\}$ and can be identified by the quotient space of $\mathcal{S}'/\mathcal{P}$ with the polynomials space $\mathcal{P}$.

The following proposition lists a few simple facts that we will use in the subsequent section. The proof is rather standard and one can refer to [12].

Proposition 2.2 *Assume that $s \in \mathbb{R}$ and $p, q \in [1,\infty]$.*

(1) *If $1 \leq q_1 \leq q_2 \leq \infty$, then $\dot{B}^s_{p,q_1} \subset \dot{B}^s_{p,q_2}$.*
(2) *(Besov embedding) If $1 \leq p_1 \leq p_2 \leq \infty$ and $s_1 = s_2 + n(\frac{1}{p_1} - \frac{1}{p_2})$, then $\dot{B}^{s_1}_{p_1,q}(\mathbb{R}^N) \subset \dot{B}^{s_2}_{p_2,q}(\mathbb{R}^N)$.*
(3) *If $1 < p < \infty$, then*

$$\dot{B}^s_{p,\min(p,2)} \subset \dot{W}^{s,p} \subset \dot{B}^s_{p,\max(p,2)},$$

where $\dot{W}^{s,p}$ denotes a standard homogeneous Sobolev space.

We next introduce the classical Bernstein inequality [13].

Lemma 2.3 *Let $\mathcal{B}$ be a ball, $\mathcal{C}$ be a ring, $0 \leq a \leq b \leq \infty$. Then $\forall k \in \mathbb{Z}^+ \cup \{0\}$, $\forall \lambda > 0$, there exists a constant $C > 0$ such that*

$$\sup_{|\alpha|=k} \|\partial^\alpha f\|_{L^b} \leq C\lambda^{k+N(\frac{1}{a}-\frac{1}{b})} \|f\|_{L^a} \quad \textit{if } \operatorname{supp} \mathcal{F}f \subset \lambda\mathcal{B},$$

$$C^{-1}\lambda^k \|f\|_{L^a} \leq \sup_{|\alpha|=k} \|\partial^\alpha f\|_{L^a} \leq C\lambda^k \|f\|_{L^a} \quad \textit{if } \operatorname{supp} \mathcal{F}f \subset \lambda\mathcal{C}.$$

Similar inequalities hold for the fractional derivative Λ^β.

The following proposition provides a lower bound for an integral that originates from the dissipative term in the process of L^p estimates (see [6, 14]).

Proposition 2.4 *Assume either $\alpha \geq 0$ and $p = 2$ or $0 \leq \alpha \leq 1$ and $2 < p < \infty$. Let j be an integer and $f \in \mathcal{S}'$. Then*

$$\int_{\mathbb{R}^N} |\Delta_j f|^{p-2} \Delta_j f \Lambda^{2\alpha} \Delta_j f \, dx \geq C 2^{2\alpha j} \|\Delta_j f\|_{L^p}^p$$

for some constant C depending on N, α and p.

The classical Calderón-Zygmund singular integrals are operators of the form

$$T_{cz} f(x) := PV \int_{\mathbb{R}^N} \frac{\Omega(y')}{|y|^N} f(x-y)\, dy = \lim_{\epsilon \to 0} \int_{|y|>\epsilon} \frac{\Omega(y')}{|y|^N} f(x-y)\, dy,$$

where Ω is defined on the unit sphere of $\mathbb{R}^N$, $\mathbb{S}^{N-1}$, and is integrable with zero average, and where $y' := \frac{y}{|y|} \in \mathcal{S}^{N-1}$. Clearly, the definition is meaningful for Schwartz functions. Moreover, if $\Omega \in C^1(\mathcal{S}^{N-1})$, T_{cz} is L^p bounded, $1 < p < \infty$.

The general version (4) of the relationship between v and θ is in fact ensured by the following result (see, *e.g.*, [15]).

Lemma 2.5 *Let $m \in C^\infty(\mathbb{R}^N \backslash \{0\})$ be a homogeneous function of degree 0, and let T_m be the corresponding multiplier operator defined by $(T_m f)^\wedge = m\hat{f}$, then there exist $a \in \mathbb{C}$ and $\Omega \in C^\infty(\mathbb{S}^{N-1})$ with zero average such that for any Schwartz function f,*

$$T_m f = af + PV \frac{\Omega(x')}{|x|^N} * f.$$

Remark 2.1 Since $-\Delta v = (\partial_1 \partial_N \theta, \ldots, -\partial_{N-1}\partial_N \theta, \partial_1^2 \theta + \cdots + \partial_{N-1}^2 \theta)$, the Fourier multiplier of the operator $\mathcal{T}$ is rather clear. In fact, each component of its multiplier is the linear combination of the term like $\frac{\xi_i \xi_j}{|\xi|^2}$, $i, j \in \{1, 2, \ldots, N\}$, which of course belongs to $C^\infty(\mathbb{R}^N \backslash \{0\})$ and is homogeneous of degree 0.

3 The main theorem and its proof

Theorem 3.1 *Let θ be a Leray-Hopf weak solution of* (1), *namely*

$$\theta \in L^\infty\big([0,\infty); L^2(\mathbb{R}^N)\big) \cap L^2\big([0,\infty); \dot{H}^{\alpha/2}(\mathbb{R}^N)\big). \tag{5}$$

Let $\delta > 1 - \alpha$ and let $0 < t < \infty$. If

$$\theta \in L^\infty\big([0,t]; C^\delta(\mathbb{R}^N)\big), \tag{6}$$

then $\theta \in C^\infty((0,t] \times \mathbb{R}^N)$ for $N = 2$ or 3.

Proof First we notice that (5) and (6) imply that

$$\theta \in L^\infty\big([0,t]; \dot{B}^{\delta_1}_{p,\infty}(\mathbb{R}^N)\big)$$

for any $p \geq 2$ and $\delta_1 = \delta(1 - \frac{N}{p})$. In fact, for any $\tau \in [0,t]$,

$$\big\|\theta(\cdot,\tau)\big\|_{\dot{B}^{\delta_1}_{p,\infty}} = \sup_j 2^{\delta_1 j} \|\Delta_j \theta\|_{L^p} \leq \sup_j 2^{\delta_1 j} \|\Delta_j \theta\|_{L^\infty}^{1-\frac{2}{p}} \|\Delta_j \theta\|_{L^2}^{\frac{2}{p}} \leq \big\|\theta(\cdot,\tau)\big\|_{C^\delta}^{1-\frac{2}{p}} \big\|\theta(\cdot,\tau)\big\|_{L^2}^{\frac{2}{p}}.$$

Since $\delta > 1 - \alpha$, we have $\delta_1 > 1 - \alpha$ when

$$p > p_0 \equiv \frac{N\delta}{\delta - (1-\alpha)}.$$

Next, we show that

$$\theta \in L^\infty\big([0,t]; \dot{B}^{\delta_1}_{p,\infty} \cap C^{\delta_1}\big)$$

implies

$$\theta(\cdot,t) \in \dot{B}^{\delta_2}_{p,\infty} \cap C^{\delta_2}$$

for some $\delta_2 > \delta_1$ to be specified. Let j be an integer. Applying Δ_j to the first equation of (1), we get

$$\partial_t \Delta_j\theta + \nu\Lambda^\alpha \Delta_j\theta = -\Delta_j(\nu\cdot\nabla\theta). \tag{7}$$

By Bony's notion of paraproduct,

$$\Delta_j(\nu\cdot\nabla\theta) = \sum_{|k-j|\le 4} \Delta_j(S_{k-1}\nu\cdot\nabla\Delta_k\theta) + \sum_{|k-j|\le 4} \Delta_j(\Delta_k\nu\cdot\nabla S_{k-1}\theta) \sum_{\substack{k\ge j-2\\ |k-l|\le 1}} \Delta_j(\Delta_k\nu\cdot\nabla\Delta_l\theta).$$

Multiplying (7) by $p|\Delta_j\theta|^{p-2}\Delta_j\theta$, integrating with respect to x and applying the lower bound

$$\int_{\mathbb{R}^N} |\Delta_j f|^{p-2}\Delta_j f\,\Lambda^{2\alpha}\Delta_j f\,dx \ge C2^{2\alpha j}\|\Delta_j f\|^p_{L^p}$$

of Proposition 2.4, we obtain

$$\begin{aligned}
\frac{d}{dt}\|\Delta_j\theta\|^p_{L^p} + C\nu 2^{\alpha j}\|\Delta_j\theta\|^p_{L^p} &\le -p\sum_{|k-j|\le 4}\int_{\mathbb{R}^N} |\Delta_j\theta|^{p-2}\Delta_j\theta\,\Delta_j(S_{k-1}\nu\cdot\nabla\Delta_k\theta)\,dx \\
&\quad - p\sum_{|k-j|\le 4}\int_{\mathbb{R}^N} |\Delta_j\theta|^{p-2}\Delta_j\theta\,\Delta_j(\Delta_k\nu\cdot\nabla S_{k-1}\theta)\,dx \\
&\quad - p\sum_{\substack{k\ge j-2\\ |k-l|\le 1}}\int_{\mathbb{R}^N} |\Delta_j\theta|^{p-2}\Delta_j\theta\,\Delta_j(\Delta_l\nu\cdot\nabla\Delta_k\theta)\,dx \\
&= I_1 + I_2 + I_3.
\end{aligned} \tag{8}$$

We now estimate I_1. The standard idea is to decompose it into three terms: one with commutator, one that becomes zero due to the divergence-free condition and the rest. That is, we rewrite I_1 as

$$\begin{aligned}
I_1 &= -p\sum_{|k-j|\le 4}\int_{\mathbb{R}^N} |\Delta_j\theta|^{p-2}\Delta_j\theta\cdot[\Delta_j, S_{k-1}\nu\cdot\nabla]\Delta_k\theta\,dx \\
&\quad - p\int_{\mathbb{R}^N} |\Delta_j\theta|^{p-2}\Delta_j\theta\cdot(S_j\nu\cdot\nabla\Delta_j\theta)\,dx
\end{aligned}$$

$$
\begin{aligned}
&- p \sum_{|k-j|\le 1} \int_{\mathbb{R}^N} |\Delta_j\theta|^{p-2}\Delta_j\theta \cdot (S_{k-1}v - S_j v)\cdot \nabla\Delta_j\Delta_k\theta\,dx \\
&= I_{11} + I_{12} + I_{13},
\end{aligned}
$$

where we have used the simple fact that $\sum_{|k-j|\le 1}\Delta_k\Delta_j = \Delta_j$, and the brackets [ ] represent the commutator, namely

$$
[\Delta_j, S_{k-1}v\cdot\nabla]\Delta_k\theta = \Delta_j(S_{k-1}v\cdot\nabla\Delta_k\theta) - S_{k-1}v\cdot\nabla\Delta_j\Delta_k\theta.
$$

Since u is divergence free, I_{12} becomes zero. We now bound I_{11} and I_{13}. By Hölder's inequality,

$$
I_{11} \le p\|\Delta_j\theta\|_{L^p}^{p-1}\sum_{|k-j|\le 4}\big\|[\Delta_j, S_{k-1}v\cdot\nabla]\Delta_k\theta\big\|_{L^p}.
$$

To bound the commutator, we have, by the definition of Δ_j,

$$
[\Delta_j, S_{k-1}v\cdot\nabla]\Delta_k\theta = 2^{kN}\int_{\mathbb{R}^N} h\big(2^k(x-y)\big)\big(S_{k-1}(v)(x) - S_{k-1}(v)(y)\big)\cdot\nabla\Delta_k\theta\,dy.
$$

Using the fact that $\theta \in C^{\delta_1}$ and thus

$$
\big\|S_{k-1}(v)(x) - S_{k-1}(v)(y)\big\|_{L^\infty} \le \|v\|_{C^{\delta_1}}|x-y|^{\delta_1},
$$

we obtain

$$
\big\|[\Delta_j, S_{k-1}v\cdot\nabla]\Delta_k\theta\big\|_{L^p} \le 2^{-\delta_1 j}\|v\|_{C^{\delta_1}}2^k\|\Delta_k\theta\|_{L^p}.
$$

Therefore,

$$
\begin{aligned}
|I_{11}| &\le Cp\|\Delta_j\theta\|_{L^p}^{p-1}2^{-\delta_1 j}\|v\|_{C^{\delta_1}}\sum_{|k-j|\le 4}2^k\|\Delta_k\theta\|_{L^p} \\
&\le Cp\|\Delta_j\theta\|_{L^p}^{p-1}2^{(1-2\delta_1)j}\|v\|_{C^{\delta_1}}\sum_{|k-j|\le 4}2^{\delta_1 k}\|\Delta_k\theta\|_{L^p}2^{(k-j)(1-\delta_1)} \\
&\le Cp\|\Delta_j\theta\|_{L^p}^{p-1}2^{(1-2\delta_1)j}\|v\|_{C^{\delta_1}}\|\theta\|_{\dot{B}^{\delta_1}_{p,\infty}}.
\end{aligned}
$$

The estimate for I_{13} is straightforward. By Hölder's inequality,

$$
\begin{aligned}
|I_{13}| &\le Cp\|\Delta_j\theta\|_{L^p}^{p-1}\sum_{|k-j|\le 4}\|S_{k-1}v - S_j v\|_{L^p}\|\nabla\Delta_j\theta\|_{L^\infty} \\
&\le Cp\|\Delta_j\theta\|_{L^p}^{p-1}2^{(1-\delta_1)j}\|\theta\|_{C^{\delta_1}}\sum_{|k-j|\le 4}\|\Delta_k v\|_{L^p} \\
&\le Cp\|\Delta_j\theta\|_{L^p}^{p-1}2^{(1-2\delta_1)j}\|\theta\|_{C^{\delta_1}}\sum_{|k-j|\le 4}2^{\delta_1 k}\|\Delta_k v\|_{L^p}2^{(j-k)\delta_1 k} \\
&\le Cp\|\Delta_j\theta\|_{L^p}^{p-1}2^{(1-2\delta_1)j}\|\theta\|_{C^{\delta_1}}\|v\|_{\dot{B}^{\delta_1}_{p,\infty}}.
\end{aligned}
$$

We then bound I_2. By Hölder's inequality, Bernstein's inequality and the fact $1 - \delta_1 > 0$, we obtain

$$\begin{aligned}
I_2 &\le Cp\|\Delta_j\theta\|_{L^p}^{p-1} \sum_{|k-j|\le 4} \|\Delta_k v\|_{L^p} \|\nabla S_{k-1}\theta\|_{L^\infty} \\
&\le Cp\|\Delta_j\theta\|_{L^p}^{p-1} \sum_{|k-j|\le 4} \|\Delta_k v\|_{L^p} \sum_{k'\le k-2} 2^{k'} \|\Delta_{k'}\theta\|_{L^\infty} \\
&\le Cp\|\Delta_j\theta\|_{L^p}^{p-1} \sum_{|k-j|\le 4} \|\Delta_k v\|_{L^p} 2^{(1-\delta_1)k} \sum_{k'\le k-2} 2^{(k'-k)(1-\delta_1)} 2^{k'\delta_1} \|\Delta_{k'}\theta\|_{L^\infty} \\
&\le Cp\|\Delta_j\theta\|_{L^p}^{p-1} \|\theta\|_{C^{\delta_1}} \sum_{|k-j|\le 4} \|\Delta_k v\|_{L^p} 2^{(1-\delta_1)k} \\
&\le Cp\|\Delta_j\theta\|_{L^p}^{p-1} \|\theta\|_{C^{\delta_1}} 2^{(1-2\delta_1)j} \sum_{|k-j|\le 4} 2^{\delta_1 k} \|\Delta_k v\|_{L^p} 2^{(k-j)(1-2\delta_1)} \\
&\le Cp\|\Delta_j\theta\|_{L^p}^{p-1} \|\theta\|_{C^{\delta_1}} 2^{(1-2\delta_1)j} \|v\|_{\dot{B}^{\delta_1}_{p,\infty}}.
\end{aligned}$$

Last, we bound I_3. By Hölder's inequality and Bernstein's inequality,

$$\begin{aligned}
I_3 &\le Cp\|\Delta_j\theta\|_{L^p}^{p-1} \left\| \Delta_j \nabla \cdot \left(\sum_{\substack{k\ge j-2 \\ |k-l|\le 1}} \Delta_l v \Delta_k\theta \right) \right\|_{L^p} \\
&\le Cp\|\Delta_j\theta\|_{L^p}^{p-1} 2^j \|v\|_{C^{\delta_1}} \sum_{k\ge j-2} 2^{-\delta_1 k} \|\Delta_k\theta\|_{L^p} \\
&\le Cp\|\Delta_j\theta\|_{L^p}^{p-1} 2^{(1-2\delta_1)j} \|v\|_{C^{\delta_1}} \sum_{k\ge j-2} 2^{-2\delta_1(k-j)} 2^{\delta_1 k} \|\Delta_k\theta\|_{L^p} \\
&\le Cp\|\Delta_j\theta\|_{L^p}^{p-1} 2^{(1-2\delta_1)j} \|v\|_{C^{\delta_1}} \|\theta\|_{\dot{B}^{\delta_1}_{p,\infty}}.
\end{aligned}$$

Inserting the estimates for I_1, I_2 and I_3 in (8) and eliminating $p\|\Delta_j\theta\|_{L^p}^{p-1}$ from both sides, we get

$$\begin{aligned}
\frac{d}{dt}\|\Delta_j\theta\|_{L^p} + C\nu 2^{\alpha j}\|\Delta_j\theta\|_{L^p} \le{}& 2^{(1-2\delta_1)j}\|v\|_{C^{\delta_1}} \|\theta\|_{\dot{B}^{\delta_1}_{p,\infty}} + 2^{(1-2\delta_1)j}\|\theta\|_{C^{\delta_1}} \|v\|_{\dot{B}^{\delta_1}_{p,\infty}} \\
&+ 2^{(1-2\delta_1)j}\|\theta\|_{C^{\delta_1}} \|v\|_{\dot{B}^{\delta_1}_{p,\infty}} + 2^{(1-2\delta_1)j}\|v\|_{C^{\delta_1}} \|\theta\|_{\dot{B}^{\delta_1}_{p,\infty}}.
\end{aligned}$$

Integrating with time t, we have

$$\begin{aligned}
\|\Delta_j\theta(t)\|_{L^p} \le{}& e^{-C\nu 2^{\alpha j}t}\|\Delta_j\theta(0)\|_{L^p} \\
&+ C\int_0^t e^{-C\nu 2^{\alpha j}(t-\tau)} 2^{(1-2\delta_1)j} \left(\|v\|_{C^{\delta_1}} \|\theta\|_{\dot{B}^{\delta_1}_{p,\infty}} + \|\theta\|_{C^{\delta_1}} \|v\|_{\dot{B}^{\delta_1}_{p,\infty}} \right) d\tau.
\end{aligned}$$

Multiplying both sides by $2^{(\alpha+2\delta_1-1)j}$ and taking the supremum with respect to j, we get

$$\begin{aligned}
\|\theta(t)\|_{\dot{B}^{\alpha+2\delta_1-1}_{p,\infty}} \le{}& \sup_j \left\{ e^{-C\nu 2^{\alpha j}t} 2^{(\alpha+\delta_1-1)j} \right\} \|\theta(0)\|_{\dot{B}^{\delta_1}_{p,\infty}} \\
&+ C\nu^{-1} \sup_j \left\{ \left(1 - e^{-C\nu 2^{\alpha j}t}\right) \right\} \max_{\tau\in[0,t]} \|\theta(\tau)\|_{\dot{B}^{\delta_1}_{p,\infty}} \|\theta(\tau)\|_{C^{\delta_1}}.
\end{aligned}$$

Here we have used the fact that

$$\|\nu\|_{C^{\delta_1}} \le \|\theta\|_{C^{\delta_1}} \quad \text{and} \quad \|\nu\|_{\dot{B}^{\delta_1}_{p,\infty}} \le \|\theta\|_{\dot{B}^{\delta_1}_{p,\infty}}.$$

Therefore, we conclude that if

$$\theta \in L^\infty\big([0,t]; \dot{B}^{\delta_1}_{p,\infty} \cap C^{\delta_1}\big),$$

then

$$\theta(\cdot,t) \in \dot{B}^{\alpha+2\delta_1-1}_{p,\infty}.$$

Since $\delta_1 > 1-\alpha$, we have $2\delta_1 + \alpha - 1 > \delta_1$ and thus gain regularity. In addition, according to the Besov embedding of Proposition 2.2,

$$\dot{B}^{\alpha+2\delta_1-1}_{p,\infty} \subset \dot{B}^{\delta_2}_{\infty,\infty},$$

where

$$\delta_2 = 2\delta_1 + \alpha - 1 - \frac{N}{p} = \delta_1 + \left(\delta_1 - \left(1 - \alpha + \frac{N}{p}\right)\right).$$

We have $\delta_2 > \delta_1$ when

$$p > p_1 \equiv \frac{N}{\delta_1 - (1-\alpha)}.$$

Noticing that

$$\dot{B}^{\delta_2}_{\infty,\infty} \cap L^\infty = C^{\delta_2},$$

we conclude that, for $p > \max\{p_0, p_1\}$,

$$\theta(\cdot,t) \in \dot{B}^{\delta_2}_{p,\infty} \cap C^{\delta_2}$$

for some $\delta_2 > \delta_1$. The above process can then be iterated with δ_1 replaced by δ_2. A finite number of iterations allow us to obtain that

$$\theta(\cdot,t) \in C^\gamma$$

for some $\gamma > 1$. The regularity in the spatial variable can then be converted into regularity in time. We have thus established that θ is a classical solution to the supercritical porous media equation. Higher regularity can be proved by well-known methods. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions
In this paper, WY carried 'Regularity of Hoder continuous solutions of the supercritical porous media equation'. YH, YT, QL and XW participated in the analysis. All authors read and approved the final manuscript.

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