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A new approach to BVPs with state-dependent impulses

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Abstract

The paper deals with the second-order Dirichlet boundary value problem with one state-dependent impulse

$$\begin{aligned}z''(t) &= f(t, z(t)) \quad \text{for a.e. } t \in [0, T], \\z'(\tau+) - z'(\tau-) &= I(z(\tau)), \quad \tau = \gamma(z(\tau)), \\z(0) &= 0, \quad z(T) = 0.\end{aligned}$$

Proofs of the main results contain a new approach to boundary value problems with state-dependent impulses which is based on a transformation to a fixed point problem of an appropriate operator in the space $C^1([0, T]) \times C^1([0, T])$. Sufficient conditions for the existence of solutions to the problem are given here. The presented approach can be extended to more impulses and to other boundary conditions.

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1 Introduction

Differential equations involving impulse effects appear as a natural description of observed evolution phenomena of several real world problems. We refer to the monographs [1–3].

Most papers in the literature on impulsive boundary value problems concern the case with fixed moments of impulsive effects. Papers dealing with state-dependent impulses, called also impulses at variable times, focus their attention on initial value problems or periodic problems. Such papers investigate the existence, stability or asymptotic properties of solutions of initial value problems [4–8] or solvability of autonomous periodic problems [9, 10] and nonautonomous ones [11–15]. We can also find papers investigating other boundary value problems with state-dependent impulses through some initial value problems for multi-valued maps [16, 17].

In this paper we provide a new approach to boundary value problems with state-dependent impulses based on a construction of proper sets and operators and the topological degree arguments. Unlike previous existing results, our approach enables us to find simple existence conditions for data functions and it can be used for other regular (and also singular) problems. We demonstrate it on the second-order Dirichlet boundary value

problem with one state-dependent impulse

$$z''(t) = f(t, z(t)) \quad \text{for a.e. } t \in [0, T], \tag{1}$$

$$z'(\tau+) - z'(\tau-) = I(z(\tau)), \quad \tau = \gamma(z(\tau)), \tag{2}$$

$$z(0) = 0, \quad z(T) = 0, \tag{3}$$

where we assume

$$f \in \text{Car}([0, T] \times \mathbb{R}), \quad I \in C(\mathbb{R}), \tag{4}$$

$$\left\{ \begin{array}{l} \text{there exists } h \in \text{Car}([0, T] \times [0, \infty)) \text{ such that} \\ h(t, \cdot) \text{ is nondecreasing for a.e. } t \in [0, T] \text{ and} \\ |f(t, x)| \leq h(t, |x|) \text{ for a.e. } t \in [0, T] \text{ and all } x \in \mathbb{R}, \end{array} \right. \tag{5}$$

$$\left\{ \begin{array}{l} \text{there exists } J \in C([0, T]) \text{ nondecreasing} \\ \text{and such that } |I(x)| \leq J(|x|) \text{ for } x \in \mathbb{R}, \end{array} \right. \tag{6}$$

$$\exists K > 0 : \frac{1}{K} \left[\int_0^T h(s, K + TJ(K)) \, ds + J(K) \right] < \min \left\{ 1, \frac{1}{T} \right\} \tag{7}$$

and

$$\left\{ \begin{array}{l} \gamma \in C^1([-K_1, K_1]), \\ 0 < \gamma(x) < T, \quad |\gamma'(x)| < \frac{T}{K_1} \quad \text{for } |x| \leq K_1, \\ \text{where } K_1 = K + TJ(K), \quad K \text{ is from (7)}. \end{array} \right. \tag{8}$$

Under assumptions (4)-(8), we prove the solvability of problem (1)-(3). In particular, we transform problem (1)-(3) to a fixed point problem for a proper operator in the space $C^1([0, T]) \times C^1([0, T])$. This approach can be also used for other types of boundary conditions and it can be easily extended to more impulses.

Here, we denote by $C(J)$ the set of all continuous functions on the interval J , by $C^1(J)$ the set of all functions having continuous derivatives on the interval J and by $L^1(J)$ the set of all Lebesgue integrable functions on J . For a compact interval J , we consider the linear space of functions from $C(J)$ or $C^1(J)$ equipped, respectively, with the norms

$$\|x\|_\infty = \max_{t \in J} |x(t)|, \quad \|x\|_1 = \|x\|_\infty + \|x'\|_\infty.$$

In this paper we work with the linear space $C^1([0, T]) \times C^1([0, T])$, where $T > 0$, equipped with the norm

$$\|(u, v)\| = \|u\|_1 + \|v\|_1 \quad \text{for } (u, v) \in C^1([0, T]) \times C^1([0, T]).$$

It is well-known that the mentioned normed spaces are Banach spaces. Recall that for $\mathcal{A} \subset \mathbb{R}$, a function $f : [a, b] \times \mathcal{A} \rightarrow \mathbb{R}$ satisfies the Carathéodory conditions on $[a, b] \times \mathcal{A}$ (we write $f \in \text{Car}([a, b] \times \mathcal{A})$) if

- $f(\cdot, x) : [a, b] \rightarrow \mathbb{R}$ is measurable for all $x \in \mathcal{A}$,
- $f(t, \cdot) : \mathcal{A} \rightarrow \mathbb{R}$ is continuous for a.e. $t \in [a, b]$,

- for each compact set $K \subset \mathcal{A}$, there exists a function $m_K \in L^1([a, b])$ such that $|f(t, x)| \leq m_K(t)$ for a.e. $t \in [a, b]$ and each $x \in K$.

We say that $z : [0, T] \rightarrow \mathbb{R}$ is a *solution of problem (1)-(3)*, if z is continuous on $[0, T]$, there exists unique $\tau \in (0, T)$ such that $\gamma(z(\tau)) = \tau$, $z|_{[0, \tau]}$ and $z|_{[\tau, T]}$ have absolutely continuous first derivatives, z satisfies equation (1) for a.e. $t \in [0, T]$ and fulfills conditions (2), (3).

2 Operators

In this section we assume that (4)-(8) are fulfilled. We introduce sets and operators corresponding to problem (1)-(3) and prove their properties which are needed for an application of the Leray-Schauder degree theory. Let us consider K of (7) and define the set

$$B_1 = \{u \in C^1([0, T]) : \|u\|_\infty < K, \|u'\|_\infty < K/T\}.$$

Lemma 1 *For each $u \in \overline{B}_1$, there exists a unique $\tau_u \in (0, T)$ such that*

$$\gamma(u(\tau_u)) = \tau_u. \tag{9}$$

Proof Let us take an arbitrary $u \in \overline{B}_1$. Obviously, the constant τ_u is a solution of the equation

$$\gamma(u(t)) = t,$$

i.e., τ_u is a root of the function

$$\sigma(t) = \gamma(u(t)) - t, \quad t \in [0, T].$$

From (8) it follows $\sigma(0) = \gamma(u(0)) > 0$, $\sigma(T) = \gamma(u(T)) - T < 0$. According to (8) and the definition of B_1 , we get

$$\sigma'(t) = \gamma'(u(t))u'(t) - 1 \leq |\gamma'(u(t))||u'(t)| - 1 < \frac{T}{K} \frac{K}{T} - 1 = 0, \quad t \in (0, T). \tag{10}$$

Therefore, σ is strictly decreasing on $[0, T]$ and hence it has exactly one root in $(0, T)$. \square

Now, define a functional $\mathcal{P} : \overline{B}_1 \rightarrow (0, T)$ by

$$\mathcal{P}u = \tau_u,$$

where τ_u fulfills (9). The next lemma provides an important result about the continuity of \mathcal{P} which is fundamental for our approach.

Lemma 2 *The functional \mathcal{P} is continuous on \overline{B}_1 .*

Proof Let us consider $u_n, u \in \overline{B}_1$ for $n \in \mathbb{N}$ such that $u_n \rightarrow u$ in $C^1([0, T])$. Let us denote

$$\sigma_n(t) = \gamma(u_n(t)) - t, \quad \sigma(t) = \gamma(u(t)) - t \quad \text{for } t \in [0, T].$$

By Lemma 1, $\sigma_n(\tau_n) = 0$ and $\sigma(\tau) = 0$, where $\tau_n = \mathcal{P}u_n$ and $\tau = \mathcal{P}u$, respectively. According to (8), we get $\sigma_n, \sigma \in C^1([0, T])$ for $n \in \mathbb{N}$ and

$$\sigma_n \rightarrow \sigma \quad \text{in } C([0, T]). \tag{11}$$

We will prove that $\lim_{n \rightarrow \infty} \tau_n = \tau$. Let us take an arbitrary $\epsilon > 0$. Since $\sigma(\tau) = 0$ and $\sigma'(\tau) < 0$ (cf. (10)), we can find $\xi \in (\tau - \epsilon, \tau)$ and $\eta \in (\tau, \tau + \epsilon)$ such that

$$\sigma(\xi) > 0 \quad \text{and} \quad \sigma(\eta) < 0.$$

From (11) it follows the existence of $n_0 \in \mathbb{N}$ such that

$$\sigma_n(\xi) > 0 \quad \text{and} \quad \sigma_n(\eta) < 0$$

for each $n \geq n_0$. By Lemma 1 and the continuity of σ_n , it follows that $\tau_n \in (\xi, \eta) \subset (\tau - \epsilon, \tau + \epsilon)$ for $n \geq n_0$. □

Further, consider K_1 of (8) and define sets B_2 and Ω by

$$B_2 = \left\{ v \in C^1([0, T]) : \|v\|_\infty < K_1, \|v'\|_\infty < \frac{K_1}{T} \right\}$$

and

$$\Omega = B_1 \times B_2 \subset C^1([0, T]) \times C^1([0, T]). \tag{12}$$

Finally, define an operator $\mathcal{F} : \overline{\Omega} \rightarrow C^1([0, T]) \times C^1([0, T])$ by $\mathcal{F}(u, v) = (x, y)$, where

$$\begin{cases} x(t) = \int_0^T G(t, s)\tilde{f}(s, u(s), v(s)) \, ds + g_1(t, \tau_u)I(u(\tau_u)), \\ y(t) = \int_0^T G(t, s)\tilde{f}(s, u(s), v(s)) \, ds + g_2(t, \tau_u)I(u(\tau_u)), \end{cases} \tag{13}$$

for $t \in [0, T]$, $\tau_u = \mathcal{P}u$,

$$\tilde{f}(t, u(t), v(t)) = \begin{cases} f(t, u(t)) & \text{for a.e. } t \in [0, \tau_u], \\ f(t, v(t)) & \text{for a.e. } t \in (\tau_u, T], \end{cases} \tag{14}$$

$$g_1(t, s) = \frac{t(s-T)}{T}, \quad g_2(t, s) = \frac{s(t-T)}{T}, \quad s, t \in [0, T], \tag{15}$$

and G is the Green function of the problem $u'' = 0$, $u(0) = u(T) = 0$, that is,

$$G(t, s) = \begin{cases} g_1(t, s) & \text{for } 0 \leq t \leq s \leq T, \\ g_2(t, s) & \text{for } 0 \leq s \leq t \leq T. \end{cases}$$

Lemma 3 *The operator \mathcal{F} is compact on $\overline{\Omega}$.*

Proof First, we will prove the continuity of the operator \mathcal{F} . Let us choose $(u_n, v_n), (u, v) \in \overline{\Omega}$ for $n \in \mathbb{N}$ such that

$$(u_n, v_n) \rightarrow (u, v) \quad \text{in } C^1([0, T]) \times C^1([0, T]). \tag{16}$$

Let us denote $\tau_n = \mathcal{P}u_n, \tau = \mathcal{P}u, (x_n, y_n) = \mathcal{F}(u_n, v_n), (x, y) = \mathcal{F}(u, v)$ for each $n \in \mathbb{N}$. We will prove that $x_n \rightarrow x$ in $C^1([0, T])$. For each $t \in [0, T]$, we get by (13)-(15)

$$\begin{aligned} x_n(t) - x(t) &= \int_0^\tau G(t, s)[f(s, u_n(s)) - f(s, u(s))] \, ds \\ &\quad + \int_\tau^T G(t, s)[f(s, v_n(s)) - f(s, v(s))] \, ds \\ &\quad + \int_\tau^{\tau_n} G(t, s)[f(s, u_n(s)) - f(s, v_n(s))] \, ds \\ &\quad + g_1(t, \tau_n)I(u_n(\tau_n)) - g_1(t, \tau)I(u(\tau)) \end{aligned}$$

and

$$\begin{aligned} x'_n(t) - x'(t) &= \int_0^\tau \frac{\partial G}{\partial t}(t, s)[f(s, u_n(s)) - f(s, u(s))] \, ds \\ &\quad + \int_\tau^T \frac{\partial G}{\partial t}(t, s)[f(s, v_n(s)) - f(s, v(s))] \, ds \\ &\quad + \int_\tau^{\tau_n} \frac{\partial G}{\partial t}(t, s)[f(s, u_n(s)) - f(s, v_n(s))] \, ds \\ &\quad + \frac{\partial g_1}{\partial t}(t, \tau_n)I(u_n(\tau_n)) - \frac{\partial g_1}{\partial t}(t, \tau)I(u(\tau)). \end{aligned}$$

Since

$$|G(t, s)| \leq T, \quad \left| \frac{\partial G}{\partial t}(t, s) \right| \leq 1 \quad \text{for } t, s \in [0, T], t \neq s,$$

we get

$$\begin{aligned} \|x_n - x\|_1 &\leq (T + 1) \int_0^T |f(s, u_n(s)) - f(s, u(s))| \, ds \\ &\quad + (T + 1) \int_0^T |f(s, v_n(s)) - f(s, v(s))| \, ds \\ &\quad + (T + 1) \int_\tau^{\tau_n} |f(s, u_n(s)) - f(s, v_n(s))| \, ds \\ &\quad + \max_{t \in [0, T]} |g_1(t, \tau_n)I(u_n(\tau_n)) - g_1(t, \tau)I(u(\tau))| \\ &\quad + \max_{t \in [0, T]} \left| \frac{\partial g_1}{\partial t}(t, \tau_n)I(u_n(\tau_n)) - \frac{\partial g_1}{\partial t}(t, \tau)I(u(\tau)) \right|. \end{aligned}$$

By (16), there exists a compact set $K \subset \mathbb{R}$ such that $u_n(t), v_n(t) \in K$ for each $t \in [0, T]$ and $n \in \mathbb{N}$. Consequently, by (4), there exists $m_K \in L^1([0, T])$ such that

$$|f(t, u_n(t))| \leq m_K(t), \quad |f(t, v_n(t))| \leq m_K(t)$$

for a.e. $t \in [0, T]$ and all $n \in \mathbb{N}$. Since

$$\lim_{n \rightarrow \infty} f(t, u_n(t)) = f(t, u(t)), \quad \lim_{n \rightarrow \infty} f(t, v_n(t)) = f(t, v(t))$$

for a.e. $t \in [0, T]$, then due to the Lebesgue dominated convergence theorem, it follows that

$$\int_0^T |f(s, u_n(s)) - f(s, u(s))| \, ds \rightarrow 0 \quad \text{and} \quad \int_0^T |f(s, v_n(s)) - f(s, v(s))| \, ds \rightarrow 0$$

as $n \rightarrow \infty$. Since $\lim_{n \rightarrow \infty} \tau_n = \tau$, the absolute continuity of the Lebesgue integral yields

$$\lim_{n \rightarrow \infty} \left| \int_{\tau}^{\tau_n} |f(s, u_n(s)) - f(s, v_n(s))| \, ds \right| \leq 2 \lim_{n \rightarrow \infty} \left| \int_{\tau}^{\tau_n} m_K(s) \, ds \right| = 0.$$

Further, we have for g_1 the inequality

$$\begin{aligned} & |g_1(t, \tau_n)I(u_n(\tau_n)) - g_1(t, \tau)I(u(\tau))| \\ & \leq T |I(u_n(\tau_n)) - I(u(\tau_n))| + |g_1(t, \tau_n)I(u(\tau_n)) - g_1(t, \tau)I(u(\tau))| \end{aligned}$$

for each $t \in [0, T]$ and the same is true for $\frac{\partial g_1}{\partial t}$. The continuity of g_1 , $\frac{\partial g_1}{\partial t}$ and I imply that

$$g_1(t, \tau_n)I(u_n(\tau_n)) \rightarrow g_1(t, \tau)I(u(\tau)), \quad \frac{\partial g_1}{\partial t}(t, \tau_n)I(u_n(\tau_n)) \rightarrow \frac{\partial g_1}{\partial t}(t, \tau)I(u(\tau))$$

as $n \rightarrow \infty$ uniformly w.r.t. $t \in [0, T]$. Therefore, x_n converges to x in $C^1([0, T])$. Similar arguments can be applied to the sequence $\{y_n\}_{n=1}^{\infty}$.

Now we will prove that $\mathcal{F}(\overline{\Omega})$ is relatively compact. The boundedness of $\overline{\Omega}$ implies the existence of $M > 0$ and $m \in L^1([0, T])$ such that for all $(u, v) \in \overline{\Omega}$,

$$|I(u(t))| \leq M \quad \text{for all } t \in [0, T]$$

and

$$|\tilde{f}(t, u(t), v(t))| \leq m(t) \quad \text{for a.e. } t \in [0, T].$$

Therefore, by (13), we get

$$\begin{aligned} |x(t)| + |x'(t)| & \leq (T + 1) \left(\int_0^T m(s) \, ds + M \right) =: M_1, \quad t \in [0, T], \\ |y(t)| + |y'(t)| & \leq M_1, \quad t \in [0, T]. \end{aligned}$$

We have proved that the set $\mathcal{F}(\overline{\Omega})$ is bounded in $C^1([0, T]) \times C^1([0, T])$. We now show that the set $\{(x', y') : (x, y) \in \mathcal{F}(\overline{\Omega})\}$ is equicontinuous on $[0, T]$. For a.e. $t \in [0, T]$ and all $(x, y) \in \mathcal{F}(\overline{\Omega})$, we have

$$|x''(t)| \leq m(t), \quad |y''(t)| \leq m(t) \quad \text{for a.e. } t \in [0, T].$$

As a result, for each $\epsilon > 0$, there exists $\delta > 0$ such that for each $t_1, t_2 \in [0, T]$ satisfying $|t_1 - t_2| < \delta$, the inequality

$$|x'(t_1) - x'(t_2)| + |y'(t_1) - y'(t_2)| \leq 2 \left| \int_{t_2}^{t_1} m(t) dt \right| < \epsilon$$

holds for all $(x, y) \in \mathcal{F}(\overline{\Omega})$. Consequently, $\mathcal{F}(\overline{\Omega})$ is relatively compact in $C^1([0, T]) \times C^1([0, T])$ by the Arzelà-Ascoli theorem. \square

Lemma 4 *Let $(u, v) \in \overline{\Omega}$ be a fixed point of \mathcal{F} . Then the function*

$$z(t) = \begin{cases} u(t), & t \in [0, \tau_u], \\ v(t), & t \in (\tau_u, T] \end{cases} \tag{17}$$

is a solution of problem (1)-(3).

Proof Let $(u, v) \in \overline{\Omega}$ be such that $(u, v) = \mathcal{F}(u, v)$, that is,

$$\begin{cases} u(t) = \int_0^T G(t, s) \tilde{f}(s, u(s), v(s)) ds + g_1(t, \tau_u) I(u(\tau_u)), \\ v(t) = \int_0^T G(t, s) \tilde{f}(s, u(s), v(s)) ds + g_2(t, \tau_u) I(u(\tau_u)), \end{cases} \tag{18}$$

$t \in [0, T]$, $\tau_u = \mathcal{P}u$.

Let us consider the function z defined in (17). Hence, $z(0) = u(0) = 0$, $z(T) = v(T) = 0$,

$$z(\tau_u) = u(\tau_u) = v(\tau_u) = z(\tau_u+), \tag{19}$$

and by Lemma 1,

$$\gamma(z(\tau_u)) = \tau_u. \tag{20}$$

In addition, by (17), τ_u is a unique point in $(0, \tau_u]$ satisfying (20). Put $\sigma(t) = \gamma(v(t)) - t$, $t \in [\tau_u, T]$. Due to (19) and (20), we get $\sigma(\tau_u) = 0$. Further,

$$\begin{aligned} \sigma'(t) &= \gamma'(v(t))v'(t) - 1 \leq |\gamma'(v(t))| |v'(t)| - 1 \\ &< \frac{T}{K_1} \frac{K_1}{T} - 1 = 0 \quad \text{for } t \in (\tau_u, T). \end{aligned}$$

Therefore, σ is strictly decreasing on $[\tau_u, T]$, which yields $\sigma(t) < 0$ for $t \in (\tau_u, T]$. Consequently, τ_u is a unique point in $(0, T)$ satisfying (20).

Further, we get

$$\begin{aligned} u'(t) &= \int_0^T \frac{\partial G}{\partial t}(t, s) \tilde{f}(s, u(s), v(s)) ds + \frac{\partial g_1}{\partial t}(t, \tau_u) I(u(\tau_u)), \\ v'(t) &= \int_0^T \frac{\partial G}{\partial t}(t, s) \tilde{f}(s, u(s), v(s)) ds + \frac{\partial g_2}{\partial t}(t, \tau_u) I(u(\tau_u)), \end{aligned}$$

$t \in [0, T]$, and by virtue of (14),

$$\begin{aligned} u''(t) &= \tilde{f}(t, u(t), v(t)) = f(t, u(t)) \quad \text{for a.e. } t \in [0, \tau_u), \\ v''(t) &= \tilde{f}(t, u(t), v(t)) = f(t, v(t)) \quad \text{for a.e. } t \in (\tau_u, T]. \end{aligned}$$

Therefore,

$$z''(t) = f(t, z(t)) \quad \text{for a.e. } t \in [0, T].$$

Finally,

$$\begin{aligned} z'(\tau_u+) &= v'(\tau_u) = \int_0^T \frac{\partial G}{\partial t}(\tau_u+, s) \tilde{f}(s, u(s), v(s)) \, ds + \frac{\partial g_2}{\partial t}(\tau_u, \tau_u) I(u(\tau_u)), \\ z'(\tau_u-) &= u'(\tau_u) = \int_0^T \frac{\partial G}{\partial t}(\tau_u-, s) \tilde{f}(s, u(s), v(s)) \, ds + \frac{\partial g_1}{\partial t}(\tau_u, \tau_u) I(u(\tau_u)). \end{aligned}$$

Since

$$\frac{\partial G}{\partial t}(\tau_u+, s) = \frac{\partial G}{\partial t}(\tau_u-, s) \quad \text{for } s \in [0, T], s \neq \tau_u,$$

we have

$$\begin{aligned} z'(\tau_u+) - z'(\tau_u-) &= \frac{\partial g_2}{\partial t}(\tau_u, \tau_u) I(u(\tau_u)) - \frac{\partial g_1}{\partial t}(\tau_u, \tau_u) I(u(\tau_u)) \\ &= \frac{\tau_u}{T} I(u(\tau_u)) - \frac{\tau_u - T}{T} I(u(\tau_u)) = I(u(\tau_u)) = I(z(\tau_u)). \quad \square \end{aligned}$$

3 Main result

Here, using the Leray-Schauder degree theory, we prove our main result about the solvability of problem (1)-(3). To this end, we will need the following lemma on *a priori* estimates.

Lemma 5 *Assume (4)-(8). Then for any $\lambda \in [0, 1]$ and any solution (u, v) of the equation*

$$(u, v) = \lambda \mathcal{F}(u, v), \tag{21}$$

the implication

$$(u, v) \in \overline{\Omega} \implies (u, v) \in \Omega \tag{22}$$

holds.

Proof Let us choose $\lambda \in (0, 1]$ and let $(u, v) \in \overline{\Omega}$ satisfy (21), *i.e.*,

$$\begin{cases} u(t) = \lambda \left(\int_0^T G(t, s) \tilde{f}(s, u(s), v(s)) \, ds + g_1(t, \tau_u) I(u(\tau_u)) \right), \\ v(t) = \lambda \left(\int_0^T G(t, s) \tilde{f}(s, u(s), v(s)) \, ds + g_2(t, \tau_u) I(u(\tau_u)) \right) \end{cases} \tag{23}$$

for $t \in [0, T]$. Then

$$\begin{cases} u'(t) = \lambda \left(\int_0^T \frac{\partial G}{\partial t}(t, s) \tilde{f}(s, u(s), v(s)) \, ds + \frac{\partial g_1}{\partial t}(t, \tau_u) I(u(\tau_u)) \right), \\ v'(t) = \lambda \left(\int_0^T \frac{\partial G}{\partial t}(t, s) \tilde{f}(s, u(s), v(s)) \, ds + \frac{\partial g_2}{\partial t}(t, \tau_u) I(u(\tau_u)) \right), \end{cases} \tag{24}$$

$t \in [0, T]$. Since $(u, v) \in \overline{\Omega}$, it follows that $(u, v) \in \overline{B}_1 \times \overline{B}_2$ and therefore $\|u'\|_\infty \leq K/T$, $\|u\|_\infty \leq K$, $\|v'\|_\infty \leq K_1/T$ and $\|v\|_\infty \leq K_1$. There are two possibilities as follows.

Case A. Let $\|u'\|_\infty < K/T$. Then $\|u\|_\infty < K$ and from (15) and (24), it follows

$$\begin{aligned} v'(t) &= u'(t) + \lambda \left(\frac{\tau_u}{T} - \frac{\tau_u - T}{T} \right) I(u(\tau_u)) \\ &= u'(t) + \lambda I(u(\tau_u)), \quad t \in (0, T), \end{aligned}$$

which implies, due to (6) and (8),

$$\|v'\|_\infty \leq \|u'\|_\infty + J(\|u\|_\infty) < \frac{K_1}{T}.$$

Then $\|v\|_\infty < K_1$, which yields $(u, v) \in \Omega$.

Case B. Let $\|u'\|_\infty = K/T$. From (24), (14), (6), (5) and (8), it follows

$$\begin{aligned} \|u'\| &\leq \int_0^{\tau_u} |f(s, u(s))| \, ds + \int_{\tau_u}^T |f(s, v(s))| \, ds + J(K) \\ &\leq \int_0^{\tau_u} h(s, K) \, ds + \int_{\tau_u}^T h(s, K_1) \, ds + J(K) \\ &\leq \int_0^T h(s, K_1) \, ds + J(K). \end{aligned}$$

This inequality together with (7) implies

$$1 \leq \frac{T}{K} \left[\int_0^T h(s, K_1) \, ds + J(K) \right] < T \min \left\{ 1, \frac{1}{T} \right\} = \min\{T, 1\},$$

which is a contradiction.

For $\lambda = 0$, the solution of (21) is $(u, v) = (0, 0)$, and it clearly belongs to Ω . □

Theorem 6 Assume (4)-(8). Then the operator \mathcal{F} has a fixed point in Ω .

Proof According to Lemma 5, the operator $I - \lambda\mathcal{F} : [0, 1] \times \overline{\Omega} \rightarrow (C^1([0, T]))^2$ is a homotopy. Therefore,

$$\deg(I - \mathcal{F}, \Omega) = \deg(I, \Omega) = 1,$$

and consequently the equation

$$(I - \mathcal{F})(u, v) = 0$$

has a solution in Ω . This solution is a fixed point of the operator \mathcal{F} . □

Theorem 7 Assume (4)-(8). Then problem (1)-(3) has a solution z such that

$$\|z\|_\infty < K_1, \quad \|z'\|_\infty < \frac{K_1}{T}. \tag{25}$$

Proof From Theorem 6 it follows that there exists a fixed point $(u, v) \in \Omega$ of the operator \mathcal{F} . Lemma 4 yields that the function z defined in (17) (with $\tau_u = \mathcal{P}u$) is a solution of problem (1)-(3). Estimates (25) follow from (17) and from the definitions of Ω and K_1 (cf. (12) and (8)). \square

Remark 8 Let us note that assumption (7) follows from the condition

$$\liminf_{x \rightarrow \infty} \frac{1}{x} \left[\int_0^T h(s, x + TJ(x)) \, ds + J(x) \right] < \min \left\{ 1, \frac{1}{T} \right\}.$$

4 Examples

In this section we demonstrate that Theorem 7 can be applied to sublinear, linear and superlinear problems.

Example 9 (Sublinear problem) Let us consider problem (1)-(3) with

$$T = 1, \quad f(t, x) = t^2 - |x|^\alpha \operatorname{sgn} x, \quad I(x) = |x|^\beta \operatorname{sgn} x, \quad \alpha, \beta \in (0, 1),$$

that is, f and I are sublinear in x . Then assumptions (5) and (6) are valid for

$$\begin{aligned} h(t, x) &= t^2 + x^\alpha, \quad t \in [0, 1], x > 0, \\ J(x) &= x^\beta, \quad x > 0. \end{aligned}$$

Since

$$\lim_{x \rightarrow \infty} \frac{1}{x} \left[\int_0^1 h(s, x + J(x)) \, ds + J(x) \right] = \lim_{x \rightarrow \infty} \frac{1}{x} \left[\frac{1}{3} + (x + x^\beta)^\alpha + x^\beta \right] = 0,$$

Remark 8 yields that condition (7) is satisfied for any sufficiently large K . In particular, let us put

$$\alpha = \beta = \frac{1}{2}.$$

If we choose $K = 10$, we see that (7) holds. Then by (8), we have

$$K_1 = 10 + \sqrt{10}.$$

For instance, if we choose $c \in (0, 1/(2K_1^2))$ and put

$$\gamma(x) = cx^2 + \frac{1}{2}, \quad x \in \mathbb{R}, \tag{26}$$

or if we choose $c \in (0, 1/2)$, $n > cK_1$ and put

$$\gamma(x) = c \sin \frac{x}{n} + \frac{1}{2}, \quad x \in \mathbb{R}, \tag{27}$$

we can check that conditions (8) are satisfied in both cases. Therefore, by Theorem 7, the corresponding problem (1)-(3) has at least one solution.

Note that (27) shows that γ need not be monotonous.

Example 10 (Linear problem) Let us consider problem (1)-(3) with f and I having the linear behavior in x and put

$$T = 1, \quad f(t, x) = a(t^\alpha - x), \quad I(x) = bx, \quad a, b \in \mathbb{R}, \alpha > 0.$$

Then assumptions (5) and (6) are valid for

$$h(t, x) = |a|(t^\alpha + x), \quad t \in [0, 1], x > 0,$$

$$J(x) = |b|x, \quad x > 0.$$

Since

$$\begin{aligned} & \lim_{x \rightarrow \infty} \frac{1}{x} \left[\int_0^1 h(s, x + J(x)) \, ds + J(x) \right] \\ &= \lim_{x \rightarrow \infty} \frac{1}{x} \left[|a| \left(\frac{1}{\alpha + 1} + x(1 + |b|) \right) + x|b| \right] = |a|(1 + |b|) + |b|, \end{aligned}$$

Theorem 7 can be applied, due to Remark 8, under the additional assumption

$$|a| < \frac{1 - |b|}{1 + |b|}. \tag{28}$$

If (28) holds, then for any sufficiently large K , condition (7) is satisfied. By (8), we have $K_1 = K(1 + |b|)$, and problem (1)-(3) has a solution for any γ satisfying (8). Consequently, if γ is given by (26) or (27), problem (1)-(3) is solvable.

Example 11 (Superlinear problem) Let us consider problem (1)-(3) with f and I superlinear in x . Put, for example,

$$T = 1, \quad f(t, x) = c_1 t^3 + c_2 x^3, \quad I(x) = \frac{1}{2} x^2, \quad c_1, c_2 \in \mathbb{R}. \tag{29}$$

Then assumptions (5) and (6) are valid for

$$h(t, x) = |c_1|t^3 + |c_2|x^3, \quad t \in [0, 1], x > 0,$$

$$J(x) = \frac{1}{2} x^2, \quad x > 0.$$

It holds

$$\frac{1}{x} \left[\int_0^1 h(s, x + J(x)) \, ds + J(x) \right] = \frac{1}{x} \left[\frac{|c_1|}{4} + |c_2| \left(x + \frac{1}{2} x^2 \right)^3 + \frac{1}{2} x^2 \right].$$

By virtue of (7), Theorem 7 can be applied provided there exists $K > 0$ such that

$$\frac{|c_1|}{4} + |c_2| \left(K + \frac{1}{2} K^2 \right)^3 + \frac{1}{2} K^2 < K. \tag{30}$$

Let us search K in the interval $(0, 1)$. Then $K^3 < K^2 < K$ and it holds

$$|c_2| \left(K + \frac{1}{2} K^2 \right)^3 + \frac{1}{2} K^2 < |c_2| \left(\frac{3}{2} K \right)^3 + \frac{1}{2} K^2 < \left(\frac{27}{8} |c_2| + \frac{1}{2} \right) K^2.$$

Consequently, each $K \in (0, 1)$ fulfilling the equation

$$\left(\frac{27}{8} |c_2| + \frac{1}{2} \right) K^2 - K + \frac{|c_1|}{4} = 0$$

satisfies (30) as well. Put, for example, $c_1 = 1$, $c_2 = -4/27$. Then we get that for $K = 1/2$ inequality (30) holds. Consequently, (8) gives $K_1 = 5/8$ and the corresponding problem (1)-(3) is solvable for any γ satisfying (8). In particular, γ given by (26) or (27) can be considered in this case as well.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

Both authors contributed equally to the manuscript and read and approved the final manuscript.

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