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The Cauchy problem for the generalized Degasperis-Procesi equation

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Abstract

In this paper, we investigate the Cauchy problem for the generalized Degasperis-Procesi equation in a Besov space. Firstly, we prove that the generalized Degasperis-Procesi equation is locally well posed in $B^s_{p,r}$ with $s > 1 + \frac{1}{p}$ (or $s \geq 1 + \frac{1}{p}$ if $r = 1$ with $p \in [1, +\infty)$). Secondly, we prove that the generalized Degasperis-Procesi equation possesses the peaked solitary wave which is the weak solution to the generalized Degasperis-Procesi equation. Thirdly, we prove that the data-to-solution map for the generalized Degasperis-Procesi equation is not uniformly continuous in $B^{3/2}_{2,\infty}$. Fourthly, we prove that the data-to-solution map for the generalized Degasperis-Procesi equation is not uniformly continuous in $H^s(\mathbf{R})$ with $s < 3/2$. Finally, we give a blow-up criterion.

MSC: 35G25; 35L05

Keywords: Cauchy problem; generalized Degasperis-Procesi equation; weak solution; blow-up criterion

1 Introduction

In this paper, we consider the Cauchy problem for the following generalized Degasperis-Procesi equation:

$$u_t - u_{txx} + u^k u_x - \left(u^k u_x\right)_{xx} + Q\left[u^{k+1}\right]_x = 0, \tag{1.1}$$

$$u(x,0) = u_0(x), \tag{1.2}$$

where $Q \in \mathbf{R}$ is a constant. When $k = 1$ and $Q = \frac{3}{2}$, (1.1) reduces to the Degasperis-Procesi equation

$$u_t - u_{txx} + 4uu_x = 3u_x u_{xx} + uu_{xxx}. \tag{1.3}$$

Equation (1.3) possesses the Lax pair and bi-Hamiltonian structures and infinite many conservation laws [1]. The Degasperis-Procesi equation [2] possesses peaked solitons which are stable [3] and shock peakons of the form $u(x,t) = -\frac{1}{t+k}\operatorname{sign}(x)e^{-|x|}$, $k > 0$. The Degasperis-Procesi equation possesses the global weak solution and blow-up structure [4–6]. Constantin and Lannes studied the relevance between the Camassa-Holm equation and the Degasperis-Procesi equation [7]. The Degasperis-Procesi equation possesses the infinite propagation speed [8]. The Degasperis-Procesi equation possesses multi-peakon solutions [9] and multisoliton [10]. Himonas and his co-authors [11, 12] proved that the data-to-solution for the Camassa-Holm equation, the Degasperis-Procesi equation is not uniformly continuous in $H^s(\mathbf{R})$ with $s > 3/2$, respectively. Himonas *et al.* proved the non-uniform continuity in H^1 of the solution map of the CH equation [13]. Recently, Gui and Liu [14] studied the Cauchy problem for the Degasperis-Procesi equation in Besov spaces. Yan *et al.* [15] studied the Cauchy problem for the Novikov equation in Besov spaces.

Let $P(D) = -\partial_x(1-\partial_x^2)^{-1}$ and $p(x) = \frac{1}{2}e^{-|x|}$, $x \in \mathbf{R}$. By using the identity $(1-\partial_x^2)^{-1}f = p * f$ for $f \in L^2$, we can rewrite (1.1)-(1.2) as follows:

$$u_t + u^k u_x = QP(D)\big[u^{k+1}\big], \tag{1.4}$$

$$u(x,0) = u_0(x). \tag{1.5}$$

In this paper, motivated by [14], we study the Cauchy problem for (1.4) in Besov spaces. Firstly, we use the standard iterative method to prove that the generalized Degasperis-Procesi equation is locally well posed in $B^s_{p,r}$ with $s > 1 + \frac{1}{p}$ (or $s \geq 1 + \frac{1}{p}$ if $r = 1$ with $p \in [1, +\infty)$). Secondly, we prove that the generalized Degasperis-Procesi equation possesses the peaked solitary wave which is the weak solution to the generalized Degasperis-Procesi equation. Thirdly, we prove that the data-to-solution map for the generalized Degasperis-Procesi equation is not uniformly continuous in $B^{3/2}_{2,\infty}$. Fourthly, we prove that the data-to-solution map for the generalized Degasperis-Procesi equation is not uniformly continuous in $H^s(\mathbf{R})$ with $s < 3/2$. Finally, we give a blow-up criterion.

Notice that the structure of (1.4) is more complicated than that of the Degasperis-Procesi equation. Thus, to prove that the sequence of smooth solutions $(u^{(n)})_{n\in\mathbf{N}}$ is uniformly bounded in $C([0,T]; B^s_{p,r}) \cap C^1([0,T]; B^{s-1}_{p,r})$ with $s > 1 + \frac{1}{p}$ (or $s \geq 1 + \frac{1}{p}$ if $r = 1$ with $p \in [1, +\infty)$), we choose that

$$\big\|u^{(n)}\big\|_{B^s_{p,r}} \leq \frac{\|u_0\|_{B^s_{p,r}}}{(1 - 2kC\|u_0\|^k_{B^s_{p,r}} t)^{1/k}}, \quad t \in [0,T]. \tag{1.6}$$

In proving Theorem 1.5, we explain why we choose (1.6). It is worthy of pointing out that we use Fatou's lemma and the upper limit as well as Gronwall's inequality to prove that $(u^{(n)})_{n\in\mathbf{N}}$ is a Cauchy sequence in $C([0,T]; B^{s-1}_{p,r})$ with $s > 1 + \frac{1}{p}$ (or $s \geq 1 + \frac{1}{p}$ if $r = 1$ with $p \in [1, +\infty)$).

To introduce the main results, we define

$$E^s_{p,r}(T) = C\big([0,T]; B^s_{p,r}\big) \cap C^1\big([0,T]; B^{s-1}_{p,r}\big).$$

The main results of this paper are as follows.

Theorem 1.1 *Let $u_0(x) \in B^s_{p,r}$ with $s > 1 + \frac{1}{p}$ (or $s \geq 1 + \frac{1}{p}$ if $r = 1$ with $p \in [1, +\infty)$). Problem* (1.4)-(1.5) *is locally well posed. Moreover,*

$$\big\|u(t)\big\|_{B^s_{p,r}} \leq \frac{\|u_0\|_{B^s_{p,r}}}{(1 - 2kC\|u_0\|^k_{B^s_{p,r}} t)^{1/k}}, \quad t \in [0,T]. \tag{1.7}$$

A function $u : [0,T) \times \mathbf{R}$ is called a weak solution to (1.1) (or (1.4)) if u belongs to $L^\infty_{\mathrm{loc}}([0,T);H^1)$ and satisfies the following identity:

$$\begin{aligned}&\int_0^T \int_{\mathbf{R}} \left[u\phi_t + \frac{1}{k+1}u^{k+1}\phi_x + p * \left[\frac{k^2+2k}{k+1}u^{k+1}\right]\phi_x\right] dx\,dt \\ &\quad + \int_{\mathbf{R}} u(x,0)\phi(x,0)\,dx = 0,\end{aligned} \tag{1.8}$$

where $p(x) = \frac{1}{2}e^{-|x|}$, for any smooth test function $\phi(x,t) \in C_c^\infty([0,T) \times \mathbf{R})$. If u is a weak solution on $[0,T)$ for every $T > 0$, then it is called a global weak solution.

Theorem 1.2 *When* $Q = \frac{k(k+2)}{k+1}$ *in* (1.4), $u_c(x,t) = c^{1/k}e^{-|x-ct|}$ *with* $c > 0$ *is a weak solution of* (1.4) *in the sense of* (1.8).

Theorem 1.3 *When* $Q = \frac{k(k+2)}{k+1}$ *in* (1.4), *the data-to-solution map for the generalized Degasperis-Procesi equation is not uniformly continuous in* $B^{3/2}_{2,\infty}$. *More precisely, there exists a global solution* $u \in L^\infty(\mathbf{R}^+;B^{3/2}_{2,\infty})$ *to the Cauchy problem for* (1.4) *such that for any* $T > 0$ *and* $\epsilon > 0$, *there exists a solution* $v \in L^\infty(0,T;B^{3/2}_{2,\infty})$ *with*

$$\left\|v(0) - u(0)\right\|_{B^{3/2}_{2,\infty}} \le \epsilon, \qquad \left\|v(t) - u(t)\right\|_{L^\infty(0,T;B^{3/2}_{2,\infty})} \ge 1.$$

Theorem 1.4 *When* $Q = \frac{k(k+2)}{k+1}$ *in* (1.4), *the data-to-solution map for the generalized Degasperis-Procesi equation is not uniformly continuous in* $H^s(\mathbf{R})$ *with* $s < 3/2$.

Theorem 1.5 *Assume that* $T^\star$ *is the maximal time of existence of the solution to problem* (1.4)-(1.5). *If* $T^\star < \infty$, *then*

$$\int_0^{T^\star} \|u_x\|^k_{L^\infty}\,d\tau = +\infty. \tag{1.9}$$

Moreover, $T^\star \ge \frac{1}{Ck\|u_0\|^k_{B^s_{p,r}}}$.

The remainder of this paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, we prove Theorem 1.1. In Section 4, we prove Theorem 1.2. In Section 5, we prove Theorem 1.3. In Section 6, we prove Theorem 1.4. In Section 7, we prove Theorem 1.5.

2 Preliminaries

In this section, we give Lemmas 2.1-2.4. The proof of Lemmas 2.1-2.4 can be seen in [16–21].

Lemma 2.1 (Littlewood-Paley decomposition) *Let* $B = \{\xi \in \mathbf{R}^n, |\xi| \le \frac{4}{3}\}$ *and* $C = \{\xi \in \mathbf{R}^n, \frac{3}{4} \le |\xi| \le \frac{8}{3}\}$. *There exists a couple of smooth radial functions* $(\chi,\phi) \in (C_c^\infty(B), C_c^\infty(C))$ *such that*

$$\forall \xi \in \mathbf{R}^n, \quad \chi(\xi) + \sum_{q\in\mathbf{N}} \phi\left(2^{-q}\xi\right) = 1$$

and

$$\operatorname{Supp}\phi(2^{-q}\cdot)\cap\operatorname{Supp}\phi(2^{-q'}\cdot)=\emptyset\quad\textit{if } |q-q'|\geq 2,$$
$$\operatorname{Supp}\chi(\cdot)\cap\operatorname{Supp}\phi(2^{-q}\cdot)=\emptyset\quad\textit{if } |q|\geq 1$$

and

$$\frac{1}{3}\leq\chi(\xi)^2+\sum_{q\geq 0}\phi(2^{-q}\xi)^2\leq 1,\quad\forall\xi\in\mathbf{R}^n. \tag{2.1}$$

Then, for $u\in\mathscr{S}'(\mathbf{R})$, *the nonhomogeneous dyadic blocks are defined as follows*:

$$\Delta_q u=0\quad\textit{if } q\leq -2,$$
$$\Delta_{-1}u=\chi(D)u=\mathscr{F}_x^{-1}\chi\mathscr{F}_x u,$$
$$\Delta_q u=\phi(2^{-q}D)=\mathscr{F}_x^{-1}\phi(2^{-q}\xi)\mathscr{F}_x u\quad\textit{if } q\geq 0.$$

Thus we obtain

$$u=\sum_{q\in\mathbf{Z}}\Delta_q u\quad\textit{in } \mathscr{S}'(\mathbf{R}),$$

and the low frequency cut-off S_q *is defined by*

$$S_q u=\sum_{p=-1}^{q-1}\Delta_p u=\chi(2^{-q}D)u=\mathscr{F}_x^{-1}\chi(2^{-q}\xi)\mathscr{F}_x u,\quad\forall q\in\mathbf{N},$$

as well as

$$\Delta_p\Delta_q u\equiv 0\quad\textit{if } |p-q|\geq 2,$$
$$\Delta_q(S_{p-1}u\Delta_p v)\equiv 0\quad\textit{if } |p-q|\geq 5, \forall u,v\in\mathscr{S}'(\mathbf{R}),$$
$$\|\Delta_p u\|_{L^p}\leq C\|u\|_{L^p},$$
$$\|S_q u\|_{L^p}\leq C\|u\|_{L^p},\quad\forall 1\leq p\leq+\infty,$$

where C is a positive constant independent of q.

Definition (Besov spaces) Let $s\in\mathbf{R}$ and $1\leq p\leq+\infty$. The nonhomogeneous Besov space $B^s_{p,r}(\mathbf{R}^n)$ is defined by

$$B^s_{p,r}(\mathbf{R}^n) = \{f\in\mathscr{S}'(\mathbf{R}^n):\|f\|_{B^s_{p,r}}=\|2^{qs}\Delta_q f\|_{l^r(L^p)}=\|(2^{qs}\|\Delta_q f\|_{L^p})_{q\geq -1}\|_{l^r}<\infty\}.$$

In particular, if $s=\infty$, then $B^s_{p,r}=\bigcap_{s\in\mathbf{R}}B^s_{p,r}$.

Lemma 2.2 *Let* $s\in\mathbf{R}$, $1\leq p,r,p_j,r_j\leq\infty$, $j=1,2$, *then*:

(1) $B^s_{p,r}$ *is a Banach space and is continuously embedded in* $\mathscr{S}'(\mathbf{R}^n)$.
(2) $B^{s_1}_{p_1,r_1} \hookrightarrow B^{s_2}_{p_2,r_2}$, *if* $p_1 \le p_2$ *and* $r_1 \le r_2$ *and* $s_2 = s_1 - n(\frac{1}{p_1} - \frac{1}{p_2})$

$$B^{s_1}_{p,r_2} \hookrightarrow B^{s_2}_{p,r_1} \quad \text{locally compact if } s_2 < s_1.$$

(3) $\forall s > 0$, $B^s_{p,r} \cap L^\infty$ *is a Banach algebra.* $B^s_{p,r}$ *is a Banach algebra iff* $B^s_{p,r} \hookrightarrow L^\infty$ *and iff* $s > \frac{1}{p}$ *or* ($s \ge \frac{1}{p}$ *and* $r = 1$).
(4) (i) *For* $s > 0$,

$$\|fg\|_{B^s_{p,r}} \le C\big(\|f\|_{B^s_{p,r}}\|g\|_{L^\infty} + \|f\|_{L^\infty}\|g\|_{B^s_{p,r}}\big), \quad \forall f,g \in B^s_{p,r} \cap L^\infty.$$

(ii) $\forall s_1 \le \frac{1}{p} < s_2$ ($s_2 \ge \frac{1}{p}$ *if* $r = 1$) *and* $s_1 + s_2 > 0$,

$$\|fg\|_{B^{s_1}_{p,r}} \le C\|f\|_{B^{s_1}_{p,r}}\|g\|_{B^{s_2}_{p,r}}, \quad \forall f \in B^{s_1}_{p,r}, g \in B^{s_2}_{p,r}.$$

(5) $\forall \theta \in [0,1]$ *and* $s = \theta s_1 + (1-\theta)s_2$,

$$\|f\|_{B^s_{p,r}} \le C\|f\|^\theta_{B^{s_1}_{p,r}}\|f\|^{1-\theta}_{B^{s_2}_{p,r}}, \quad \forall f \in B^{s_1}_{p,r} \cap B^{s_2}_{p,r}.$$

(6) *If* $(u_n)_{n\in\mathbf{N}}$ *is bounded in* $B^s_{p,r}$ *and* $u_n \to u$ *in* $\mathscr{S}'(\mathbf{R}^n)$, *then* $u \in B^s_{p,r}$ *and*

$$\|u\|_{B^s_{p,r}} \le \liminf_{n\to\infty} \|u_n\|_{B^s_{p,r}}.$$

(7) *Let* $m \in \mathbf{R}$ *and* Ψ *be an* S^m*-multiplier. Then the operator* $\Psi(D)$ *is continuous from* $B^s_{p,r}$ *into* $B^{s-m}_{p,r}$. *In particular,* $-\partial_x(1-\partial_x^2)^{-1}$ *is continuous from* $B^s_{p,r}$ *into* $B^{s-1}_{p,r}$.

Lemma 2.3 (*A priori* estimates in Besov spaces) *Let* $1 \le p, r \le \infty$ *and* $s > -\min\{\frac{1}{p}, 1 - \frac{1}{p}\}$. *Assume that* $f_0 \in B^s_{p,r}$, $F \in L^1(0,T;B^s_{p,r})$ *and that* $\partial_x v$ *belongs to* $L^1(0,T;B^{s-1}_{p,r})$ *if* $s > 1 + \frac{1}{p}$ *or to* $L^1(0,T;B^{1/p}_{p,r} \cap L^\infty)$ *otherwise. If* $f \in L^\infty(0,T;B^s_{p,r}) \cap C([0,T];\mathscr{S}'(\mathbf{R}))$ *solves the following* 1*-D linear transport equation*:

$$f_t + vf_x = F, \tag{2.2}$$

$$f(x,0) = f_0, \tag{2.3}$$

then there exists a constant C depending only on s, p, r such that the following statements hold:
(1) *If* $r = 1$ *or* $s \ne 1 + \frac{1}{p}$, *then*

$$\|f\|_{B^s_{p,r}} \le \|f_0\|_{B^s_{p,r}} + \int_0^t \big\|F(\tau)\big\|_{B^s_{p,r}}\,d\tau + C\int_0^t V'(\tau)\big\|f(\tau)\big\|_{B^s_{p,r}}\,d\tau$$

or

$$\|f\|_{B^s_{p,r}} \le e^{CV(t)}\left(\|f_0\|_{B^s_{p,r}} + \int_0^t e^{-CV(\tau)}\big\|F(\tau)\big\|_{B^s_{p,r}}\,d\tau\right) \tag{2.4}$$

with $V(t) = \int_0^t \|v_x(\tau)\|_{B^{1/p}_{p,r}\cap L^\infty}\,d\tau$ *if* $s < 1 + \frac{1}{p}$ *and* $V(t) = \int_0^t \|v_x(\tau)\|_{B^{s-1}_{p,r}}\,d\tau$ *else.*

(2) *If* $s \le 1 + \frac{1}{p}$, $f_0' \in L^\infty$ *and* $f_x \in L^\infty((0,T) \times \mathbf{R})$ *and* $F_x \in L^1(0,T;L^\infty)$, *then*

$$\begin{aligned}
&\|f(t)\|_{B^s_{p,r}} + \|f_x(t)\|_{L^\infty} \\
&\quad \le e^{CV(t)}\left(\|f_0\|_{B^s_{p,r}} + \|f_0'\|_{L^\infty} + \int_0^t e^{-CV(\tau)}\left[\|F(\tau)\|_{B^s_{p,r}} + \|F_x(\tau)\|_{L^\infty}\right] d\tau\right)
\end{aligned}$$

with

$$V(t) = \int_0^t \|\partial_x v(\tau)\|_{B^{1/p}_{p,r} \cap L^\infty} \, d\tau.$$

(3) *If* $f = v$, *then for all* $s > 0$, (1) *holds true when* $V(t) = \int_0^t \|v_x(\tau)\|_{L^\infty} \, d\tau$.
(4) *If* $r < \infty$, *then* $f \in C([0,T];B^s_{p,r})$. *If* $r = \infty$, *then* $f \in C([0,T];B^{s'}_{p,1})$ *for all* $s' < s$.

Lemma 2.4 (Existence and uniqueness) *Let* p, r, s, f_0 *and* F *be as in the statement of Lemma* 2.3. *Assume that* $v \in L^\rho(0,T;B^{-M}_{\infty,\infty})$ *for some* $\rho > 1$ *and* $M > 0$ *and* $v_x \in L^1(0,T;B^{s-1}_{p,r})$ *if* $s > 1 + \frac{1}{p}$ *or* $s = 1 + \frac{1}{p}$ *and* $r = 1$ *and* $v_x \in L^1(0,T;B^{1/p}_{p,\infty} \cap L^\infty)$ *if* $s < 1 + \frac{1}{p}$. *Then problem* (2.1)-(2.2) *has a unique solution* $f \in L^\infty(0,T;B^s_{p,r}) \cap (\bigcap_{s'<s} C([0,T];B^{s'}_{p,1}))$ *and the inequalities of Lemma* 2.3 *can hold true. Moreover, if* $r < \infty$, *then* $f \in C([0,T];B^s_{p,r})$.

3 Proof of Theorem 1.1

In this section, we complete the proof of Theorem 1.1 and suppose that $s > 1 + \frac{1}{p}$ (or $s \ge 1 + \frac{1}{p}$ if $r = 1$ with $p \in [1,+\infty)$).

First step: approximate solution

By using the standard iterative process, we construct a sequence of smooth solutions $(u^{(n)})_{n\in\mathbf{N}} \in C(\mathbf{R}^+;B^\infty_{p,r})$. Assume that $u^{(0)} := 0$, by induction we define a sequence of smooth functions $(u^{(n)})_{n\in\mathbf{N}}$ by solving the following linear transport equation:

$$u_t^{(n+1)} + \left[u^{(n)}\right]^k u_x^{(n+1)} = QP(D)\left[\left(u^{(n)}\right)^{k+1}\right], \tag{3.1}$$

$$u^{(n+1)}(x,0) = u_0^{(n+1)}(x) = S_{n+1}u_0(x). \tag{3.2}$$

By using the fact that $S_{n+1}u_0$ belong to $B^\infty_{p,r}$, from Lemma 2.4, for all $n \in \mathbf{N}$, we can show by induction that problem (3.1)-(3.2) has a global solution $(u^{(n)})_{n\in\mathbf{N}} \in C(\mathbf{R}^+, B^\infty_{p,r})$.

Second step: uniform bounds

For $s > 1 + \frac{1}{p}$ (or $s \ge 1 + \frac{1}{p}$ if $r = 1$ with $p \in [1,+\infty)$) and $n \in \mathbf{N}$, we prove that

$$\begin{aligned}
&\|u^{(n+1)}\|_{B^s_{p,r}} \\
&\quad \le e^{CU^n(t)}\left(\|u_0\|_{B^s_{p,r}} + C\int_0^t e^{-CU^n(\tau)} \|u^{(n)}\|^{k+1}_{B^s_{p,r}} \, d\tau\right),
\end{aligned} \tag{3.3}$$

with $U^n = \int_0^t \|u^{(n)}\|^k_{B^s_{p,r}} \, d\tau$.

From (2.3) of Lemma 2.3 and (3.1), we derive that

$$\begin{aligned}\|u^{(n+1)}(t)\|_{B^s_{p,r}} &\le e^{C\int_0^t \|u_x^{(n)}(t')\|_{B^{s-1}_{p,r}}\,dt'}\|u_0\|_{B^s_{p,r}} \\ &\quad + C\int_0^t e^{C\int_\tau^t \|u_x^{(n)}(t')\|_{B^{s-1}_{p,r}}\,dt'}\|F(u^{(n)})\|_{B^s_{p,r}}\,d\tau,\end{aligned} \tag{3.4}$$

where

$$F(u^{(n)}) = P(D)\big[(u^{(n)})^{k+1}\big]. \tag{3.5}$$

By using the S^{-1} multiplier property of $P(D)$ and the fact $B^s_{p,r}$ with $s > 1 + \frac{1}{p}$ (or $s \ge 1 + \frac{1}{p}$ if $r = 1$ with $p \in [1, +\infty)$) is a Banach algebra, we have

$$\begin{aligned}\|F(u^{(n)})\|_{B^{s-1}_{p,r}} &\le C\|P(D)\big[(u^{(n)})^{k+1}\big](t')\|_{B^s_{p,r}} \\ &\le C\|(u^{(n)})^{k+1}(t')\|_{B^{s-1}_{p,r}} \le C\|u^{(n)}(t')\|^{k+1}_{B^s_{p,r}}.\end{aligned} \tag{3.6}$$

Inserting (3.6) into (3.4) yields (3.3).

Let us fix $T > 0$ such that

$$T \le \frac{1}{4kC\|u_0\|^k_{B^s_{p,r}}}, \tag{3.7}$$

and suppose that

$$\|u^{(n)}(t)\|_{B^s_{p,r}} \le \frac{\|u_0\|_{B^s_{p,r}}}{(1 - 2kC\|u_0\|^k_{B^s_{p,r}}t)^{1/k}}, \quad t \in [0, T]. \tag{3.8}$$

Since $U^n(t) = \int_0^t \|u^{(n)}\|^k_{B^s_{p,r}}\,d\tau$, by using (3.8), we have

$$\begin{aligned}e^{C(U^n(t) - U^n(\tau))} &= e^{C\int_\tau^t \|u^{(n)}(t')\|^k_{B^s_{p,r}}\,dt'} \le e^{-\frac{1}{2k}\int_\tau^t \frac{d(1-2kC\|u_0\|^k_{B^s_{p,r}}t')}{(1-2kC\|u_0\|^k_{B^s_{p,r}}t')}} \\ &= \left(\frac{1 - 2kC\|u_0\|^k_{B^s_{p,r}}\tau}{1 - 2kC\|u_0\|^k_{B^s_{p,r}}t}\right)^{\frac{1}{2k}}.\end{aligned} \tag{3.9}$$

When $\tau = 0$ in (3.9), we have

$$e^{CU^n(t)} \le \left(\frac{1}{1 - 2kC\|u_0\|^k_{B^s_{p,r}}t}\right)^{\frac{1}{2k}}. \tag{3.10}$$

Inserting (3.9), (3.10) into (3.3) yields

$$\begin{aligned}\|u^{(n+1)}(t)\|_{B^s_{p,r}} &\le \frac{\|u_0\|_{B^s_{p,r}}}{(1 - 2kC\|u_0\|^k_{B^s_{p,r}}t)^{1/2k}}\left[1 - \frac{1}{2k}\int_0^t \frac{d(1 - 2kC\|u_0\|^k_{B^s_{p,r}}t)}{(1 - 2kC\|u_0\|^k_{B^s_{p,r}}t)^{1+\frac{1}{2k}}}\right] \\ &\le \frac{\|u_0\|_{B^s_{p,r}}}{(1 - 2kC\|u_0\|^k_{B^s_{p,r}}t)^{1/k}}.\end{aligned} \tag{3.11}$$

Thus, $(u^{(n)})_{n\in\mathbf{N}}$ is uniformly bounded in $C([0,T];B^s_{p,r})$. By using (3.9) and the fact that $B^s_{p,r}$ is a Banach algebra and (3.5) and using the S^{-1}-multiplier property of $P(D)$, we have that

$$\big\|\big[u^{(n)}\big]^k u_x^{(n+1)}\big\|_{B^s_{p,r}} \le C\big\|u^{(n)}\big\|^k_{B^s_{p,r}}\big\|u^{(n+1)}\big\|_{B^s_{p,r}} \le \frac{C\|u_0\|^{k+1}_{B^s_{p,r}}}{(1-2C\|u_0\|^k_{B^s_{p,r}}t)^{\frac{k+1}{k}}} \tag{3.12}$$

and

$$\begin{aligned}\big\|P(D)\big[\big(u^{(n)}\big)^{k+1}\big]\big\|_{B^{s-1}_{p,r}} &\le C\big\|\big(u^{(n)}\big)^{k+1}\big\|_{B^{s-1}_{p,r}} \le C\big\|u^{(n)}\big\|^{k+1}_{B^s_{p,r}} \\ &\le \frac{C\|u_0\|^{k+1}_{B^s_{p,r}}}{(1-2kC\|u_0\|_{B^s_{p,r}}t)^{\frac{k+1}{k}}}.\end{aligned} \tag{3.13}$$

Consequently,

$$\big(u^{(n)}\big)_n \subset C\big([0,T];B^s_{p,r}\big)\cap C^1\big([0,T];B^{s-1}_{p,r}\big). \tag{3.14}$$

Remark Inserting (3.7) into (3.8) yields

$$\big\|u^{(n)}\big\|_{B^s_{p,r}} \le 4\|u_0\|_{B^s_{p,r}} \tag{3.15}$$

for $n\in\mathbf{N}$. From (3.11) and (3.7), $\forall n\in\mathbf{N}^+$, we have that

$$e^{CU^n} \le \exp\left[\int_0^t \frac{C\|u_0\|^k_{B^s_{p,r}}}{1-2kC\|u_0\|^k_{B^s_{p,r}}t}\,d\tau\right] \le 4 \tag{3.16}$$

and

$$\big\|u(t)\big\|_{B^s_{p,r}} \le \frac{\|u_0\|_{B^s_{p,r}}}{(1-2kC\|u_0\|^k_{B^s_{p,r}}t)^{1/k}} \tag{3.17}$$

with the aid of Fatou's lemma. We define

$$L = 4\big(\|u_0\|_{B^s_{p,r}}+1\big). \tag{3.18}$$

Thus,

$$\big\|u^{(n)}\big\|_{B^s_{p,r}}+1 \le L \tag{3.19}$$

for $n\in\mathbf{N}^+$.

Third step: convergence

We prove that $(u^{(n)})_{n\in\mathbf{N}}$ is a Cauchy sequence in $C([0,T];B^{s-1}_{p,r})$. For $(m,n)\in\mathbf{N}^2$, we have

$$\begin{aligned}&\big[\partial_t+\big(u^{(n+m)}\big)^k\partial_x\big]\big(u^{(n+1+m)}-u^{(n+1)}\big) \\ &\quad= \big(\big(u^{(n)}\big)^k-\big(u^{(n+m)}\big)^k\big)\partial_x u^{(n+1)} \\ &\qquad+ QP(D)\left[\big(u^{(n+m)}-u^{(n)}\big)\sum_{j=0}^k\big(u^{(n+m)}\big)^{k-j}\big(u^{(n)}\big)^j\right].\end{aligned} \tag{3.20}$$

Combining (2.3) with (3.20), we have that

$$\begin{aligned}
&\big\|\big[u^{(n+1+m)} - u^{(n+1)}\big](t)\big\|_{B^{s-1}_{p,r}} \\
&\quad\le e^{U^n(t)}\big(\big\|u_0^{(n+1+m)} - u_0^{(n+1)}\big\|_{B^{s-1}_{p,r}}\big) \\
&\qquad + \int_0^t e^{U^n(t)-U^n(\tau)}\big\|F\big(u^{(n)}, u^{(n+m)}, \partial_x u^{(n+1)}\big)\big\|_{B^{s-1}_{p,r}}\,d\tau,
\end{aligned}\tag{3.21}$$

where

$$\begin{aligned}
&U^n(t) = \int_0^t \big\|\big(u^{(n+m)}\big)^k\big\|_{B^s_{p,r}}\,d\tau, \\
&F\big(u^{(n)}, u^{(n+m)}, \partial_x u^{(n+1)}\big) \\
&\quad= \big(\big(u^{(n)}\big)^k - \big(u^{(n+m)}\big)^k\big)\partial_x u^{(n+1)} + QP(D)\Bigg[\big(u^{(n+m)} - u^{(n)}\big)\sum_{j=0}^k \big(u^{(n+m)}\big)^{k-j}\big(u^{(n)}\big)^j\Bigg].
\end{aligned}$$

By using (3) and (7) of Lemma 2.2, we have that

$$\begin{aligned}
&\big\|F\big(u^{(n)}, u^{(n+m)}, \partial_x u^{(n+1)}\big)\big\|_{B^{s-1}_{p,r}} \\
&\quad\le C\big\|\big(u^{(n)}\big)^k - \big(u^{(n+m)}\big)^k\big\|_{B^{s-1}_{p,r}}\big\|\partial_x u^{(n+1)}\big\|_{B^{s-1}_{p,r}} \\
&\qquad + C\Bigg\|P(D)\Bigg[\big(u^{(n+m)} - u^{(n)}\big)\sum_{j=0}^k \big(u^{(n+m)}\big)^{k-j}\big(u^{(n)}\big)^j\Bigg]\Bigg\|_{B^{s-1}_{p,r}} \\
&\quad\le C\big\|u^{(n)} - u^{(n+m)}\big\|_{B^{s-1}_{p,r}}\big[\big\|u^{(n)}\big\|_{B^s_{p,r}} + \big\|u^{(n+1)}\big\|_{B^s_{p,r}} + \big\|u^{(n+m)}\big\|_{B^s_{p,r}} + 1\big]^k \\
&\quad\le CL^k\big\|u^{(n)} - u^{(n+m)}\big\|_{B^{s-1}_{p,r}}.
\end{aligned}\tag{3.22}$$

Inserting (3.22) into (3.21) yields

$$\begin{aligned}
&\big\|\big[u^{(n+1+m)} - u^{(n+1)}\big](t)\big\|_{B^{s-1}_{p,r}} \\
&\quad\le e^{U^n(t)}\big(\big\|u_0^{(n+1+m)} - u_0^{(n+1)}\big\|_{B^{s-1}_{p,r}}\big) + CL^k\int_0^t e^{U^n(t)-U^n(\tau)}\big\|u^{(n)} - u^{(n+m)}\big\|_{B^{s-1}_{p,r}}\,d\tau.
\end{aligned}\tag{3.23}$$

From (3.2) and Lemma 2.1, we can easily obtain that

$$\big\|u_0^{(n+1+m)} - u_0^{(n+1)}\big\|_{B^{s-1}_{p,r}} \le C2^{-n}.\tag{3.24}$$

Obviously,

$$e^{U^n(t)} \le 4, \qquad e^{U^n(t)-U^n(\tau)} \le 4.\tag{3.25}$$

Inserting (3.24) and (3.25) into (3.23) leads to

$$\begin{aligned}
&\big\|\big(u^{(n+1+m)} - u^{(n+1)}\big)(t)\big\|_{B^{s-1}_{p,r}} \\
&\quad\le C2^{-n} + CL^k\int_0^t \big\|\big(u^{(n)} - u^{(n+m)}\big)(\tau)\big\|_{B^{s-1}_{p,r}}\,d\tau.
\end{aligned}\tag{3.26}$$

We define

$$A_{(n,m)}(t) = \left\| \left(u^{(n+m)} - u^{(n)}\right) \right\|_{B^{s-1}_{p,r}}. \tag{3.27}$$

Inserting (3.27) into (3.26) leads to

$$A_{(n+1,m)}(t) \le C2^{-n} + CL^k \int_0^t A_{(n,m)}(\tau)\,d\tau. \tag{3.28}$$

We define

$$\rho_n(t) = \sup_{m\in\mathbf{N}^+} A_{(n,m)}(t) = \sup_{m\in\mathbf{N}^+} \left\| \left(u^{(n+m)} - u^{(n)}\right)(t) \right\|_{B^s_{p,r}} \tag{3.29}$$

and

$$\widetilde{\rho}(t) = \limsup_{n\to+\infty} \rho_n(t). \tag{3.30}$$

Combining (3.30) with (3.29), (3.28), by using Fatou's lemma, we have that

$$\widetilde{\rho}(t) = \limsup_{n\to+\infty} \rho_{n+1}(t) \le C^k \int_0^t \widetilde{\rho}(\tau)\,d\tau. \tag{3.31}$$

Applying Gronwall's inequality to (3.31) yields

$$\widetilde{\rho}(t) \le e^{tC^k} \widetilde{\rho}(0) \tag{3.32}$$

for $t \in [0,T]$. According to the definition of $\widetilde{\rho}(t)$, we can easily obtain that

$$\widetilde{\rho}(0) = 0. \tag{3.33}$$

Combining (3.32) with (3.33), we have that

$$\widetilde{\rho}(t) = 0. \tag{3.34}$$

Hence, $(u^n)_n$ is a Cauchy sequence in $C([0,T]; B^{s-1}_{p,r})$.

Fourth step: existence in $E^s_{p,r}(T)$

Now we prove that $u \in E^s_{p,r}(T)$ and satisfies (1.4)-(1.5) since $(u^n)_{n\in\mathbf{N}}$ is uniformly bounded in $L^\infty(0,T;B^s_{p,r})$. From (6) in Lemma 2.2, we have that $u \in L^\infty(0,T;B^s_{p,r})$. From (1.4), we can easily prove that $u_t \in L^\infty(0,T;B^{s-1}_{p,r})$. It is easily checked that u is indeed a solution to (1.4)-(1.5) by passing to the limit in (3.1)-(3.2).

Now we prove that $u \in E^s_{p,r}(T)$. Since $u \in B^s_{p,r}$, $\forall\epsilon > 0$, there exists $q_0 \in \mathbf{N}^+$ such that

$$\sum_{q\ge q_0} 2^{qsr} \|\Delta_q u\|^r_{L^\infty(0,T;L^p)} \le \frac{\epsilon}{4}. \tag{3.35}$$

From the definition of Besov spaces, we have that

$$\begin{aligned}
&\|u(t+\delta)-u(t)\|_{B^s_{p,r}}^r \\
&\quad=\sum_{q<q_0}2^{qsr}\|\Delta_q u(t)-\Delta_q u(t+\delta)\|_{L^p}^r+\sum_{q\geq q_0}2^{qsr}\|\Delta_q u-\Delta_q u(t+\delta)\|_{L^p}^r \\
&\quad\leq\sum_{q<q_0}2^{qsr}\|\Delta_q u(t)-\Delta_q u(t+\delta)\|_{L^p}^r+2\sum_{q\geq q_0}2^{qsr}\|\Delta_q u(t)\|_{L^\infty(0,T;L^p)}^r \\
&\quad\leq\sum_{q<q_0}2^{qsr}\|\Delta_q u(t)-\Delta_q u(t+\delta)\|_{L^p}^r+\frac{\epsilon}{2}.
\end{aligned}\tag{3.36}$$

By using the mean value theorem, we have that

$$\|\Delta_q u(t)-\Delta_q u(t+\delta)\|_{L^p}=\|\Delta_q u_t(t+\theta\delta)\|_{L^p}|\delta|\leq\|\Delta_q u_t(t)\|_{L^\infty(0,T;L^p)}|\delta|,\tag{3.37}$$

where $0<\theta<1$. Inserting (3.37) into (3.36) yields

$$\begin{aligned}
&\|u(t+\delta)-u(t)\|_{B^s_{p,r}}^r \\
&\quad=\sum_{q<q_0}2^{qsr}\|\Delta_q u(t)-\Delta_q u(t+\delta)\|_{L^p}^r+\sum_{q\geq q_0}2^{qsr}\|\Delta_q u(t)-\Delta_q u(t+\delta)\|_{L^p}^r \\
&\quad\leq|\delta|\sum_{q<q_0}\|\Delta_q u_t(t)\|_{L^\infty(0,T;L^p)}^r+\frac{\epsilon}{2} \\
&\quad\leq|\delta|\|u_t\|_{L^\infty(0,T;B^{s-1}_{p,r})}^r+\frac{\epsilon}{2}.
\end{aligned}\tag{3.38}$$

We may choose δ sufficiently small such that

$$|\delta|\|u_t\|_{L^\infty(0,T;B^{s-1}_{p,r})}^r\leq\frac{\epsilon}{2}.\tag{3.39}$$

Inserting (3.39) into (3.38) leads to

$$\|u(t+\delta)-u(t)\|_{B^s_{p,r}}\leq\epsilon^{1/r}.\tag{3.40}$$

Thus, we derive that

$$u\in C\big([0,T];B^s_{p,r}\big).\tag{3.41}$$

Combining (3.41) with (1.4), we can easily obtain

$$u_t\in C\big([0,T];B^{s-1}_{p,r}\big).\tag{3.42}$$

From (3.41) and (3.42), we have that $u\in E^s_{p,r}(T)$.

Fifth step: uniqueness of solution

The uniqueness of the solution to the Cauchy problem for (1.4) can be proved similarly to Proposition 3.1 of [14].

Sixth step: continuity in $E^s_{p,r}(T)$ with $s > 1 + \frac{1}{p}$ (or $s \geq 1 + \frac{1}{p}$ if $r = 1$ with $p \in [1, +\infty)$)

The continuity of the solution to the Cauchy problem for (1.4) can be proved similarly to the continuity of the solution to the Degasperis-Procesi equation which can be seen in [14].

4 Proof of Theorem 1.2

From (6.5) and (6.7) of [22], we have that

$$\partial_x u_c(x,t) = -\operatorname{sign}(x-ct)u_c(x,t), \qquad \partial_t u_c(x,t) = c\operatorname{sign}(x-ct)u_c(x,t), \tag{4.1}$$

where $u_c(x,t) = c^{1/k}e^{-|x-ct|}$. By using integration by parts and (4.1), we have that

$$\begin{aligned}
&\int_0^{+\infty}\int_{\mathbf{R}}\left(u_c\partial_t\phi + \frac{1}{k+1}u^{k+1}\partial_x\phi\right)dx\,dt + \int_{\mathbf{R}} u_c(x,0)\phi(x,0)\,dx \\
&\quad = -\int_0^{+\infty}\int_{\mathbf{R}}\phi\left[\partial_t u_c + u_c^k\partial_x u_c\right]dx\,dt \\
&\quad = -\int_0^{+\infty}\int_{\mathbf{R}}\phi\operatorname{sign}(x-ct)\left[cu_c - u_c^{k+1}\right]dx\,dt.
\end{aligned} \tag{4.2}$$

Since $u_c(x,t) = c^{1/k}e^{-|x-ct|}$, we have that when $x > ct$,

$$\operatorname{sign}(x-ct)\left[cu_c - u_c^{k+1}\right] = c^{1+\frac{1}{k}}\left[e^{ct-x} - e^{(k+1)(ct-x)}\right] \tag{4.3}$$

and when $x \leq ct$,

$$\operatorname{sign}(x-ct)\left[cu_c - u_c^{k+1}\right] = c^{1+\frac{1}{k}}\left[e^{(x-ct)} - e^{(k+1)(x-ct)}\right]. \tag{4.4}$$

By using $(1-\partial_x^2)^{-1}f = p * f$ and (4.1), we have that

$$\begin{aligned}
&\int_0^{+\infty}\int_{\mathbf{R}}\phi\partial_x\left(1-\partial_x^2\right)^{-1}\left[\frac{k^2+2k}{k+1}u_c^{k+1}\right]dx\,dt \\
&\quad = \int_0^{+\infty}\int_{\mathbf{R}}\phi\partial_x p * \left[\frac{k^2+2k}{k+1}u_c^{k+1}\right]dx\,dt \\
&\quad = -\frac{k^2+2k}{2(k+1)}c^{1+\frac{1}{k}}\int_0^{+\infty}\int_{\mathbf{R}_x}\int_{\mathbf{R}_y}\phi\operatorname{sign}(x-y)e^{-|x-y|}e^{-(k+1)|y-ct|}\,dy\,dx\,dt.
\end{aligned} \tag{4.5}$$

Now, we compute

$$\begin{aligned}
&-\frac{k^2+2k}{2(k+1)}c^{1+\frac{1}{k}}\int_{\mathbf{R}_y}\operatorname{sign}(x-y)e^{-|x-y|}e^{-(k+1)|y-ct|}\,dy \\
&\quad := I_1 + I_2 + I_3.
\end{aligned} \tag{4.6}$$

When $x > ct$, we have that

$$\begin{aligned} I_1 &= -\frac{k^2+2k}{2(k+1)} c^{1+\frac{1}{k}} \int_{-\infty}^{ct} e^{-(x-y)} e^{(k+1)(y-ct)}\,dy \\ &= -\frac{k^2+2k}{2(k+1)} c^{1+\frac{1}{k}} e^{-(x+(k+1)ct)} \int_{-\infty}^{ct} e^{(k+2)y}\,dy \\ &= -\frac{k}{2(k+1)} c^{1+\frac{1}{k}} e^{ct-x} \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} I_2 &= -\frac{k^2+2k}{2(k+1)} c^{1+\frac{1}{k}} \int_{ct}^{x} e^{-(x-y)} e^{-(k+1)(y-ct)}\,dy \\ &= -\frac{k^2+2k}{2(k+1)} c^{1+\frac{1}{k}} e^{-(x-(k+1)ct)} \int_{ct}^{x} e^{-\frac{k(k+2)}{k+1}y}\,dy \\ &= -\frac{k+2}{2(k+1)} c^{1+\frac{1}{k}} \left[e^{ct-x} - e^{(k+1)(ct-x)}\right] \end{aligned} \tag{4.8}$$

and

$$\begin{aligned} I_3 &= \frac{k^2+2k}{2(k+1)} c^{1+\frac{1}{k}} \int_{x}^{\infty} e^{(x-y)} e^{-(k+1)(y-ct)}\,dy \\ &= \frac{k^2+2k}{2(k+1)} c^{1+\frac{1}{k}} e^{(x+(k+1)ct)} \int_{x}^{\infty} e^{-(k+2)y}\,dy \\ &= \frac{k}{2(k+1)} c^{1+\frac{1}{k}} e^{(k+1)(ct-x)}. \end{aligned} \tag{4.9}$$

Thus, when $x > ct$, from (4.7)-(4.9), we have that

$$\begin{aligned} &I_1 + I_2 + I_3 \\ &\quad = -\frac{k}{2(k+1)} c^{1+\frac{1}{k}} e^{ct-x} - \frac{k+2}{2(k+1)} c^{1+\frac{1}{k}} \left[e^{ct-x} - e^{(k+1)(ct-x)}\right] \\ &\qquad + \frac{k}{2(k+1)} c^{1+\frac{1}{k}} e^{(k+1)(ct-x)} \\ &\quad = -c^{1+\frac{1}{k}} \left[e^{(k+1)(ct-x)} - e^{ct-x}\right]. \end{aligned} \tag{4.10}$$

Similarly, when $x < ct$, we can obtain

$$I_1 + I_2 + I_3 = c^{1+\frac{1}{k}} \left[e^{(k+1)(x-ct)} - e^{x-ct}\right]. \tag{4.11}$$

From (4.3), (4.4) and (4.10) as well as (4.11), we have that

$$\begin{aligned} &\int_0^{+\infty} \int_{\mathbf{R}} \left[u_c \phi_t + \frac{1}{k+1} u_c^{k+1} \phi_x + p * \left[\frac{k^2+2k}{k+1} u_c^{k+1} \right] \phi_x \right] dx\,dt \\ &\quad + \int_{\mathbf{R}} u_c(x,0) \phi(x,0)\,dx = 0. \end{aligned} \tag{4.12}$$

Thus, $u_c = c^{1/k}e^{-|x-ct|}$ is the solution in the sense of (4.12). For $c > 0$, let $u_c(x,t) = c^{1/k}e^{-|x-ct|}$. Thus $u_c(x,t)$ is the solitary wave for (1.1) (or (1.4)).

5 Proof of Theorem 1.3

Since when $Q = \frac{k(k+2)}{k+1}$ in (1.4), (1.4) possesses the peaked solitary wave $c^{1/k}e^{|x-ct|}$, Theorem 1.3 can be proved similarly to Proposition 4 of [19].

6 Proof of Theorem 1.4

Since when $Q = \frac{k(k+2)}{k+1}$ in (1.4), (1.4) possesses the peaked solitary wave $c^{1/k}e^{|x-ct|}$, Theorem 1.4 can be proved similarly to Theorem 3 of [23].

7 Proof of Theorem 1.5

In this section, we always assume that $s > 1 + \frac{1}{p}$ (or $s \geq 1 + \frac{1}{p}$ if $r = 1$ with $p \in [1, +\infty)$).

Proof of Theorem 1.5 Applying Δ_q to (1.4) yields

$$\left(\partial_t + u^k\partial_x\right)\Delta_q u = \left[u^k, \Delta_q\right]\partial_x u + QP(D)\Delta_q\left[\frac{k^2+2k}{k+1}u^{k+1}\right]. \tag{7.1}$$

From (2.54) of page 112 in [24], since $s > 1 + \frac{1}{p}$ (or $s \geq 1 + \frac{1}{p}$ if $r = 1$ with $p \in [1, +\infty)$), we have that

$$\left\|2^{sq}\left\|\left[u^k, \Delta_q\right]\partial_x u\right\|_{L^p}\right\|_{\ell^r} \leq C\|u_x\|_{L^\infty}^k\|u\|_{B^s_{p,r}} \leq C\|u\|_{B^s_{p,r}}^{k+1}. \tag{7.2}$$

By using (4) of Lemma 2.2 and $P(D)$ is an S^{-1}-multiplier, since $s > 1 + \frac{1}{p}$ (or $s \geq 1 + \frac{1}{p}$ if $r = 1$ with $p \in [1, +\infty)$), we have that

$$\left\|P(D)\left[\frac{k^2+2k}{k+1}u^{k+1}\right]\right\|_{B^s_{p,r}} \leq C\|u_x\|_{L^\infty}^k\|u\|_{B^s_{p,r}} \leq C\|u\|_{B^s_{p,r}}^{k+1}. \tag{7.3}$$

Going along the lines of the proof of Proposition A.1 of [18], from (7.2) and (7.3), we have that

$$\|u\|_{B^s_{p,r}} \leq \|u_0\|_{B^s_{p,r}} + C\int_0^t \|u_x\|_{L^\infty}^k\|u\|_{B^s_{p,r}}\,d\tau \tag{7.4}$$

$$\leq \|u_0\|_{B^s_{p,r}} + C\int_0^t \|u\|_{B^s_{p,r}}^{k+1}\,d\tau. \tag{7.5}$$

Solving (7.4) yields

$$\|u\|_{B^s_{p,r}} \leq e^{c\int_0^t \|u_x\|_{L^\infty}^k\,d\tau}\|u_0\|_{B^s_{p,r}}. \tag{7.6}$$

Solving (7.5) yields

$$\|u\|_{B^s_{p,r}} \leq \frac{\|u_0\|_{B^s_{p,r}}}{(1 - Ckt\|u_0\|_{B^s_{p,r}}^k)^{1/k}}. \tag{7.7}$$

Assume that $T^\star$ is the maximal time of existence of the solution to problem (1.4)-(1.5). If $T^\star < \infty$, we claim that

$$\int_0^{T^\star} \|u_x\|_{L^\infty}^k \, d\tau = +\infty. \tag{7.8}$$

We prove the claim (7.8) by contradiction. If (7.8) is untrue, then from (7.8), we have that

$$\|u(T^\star)\|_{B^s_{p,r}} < \infty, \tag{7.9}$$

which contradicts the fact that $T^\star$ is the maximal time of existence of the solution to problem (1.4)-(1.5). Consequently, (7.8) is true. From (7.7), we know that $T^\star \geq \frac{1}{Ck\|u_0\|^k_{B^s_{p,r}}}$. Moreover, (7.7) ensures the validity of (3.8).

The proof of Theorem 1.5 is completed. □

Competing interests
We declare that we have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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