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Extinction and decay estimates of solutions for a porous medium equation with nonlocal source and strong absorption

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Abstract

In this paper, we investigate extinction properties of the solutions for the initial Dirichlet boundary value problem of a porous medium equation with nonlocal source and strong absorption terms. We obtain some sufficient conditions for the extinction of nonnegative nontrivial weak solutions and the corresponding decay estimates which depend on the initial data, coefficients, and domains.

1 Introduction

We consider the initial Dirichlet boundary value problem for a class of porous medium equations with nonlocal source and strong absorption terms

$$u_t = d\Delta u^m + \lambda \int_{\Omega} u^q(x, t) dx - \beta u^k, \quad x \in \Omega, t > 0, \quad (1)$$

$$u(x, t) = 0, \quad x \in \partial\Omega, t > 0, \quad (2)$$

$$u(x, 0) = u_0(x), \quad x \in \Omega, \quad (3)$$

where $0 < m, k < 1$, $d, \lambda, \beta, q > 0$, $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) is a bounded domain with smooth boundary, and $u_0(x) \in L^\infty(\Omega) \cap W_0^{1,2}(\Omega)$ is a nonnegative function. The symbols $\|\cdot\|_p$ and $\|\cdot\|_{1,p}$, where $p \geq 1$, denote $L^p(\Omega)$ - and $W^{1,p}(\Omega)$ -norm, respectively, and $|\Omega|$ denotes the measure of Ω .

Equation (1) describes the fast diffusion of concentration of some Newtonian fluids through a porous medium or the density of some biological species in many physical phenomena and biological species theories, while nonlocal source and absorption terms cooperate and interact with each other during the diffusion. It has been known that the nonlocal source term presents a more realistic model for population dynamics; see [1–3]. In the nonlinear diffusion theory, obvious differences exist among the situations of slow ($m > 1$), fast ($0 < m < 1$), and linear ($m = 1$) diffusions. For example, there is a finite speed propagation in the slow and linear diffusion situations, whereas an infinite speed propagation exists in the fast diffusion situation.

Recently, many scholars have been devoted to the study of blow-up and extinction properties of solutions for nonlinear parabolic equations with nonlocal terms. The blow-up rates and blow-up sets of solutions to equation (1) have been investigated when $d = \lambda = 1$,

$m \geq 1$, and the linear absorption term is replaced with a nonlinear term with exponent (cf. [4–9]). Extinction is the phenomenon whereby there exists a finite time $T > 0$ such that the solution is nontrivial for $0 < t < T$ and then $u(x, t) \equiv 0$ for all $(x, t) \in \Omega \times [T, +\infty)$. In this case, T is called the extinction time. It is also an important property of solutions for nonlinear parabolic equations which have been studied by many researchers. For instance, Evans and Knerr [10] investigated the extinction behaviors of solutions for the Cauchy problem of the semilinear parabolic equation

$$u_t(x, t) = \Delta u(x, t) - \beta(u(x, t)), \quad x \in R^n, t > 0,$$

by constructing a suitable comparison function. Ferreira and Vazquez [11] studied the extinction phenomena of solutions for the Cauchy problem of the porous medium equation with an absorption term

$$u_t = (u^m)_{xx} - u^p, \quad x \in R, t > 0,$$

by using the analysis of self-similar solutions. Li and Wu [12] considered the problem of the porous medium equation with a source term

$$u_t = \Delta u^m + \lambda u^p, \quad x \in \Omega, t > 0, \tag{4}$$

subject to (2) and (3). They obtained some conditions for the extinction and non-extinction of solutions to equation (4) and decay estimates by the upper and lower solutions method. On extinctions of solutions to the p -Laplacian equations or the doubly degenerate equations, we refer readers to [13, 14] and the references therein.

Replacing the nonlocal term in equation (1) with a local term, Liu [15] *et al.* considered the initial Dirichlet boundary value problem for a class of porous medium equations

$$u_t = \Delta u^m + u^q - \beta u, \quad x \in \Omega, t > 0,$$

and obtained sufficient conditions for the extinction and non-extinction of solutions to that equation. Thereafter, Fang and Li [16] extended their results to the doubly degenerate equation in the whole dimensional space.

For equation (1) with $\beta = 0$, $q > 0$, and $N > 2$, Han and Gao [17] showed that $q = m$ is the critical exponent for the occurrence of extinction or non-extinction. When $m = 1$, $q > 0$, and $N > 2$, the conditions for the extinction and non-extinction of solutions and corresponding decay estimates were obtained (cf. [18]). Recently, Fang and Xu [19] considered equation (1) with $k = 1$, when the diffusion term was replaced with a p -Laplacian operator in the whole dimensional space, and showed that the extinction of the weak solution is determined by the competition of two nonlinear terms. They also obtained the exponential decay estimates which depend on the initial data, coefficients, and domains. The extinctions of solutions to equation (1) with nonlocal source terms do not depend on the first eigenvalue of the corresponding operator, which is different from the case of local source terms. The extinction and decay estimates for solutions to the nonlocal fast diffusion equations with nonzero coefficients and strong absorption terms, like equation (1), are still being investigated.

Motivated by the above works, we investigate whether the existence of strong absorption can change extinction behaviors for solutions to problem (1)-(3) in the whole dimensional space. The main tools we use are the integral estimate method and the Gagliardo-Nirenberg inequality. This technique has a wide application, especially for equations that do not satisfy the maximum principle (cf. [20]). Our goals are to show that the extinction of nonnegative nontrivial weak solutions to problem (1)-(3) occurs when $0 < k \leq q < 1$ and to find the decay estimates depending on the initial data, coefficients, and domains.

Our paper is organized as follows. In Section 2, we give preliminary knowledge including lemmas that are required in the proofs of our results and present the proofs for the results in Section 3.

2 Preliminary knowledge

Due to the singularity of equation (1), problem (1)-(3) has no classical solutions in general. To state the definition of the weak solution, we let $Q_T = \Omega \times (0, T)$ and firstly define the class of nonnegative testing functions

$$F = \{ \xi : \xi \in C(\overline{Q_T}) \cap C^{2,1}(Q_T), \xi_t, \Delta \xi \in L^2(Q_T); \xi \geq 0, \xi|_{\partial\Omega \times (0, T)} = 0 \}.$$

Definition 1 A function $u \in L^\infty(Q_T)$ is called a weak subsolution (supersolution) of problem (1)-(3) in Q_T if the following conditions hold:

- $u(x, 0) \leq (\geq) u_0(x)$ in Ω ;
- $u(x, t) \leq (\geq) 0$ on $\partial\Omega \times (0, T)$;
- For every $t \in (0, T)$ and every $\xi \in F$,

$$\int_{\Omega} u(x, t) \xi(x, t) dx \leq \int_{\Omega} u_0(x) \xi(x, 0) dx + \int_0^t \int_{\Omega} \left\{ u \xi_s + du^m \Delta \xi + \lambda \int_{\Omega} u^q(y, s) dy \xi(x, s) - \beta u^k(x, s) \xi(x, s) \right\} dx ds.$$

A function u is called a locally weak solution of problem (1)-(3) if it is both a subsolution and a supersolution for some $T > 0$.

Remark 1 The existence and uniqueness of locally nonnegative solutions in time to problem (1)-(3) can be obtained by the standard parabolic regular theory that can be applied to get suitable estimates in the standard limiting process (cf. [2, 21, 22]). The proof is similar to the ones in the cited references, and so it is omitted here.

Lemma 1 Let k, α be positive constants and $k < 1$. If $y(t)$ is a nonnegative absolutely continuous function on $[0, +\infty)$ satisfying the problem

$$\frac{dy}{dt} + \alpha y^k \leq 0, \quad t \geq 0; \quad y(0) \geq 0,$$

then we have the decay estimate

$$y(t) \leq [y^{1-k}(0) - \alpha(1-k)t]^{\frac{1}{1-k}}, \quad t \in [0, T_*),$$

$$y(t) = 0, \quad t \in [T_*, +\infty),$$

where $T_* = \frac{y^{1-k}(0)}{\alpha(1-k)}$.

Proof By solving the initial problem

$$\frac{dy}{dt} + \alpha y^k = 0, \quad t > 0; \quad y|_{t=0} = y(0),$$

and using the comparison principle, one can easily obtain the result. \square

Lemma 2 (The Gagliardo-Nirenberg inequality) [23] *Suppose that $u \in W_0^{k,m}(\Omega)$, $1 \leq m \leq +\infty$, $0 \leq j < k$, and $1 \geq \frac{1}{r} \geq \frac{1}{m} - \frac{k}{N}$. We then have the inequality*

$$\|D^j u\|_q \leq C \|D^k u\|_m^\theta \|u\|_r^{1-\theta},$$

where C is a constant depending on N, m, r, j, k , and q such that $\frac{1}{q} = \frac{j}{N} + \theta(\frac{1}{m} - \frac{k}{N}) + \frac{1-\theta}{r}$ and $\theta \in [0, 1)$. While if $m < \frac{N}{k-j}$, then $q \in [\frac{Nr}{N+j}, \frac{Nm}{N-(k-j)m}]$, and if $m \geq \frac{N}{k-j}$, then $q \in [\frac{Nr}{N+j}, +\infty]$.

3 Main results

In this section, we give some extinction properties of nonnegative nontrivial weak solutions of problem (1)-(3) stated in the following theorems. The corresponding decay estimates to the solutions will be presented in the proofs of the theorems for brief expressions instead of in the statements.

Theorem 1 *Suppose that $0 < k < 1$ and $0 < m = q < 1$. Then the nonnegative nontrivial weak solution of problem (1)-(3) vanishes in finite time for any nonnegative initial data provided that either $|\Omega|$ or λ is sufficiently small.*

Proof We first consider the case that $N = 1$ or 2 . Multiplying both sides of (1) by u and integrating the result over Ω , we have

$$\frac{1}{2} \frac{d}{dt} \|u\|_2^2 + \frac{4md}{(m+1)^2} \|\nabla u^{\frac{m+1}{2}}\|_2^2 + \beta \|u\|_{k+1}^{k+1} = \lambda \int_{\Omega} u^m dx \int_{\Omega} u dx.$$

By Hölder's inequality, we get the inequality

$$\int_{\Omega} u^m dx \int_{\Omega} u dx \leq |\Omega|^{\frac{2s_1-1-m}{s_1}} \|u\|_{s_1}^{m+1}.$$

In particular, if $s_1 = 2$, we get the inequality

$$\frac{1}{2} \frac{d}{dt} \|u\|_2^2 + \frac{4md}{(m+1)^2} \|\nabla u^{\frac{m+1}{2}}\|_2^2 + \beta \|u\|_{k+1}^{k+1} \leq \lambda |\Omega|^{\frac{3-m}{2}} \|u\|_2^{m+1}, \tag{5}$$

from the two expressions above. By using the Sobolev embedding inequality, one can show that there exists an embedding constant $\gamma(N, \Omega) > 0$ such that

$$\|u^{\frac{m+1}{2}}\|_{s_2} \leq \gamma(N, \Omega) \|\nabla u^{\frac{m+1}{2}}\|_2,$$

i.e.,

$$\|u\|_{\frac{(m+1)s_2}{2}}^{m+1} \leq \gamma(N, \Omega)^2 \|\nabla u^{\frac{m+1}{2}}\|_2^2.$$

In particular, if $s_2 = \frac{4}{m+1}$, then the inequality above turns out to be

$$\gamma(N, \Omega)^{-2} \|u\|_2^{m+1} \leq \|\nabla u^{\frac{m+1}{2}}\|_2^2. \tag{6}$$

Since $0 < m < 1$, we get $\frac{4}{m+1} \geq 2$, and hence, inequality (5) becomes

$$\frac{1}{2} \frac{d}{dt} \|u\|_2^2 + \left[\frac{4md}{(m+1)^2} - \lambda |\Omega|^{\frac{3-m}{2}} \gamma(N, \Omega)^2 \right] \|\nabla u^{\frac{m+1}{2}}\|_2^2 + \beta \|u\|_{k+1}^{k+1} \leq 0. \tag{7}$$

By Lemma 2, we get the inequality

$$\|u\|_2 \leq C(N, m, k) \|u\|_{k+1}^{1-\theta_1} \|\nabla u^{\frac{m+1}{2}}\|_2^{\frac{2\theta_1}{m+1}}, \tag{8}$$

where $\theta_1 = \frac{m+1}{2} \left(\frac{1}{k+1} - \frac{1}{2} \right) \left(\frac{1}{N} - \frac{1}{2} + \frac{m+1}{2} \frac{1}{k+1} \right)^{-1} = \frac{N(1-k)(m+1)}{2[2(k+1)+N(m-k)]}$. Since $N = 1$ or 2 , and $0 < k < 1$, it can be easily seen that $0 < \theta_1 < 1$.

It then follows from (8) and Young's inequality that

$$\begin{aligned} \|u\|_2^{k_1} &\leq C(N, m, k)^{k_1} \|u\|_{k+1}^{k_1(1-\theta_1)} \|\nabla u^{\frac{m+1}{2}}\|_2^{\frac{2k_1\theta_1}{m+1}} \\ &\leq C(N, m, k)^{k_1} \left(\eta_1 \|\nabla u^{\frac{m+1}{2}}\|_2^2 + C(\eta_1) \|u\|_{k+1}^{\frac{k_1(1-\theta_1)(m+1)}{m+1-k_1\theta_1}} \right), \end{aligned} \tag{9}$$

where $k_1 > 1$ and $\eta_1 > 0$ will be determined later. If we choose

$$k_1 = \frac{(k+1)(m+1)}{(1-\theta_1)(m+1) + \theta_1(k+1)} = \frac{4(k+1) + 2N(m-k)}{4 + N(m-k)},$$

then $1 < k_1 < 2$ and $\frac{k_1(1-\theta_1)(m+1)}{m+1-k_1\theta_1} = k+1$. From (9) we have

$$\frac{C(N, m, k)^{-k_1} \beta}{C(\eta_1)} \|u\|_2^{k_1} \leq \frac{\eta_1 \beta}{C(\eta_1)} \|\nabla u^{\frac{m+1}{2}}\|_2^2 + \beta \|u\|_{k+1}^{k+1}. \tag{10}$$

By inequalities (7) and (10), we get the inequality

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u\|_2^2 + \left[\frac{4md}{(m+1)^2} - \lambda |\Omega|^{\frac{3-m}{2}} \gamma(N, \Omega)^2 - \frac{\eta_1 \beta}{C(\eta_1)} \right] \|\nabla u^{\frac{m+1}{2}}\|_2^2 \\ + \frac{C(N, m, k)^{-k_1} \beta}{C(\eta_1)} \|u\|_2^{k_1} \leq 0. \end{aligned}$$

Here, we can choose η_1 and λ or $|\Omega|$ small enough so that

$$\frac{4md}{(m+1)^2} - \lambda |\Omega|^{\frac{3-m}{2}} \gamma(N, \Omega)^2 - \frac{\eta_1 \beta}{C(\eta_1)} \geq 0.$$

Setting $C_1 = \frac{C(N, m, k)^{-k_1} \beta}{C(\eta_1)}$, we have $\frac{d}{dt} \|u\|_2 + C_1 \|u\|_2^{k_1-1} \leq 0$.

By Lemma 1, we then obtain

$$\begin{aligned} \|u\|_2 &\leq [\|u_0\|_2^{2-k_1} - C_1(2-k_1)t]^{\frac{1}{2-k_1}}, \quad t \in [0, T_1), \\ \|u\|_2 &= 0, \quad t \in [T_1, +\infty), \end{aligned}$$

where $T_1 = \frac{\|u_0\|_2^{2-k_1}}{C_1(2-k_1)}$, which give the decay estimates in finite time for $N = 1$ or 2 .

Secondly, we consider the case that $N > 2$. If $\frac{N-2}{N+2} \leq m < 1$, multiplying both sides of (1) by u^s ($s > m$) and integrating the result over Ω , we get

$$\frac{1}{s+1} \frac{d}{dt} \|u\|_{s+1}^{s+1} + \frac{4msd}{(m+s)^2} \|\nabla u^{\frac{m+s}{2}}\|_2^2 + \beta \|u\|_{k+s}^{k+s} = \lambda \int_{\Omega} u^m dx \int_{\Omega} u^s dx.$$

By Hölder's inequality, we have the inequality

$$\int_{\Omega} u^m dx \int_{\Omega} u^s dx \leq |\Omega|^{\frac{2s_3-m-s}{s_3}} \|u\|_{s_3}^{m+s}.$$

In particular, if $s_3 = s + 1$, we then get the inequality

$$\frac{1}{s+1} \frac{d}{dt} \|u\|_{s+1}^{s+1} + \frac{4msd}{(m+s)^2} \|\nabla u^{\frac{m+s}{2}}\|_2^2 + \beta \|u\|_{k+s}^{k+s} \leq \lambda |\Omega|^{\frac{s-m+2}{s+1}} \|u\|_{s+1}^{m+s} \tag{11}$$

by the two expressions above. By the Sobolev embedding inequality, one can see that there exists an embedding constant $C_0 > 0$ such that

$$\|u^{\frac{m+s}{2}}\|_{\frac{2N}{N-2}}^2 \leq C_0^2 \|\nabla u^{\frac{m+s}{2}}\|_2^2. \tag{12}$$

Using Hölder's inequality again, we have the inequality

$$\|u\|_{s+1}^{m+s} \leq |\Omega|^{\frac{m+s}{s+1} - \frac{N-2}{N}} \|u^{\frac{m+s}{2}}\|_{\frac{2N}{N-2}}^2. \tag{13}$$

From inequalities (11), (12), and (13), we then obtain the inequality

$$\frac{1}{s+1} \frac{d}{dt} \|u\|_{s+1}^{s+1} + \left[\frac{4msd}{(m+s)^2} - \lambda C_0^2 |\Omega|^{1+\frac{2}{N}} \right] \|\nabla u^{\frac{m+s}{2}}\|_2^2 + \beta \|u\|_{k+s}^{k+s} \leq 0. \tag{14}$$

By Lemma 2, we can also have

$$\|u\|_{s+1} \leq C(N, k, s) \|u\|_{k+s}^{1-\theta_2} \|\nabla u^{\frac{m+s}{2}}\|_2^{\frac{2\theta_2}{m+s}}, \tag{15}$$

where $\theta_2 = \frac{m+s}{2} (\frac{1}{k+s} - \frac{1}{s+1}) (\frac{1}{N} - \frac{1}{2} + \frac{m+s}{2} \frac{1}{k+s})^{-1} = \frac{N(1-k)(m+s)}{(s+1)[2(k+s)+N(m-k)]}$. Since $\frac{N-2}{N+2} \leq m < 1$ and $0 < k < 1$, one can easily see that $0 < \theta_2 < 1$. Then it follows from (15) and Young's inequality that

$$\begin{aligned} \|u\|_{s+1}^{k_2} &\leq C(N, k, s)^{k_2} \|u\|_{k+s}^{k_2(1-\theta_2)} \|\nabla u^{\frac{m+s}{2}}\|_2^{\frac{2k_2\theta_2}{m+s}} \\ &\leq C(N, k, s)^{k_2} (\eta_2 \|\nabla u^{\frac{m+s}{2}}\|_2^2 + C(\eta_2) \|u\|_{k+s}^{\frac{k_2(1-\theta_2)(m+s)}{m+s-k_2\theta_2}}), \end{aligned} \tag{16}$$

where $k_2 > 0$ and $\eta_2 > 0$ will be determined later. If we choose

$$k_2 = \frac{(k+s)(m+s)}{(1-\theta_2)(m+s) + \theta_2(k+s)} = \frac{2(s+1)(k+s) + N(s+1)(m-k)}{2(s+1) + N(m-k)},$$

then $s < k_2 < s+1$ and $\frac{k_2(1-\theta_2)(m+s)}{m+s-k_2\theta_2} = k+s$. We then have the inequality

$$\frac{C(N, k, s)^{-k_2} \beta}{C(\eta_2)} \|u\|_{s+1}^{k_2} \leq \frac{\eta_2 \beta}{C(\eta_2)} \|\nabla u^{\frac{m+s}{2}}\|_2^2 + \beta \|u\|_{k+s}^{k+s} \tag{17}$$

by (16). From inequalities (14) and (17), we can also obtain the inequality

$$\begin{aligned} \frac{1}{s+1} \frac{d}{dt} \|u\|_{s+1}^{s+1} + \left[\frac{4msd}{(m+s)^2} - \lambda C_0^2 |\Omega|^{1+\frac{2}{N}} - \frac{\eta_2 \beta}{C(\eta_2)} \right] \|\nabla u^{\frac{m+s}{2}}\|_2^2 \\ + \frac{C(N, k, s)^{-k_2} \beta}{C(\eta_2)} \|u\|_{s+1}^{k_2} \leq 0. \end{aligned}$$

Here, we can choose η_2 and λ or $|\Omega|$ small enough so that $\frac{4msd}{(m+s)^2} - \lambda C_0^2 |\Omega|^{1+\frac{2}{N}} - \frac{\eta_2 \beta}{C(\eta_2)} \geq 0$. Setting $C_2 = \frac{C(N, k, s)^{-k_2} \beta}{C(\eta_2)}$, we have $\frac{d}{dt} \|u\|_{s+1} + C_2 \|u\|_{s+1}^{k_2-s} \leq 0$ from the inequality above. By Lemma 1, we obtain that

$$\begin{aligned} \|u\|_{s+1} \leq \left[\|u_0\|_{s+1}^{s+1-k_2} - C_2(s+1-k_2)t \right]^{\frac{1}{s+1-k_2}}, \quad t \in [0, T_2), \\ \|u\|_{s+1} = 0, \quad t \in [T_2, +\infty), \end{aligned}$$

where $T_2 = \frac{\|u_0\|_{s+1}^{s+1-k_2}}{C_2(s+1-k_2)}$, which give the decay estimates in finite time for $N > 2$ such that $\frac{N-2}{N+2} \leq m < 1$.

If $0 < m < \frac{N-2}{N+2}$, one can show that there exists an embedding constant $C_{00} > 0$ such that

$$\|u\|_{s+1}^{m+s} \leq |\Omega|^{1-\frac{(s+1)(N+2)}{N(m+s)}} \|u\|_{\frac{N(m+s)}{N-2}}^{m+s} \leq C_{00}^2 |\Omega|^{1-\frac{(s+1)(N+2)}{N(m+s)}} \|\nabla u^{\frac{m+s}{2}}\|_2^2$$

by multiplying both sides of (1) by u^s ($s > \frac{N(1-m)}{2} - 1$) and integrating the result over Ω , and the Sobolev embedding inequality. By using the inequality above and a similar argument as above, the following decay estimates can be obtained:

$$\begin{aligned} \|u\|_{s+1} \leq \left[\|u_0\|_{s+1}^{s+1-k_2} - \frac{C(N, k, s)^{-k_2} \beta}{C(\eta_2)} (s+1-k_2)t \right]^{\frac{1}{s+1-k_2}}, \quad t \in (0, T_2^*), \\ \|u\|_{s+1} = 0, \quad t \in [T_2^*, +\infty), \end{aligned}$$

provided that

$$\frac{4msd}{(m+s)^2} - \lambda C_{00}^2 |\Omega|^{1-\frac{(s+1)(N+2)}{N(m+s)} + \frac{s-m+2}{s+1}} - \frac{\eta_2 \beta}{C(\eta_2)} \geq 0,$$

where $T_2^* = \frac{C(\eta_2) \|u_0\|_{s+1}^{s+1-k_2}}{C(N, k, s)^{-k_2} \beta (s+1-k_2)}$. □

Theorem 2 *If $0 < k < 1$, then the nonnegative nontrivial weak solution of problem (1)-(3) vanishes in finite time provided that $u_0, |\Omega|$ or λ is sufficiently small, and $q > \frac{2k(s+1)+N(m-k)}{2(s+1)+N(m-k)}$, where if $N = 1$ or 2 , then $s = 1$, and if $N > 2$, then $s > m$.*

Proof Assume that $q \leq 1$. If $N = 1$ or 2 , multiplying both sides of (1) by u and integrating the result over Ω , we have the equation

$$\frac{1}{2} \frac{d}{dt} \|u\|_2^2 + \frac{4md}{(m+1)^2} \|\nabla u^{\frac{m+1}{2}}\|_2^2 + \beta \|u\|_{k+1}^{k+1} = \lambda \int_{\Omega} u^q dx \int_{\Omega} u dx. \tag{18}$$

By (10) and (18), and using Hölder’s inequality, we get the inequality

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u\|_2^2 + \left[\frac{4md}{(m+1)^2} - \frac{\eta_1 \beta}{C(\eta_1)} \right] \|\nabla u^{\frac{m+1}{2}}\|_2^2 + \frac{C(N, m, k)^{-k_1} \beta}{C(\eta_1)} \|u\|_2^{k_1} \\ \leq \lambda |\Omega|^{\frac{3-q}{2}} \|u\|_2^{q+1}. \end{aligned}$$

Choosing η_1 small enough so that $\frac{4md}{(m+1)^2} - \frac{\eta_1 \beta}{C(\eta_1)} \geq 0$, we obtain the inequality

$$\frac{d}{dt} \|u\|_2 + \|u\|_2^{k_1-1} \left[\frac{C(N, m, k)^{-k_1} \beta}{C(\eta_1)} - \lambda |\Omega|^{\frac{3-q}{2}} \|u\|_2^{q-k_1+1} \right] \leq 0.$$

Hence, we have the inequality

$$\frac{d}{dt} \|u\|_2 + C_3 \|u\|_2^{k_1-1} \leq 0$$

provided that

$$\|u_0\|_2 < \left[\frac{C(N, m, k)^{-k_1} \beta}{C(\eta_1) \lambda |\Omega|^{\frac{3-q}{2}}} \right]^{\frac{1}{q-k_1+1}},$$

and

$$q > k_1 - 1 = \frac{4k + N(m - k)}{4 + N(m - k)},$$

where $C_3 = \frac{C(N, m, k)^{-k_1} \beta}{C(\eta_1)} - \lambda |\Omega|^{\frac{3-q}{2}} \|u_0\|_2^{q-k_1+1} > 0$, from which and a similar argument as the one used in the proof of Theorem 1, the following decay estimates can be obtained:

$$\begin{aligned} \|u\|_2 \leq \left[\|u_0\|_2^{2-k_1} - C_3(2 - k_1)t \right]^{\frac{1}{2-k_1}}, \quad t \in [0, T_3), \\ \|u\|_2 = 0, \quad t \in [T_3, +\infty), \end{aligned}$$

where $T_3 = \frac{\|u_0\|_2^{2-k_1}}{C_3(2-k_1)}$.

If $N > 2$ and $0 < m < 1$, multiplying both sides of (1) by u^s ($s > m$) and integrating the result over Ω , we get the equation

$$\frac{1}{s+1} \frac{d}{dt} \|u\|_{s+1}^{s+1} + \frac{4msd}{(m+s)^2} \|\nabla u^{\frac{m+s}{2}}\|_2^2 + \beta \|u\|_{k+s}^{k+s} = \lambda \int_{\Omega} u^q dx \int_{\Omega} u^s dx. \tag{19}$$

By (17) and (19), and using Hölder's inequality, we obtain the inequality

$$\begin{aligned} & \frac{1}{s+1} \frac{d}{dt} \|u\|_{s+1}^{s+1} + \left[\frac{4msd}{(m+s)^2} - \frac{\eta_2\beta}{C(\eta_2)} \right] \|\nabla u^{\frac{m+s}{2}}\|_2^2 \\ & + \frac{C(N, k, s)^{-k_2}\beta}{C(\eta_2)} \|u\|_{s+1}^{k_2} \leq \lambda |\Omega|^{\frac{s-q+2}{s+1}} \|u\|_{s+1}^{q+s}. \end{aligned}$$

Choosing η_2 small enough so that $\frac{4msd}{(m+s)^2} - \frac{\eta_2\beta}{C(\eta_2)} \geq 0$, we have

$$\frac{d}{dt} \|u\|_{s+1} + \|u\|_{s+1}^{k_2-s} \left[\frac{C(N, k, s)^{-k_2}\beta}{C(\eta_2)} - \lambda |\Omega|^{\frac{s-q+2}{s+1}} \|u\|_{s+1}^{q-k_2+s} \right] \leq 0.$$

Therefore, we obtain the inequality

$$\frac{d}{dt} \|u\|_{s+1} + C_4 \|u\|_{s+1}^{k_2-s} \leq 0$$

provided that

$$\|u_0\|_{s+1} < \left[\frac{C(N, k, s)^{-k_2}\beta}{C(\eta_2)\lambda |\Omega|^{\frac{s-q+2}{s+1}}} \right]^{\frac{1}{q-k_2+s}},$$

and

$$q > k_2 - s = \frac{2k(s+1) + N(m-k)}{2(s+1) + N(m-k)},$$

where $C_4 = \frac{C(N, k, s)^{-k_2}\beta}{C(\eta_2)} - \lambda |\Omega|^{\frac{s-q+2}{s+1}} \|u_0\|_{s+1}^{q-k_2+s} > 0$, which yields the following decay estimates:

$$\begin{aligned} & \|u\|_{s+1} \leq \left[\|u_0\|_{s+1}^{s+1-k_2} - C_4(s+1-k_2)t \right]^{\frac{1}{s+1-k_2}}, \quad t \in [0, T_4), \\ & \|u\|_{s+1} = 0, \quad t \in [T_4, +\infty), \end{aligned}$$

where $T_4 = \frac{\|u_0\|_{s+1}^{s+1-k_2}}{C_4(s+1-k_2)}$.

Since $s > m$, we have $2(s+1) > 2m+2$, and hence, if $k \geq m$, then $q > k_2 - s \geq m$.

Assume that $q > 1$. If λ_1 is the first eigenvalue of the boundary problem

$$-\Delta \psi(x) = \lambda \psi(x), \quad x \in \Omega; \quad \psi(x) = 0, \quad x \in \partial\Omega,$$

and $\varphi_1(x) \geq 0$, $\|\varphi_1\|_\infty = 1$, is an eigenfunction corresponding to the eigenvalue λ_1 , then for sufficiently small $a > 0$, it can be easily shown that $a\varphi_1^{\frac{1}{m}}(x)$ is an upper solution of problem (1)-(3) provided that $u_0(x) \leq a\varphi_1^{\frac{1}{m}}(x)$, $x \in \Omega$. We then have $u(x, t) \leq a\varphi_1(x)$ for $t > 0$ by the comparison principle. Therefore, from equation (19), we can obtain the inequality

$$\begin{aligned} & \frac{1}{s+1} \frac{d}{dt} \|u\|_{s+1}^{s+1} + \left[\frac{4msd}{(m+s)^2} - \lambda a^{q-m} C_0^2 |\Omega|^{1+\frac{2}{N}} - \frac{\eta_2\beta}{C(\eta_2)} \right] \|\nabla u^{\frac{m+s}{2}}\|_2^2 \\ & + \frac{C(N, k, s)^{-k_2}\beta}{C(\eta_2)} \|u\|_{s+1}^{k_2} \leq 0, \end{aligned}$$

from which the following decay estimates can be obtained:

$$\|u\|_{s+1} \leq \left[\|u_0\|_{s+1}^{s+1-k_2} - \frac{C(N, k, s)^{-k_2} \beta}{C(\eta_2)} (s+1-k_2)t \right]^{\frac{1}{s+1-k_2}}, \quad t \in (0, T_4^*),$$

$$\|u\|_{s+1} = 0, \quad t \in [T_4^*, +\infty),$$

provided that

$$\frac{4msd}{(m+s)^2} - \lambda a^{q-m} C_0^2 |\Omega|^{1+\frac{2}{N}} - \frac{\eta_2 \beta}{C(\eta_2)} \geq 0,$$

where $T_4^* = \frac{C(\eta_2) \|u_0\|_{s+1}^{s+1-k_2}}{C(N, k, s)^{-k_2} \beta (s+1-k_2)}$. □

Remark 2 Since the Sobolev embedding inequality cannot be used in the proof of Theorem 2, it is not necessary to consider the cases that $\frac{N-2}{N+2} \leq m < 1$ and $0 < m < \frac{N-2}{N+2}$, when $N > 2$. In addition, if $k \geq m$, the conditions in Theorem 2 imply that $q > m$.

Theorem 3 Suppose that $0 < k < 1$ and $m > q \geq k$. Then the nonnegative nontrivial weak solution of problem (1)-(3) vanishes in finite time for any nonnegative initial data provided that β is sufficiently large.

Proof We first consider the case that $N = 1$ or 2 . Multiplying both sides of (1) by u and integrating the result over Ω , and using Hölder's inequality, we get

$$\frac{1}{2} \frac{d}{dt} \|u\|_2^2 + \frac{4md}{(m+1)^2} \|\nabla u^{\frac{m+1}{2}}\|_2^2 + \beta \|u\|_{k+1}^{k+1}$$

$$= \lambda \int_{\Omega} u^q dx \int_{\Omega} u dx \leq \lambda |\Omega| \|u\|_{q+1}^{q+1}.$$

By Lemma 2, we have the inequality

$$\|u\|_{q+1} \leq C(N, k, q) \|u\|_{k+1}^{1-\theta_3} \|\nabla u^{\frac{m+1}{2}}\|_2^{\frac{2\theta_3}{m+1}}, \tag{20}$$

where $\theta_3 = \frac{m+1}{2} (\frac{1}{k+1} - \frac{1}{q+1}) (\frac{1}{N} - \frac{1}{2} + \frac{m+1}{2} \frac{1}{k+1})^{-1} = \frac{N(q-k)(m+1)}{(q+1)[2(k+1)+N(m-k)]} \in [0, 1)$. Since $q < m$, we have $m+1 - (q+1)\theta_3 > 0$. It then follows from (20) and Young's inequality that

$$\lambda |\Omega| \|u\|_{q+1}^{q+1} \leq \lambda |\Omega| C(N, k, q)^{q+1} \|u\|_{k+1}^{(q+1)(1-\theta_3)} \|\nabla u^{\frac{m+1}{2}}\|_2^{\frac{2(q+1)\theta_3}{m+1}}$$

$$\leq \lambda |\Omega| C(N, k, q)^{q+1} (\eta_3 \|\nabla u^{\frac{m+1}{2}}\|_2^2 + C(\eta_3) \|u\|_{k+1}^{\frac{(q+1)(1-\theta_3)(m+1)}{m+1-(q+1)\theta_3}}), \tag{21}$$

where η_3 will be determined later. From (18) and (21), one can see that

$$\frac{1}{2} \frac{d}{dt} \|u\|_2^2 + \left[\frac{4md}{(m+1)^2} - \eta_3 \lambda |\Omega| C(N, k, q)^{q+1} \right] \|\nabla u^{\frac{m+1}{2}}\|_2^2 + \beta \|u\|_{k+1}^{k+1}$$

$$\leq C(\eta_3) \lambda |\Omega| C(N, k, q)^{q+1} \|u\|_{k+1}^{\frac{(q+1)(1-\theta_3)(m+1)}{m+1-(q+1)\theta_3}}.$$

We then obtain the inequality

$$\frac{1}{2} \frac{d}{dt} \|u\|_2^2 + \left[\frac{4md}{(m+1)^2} - \eta_3 \lambda |\Omega| C(N, k, q)^{q+1} \right] \gamma^{-2} \|u\|_2^{m+1} + \|u\|_{k+1}^{k+1} [\beta - C(\eta_3) \lambda |\Omega| C(N, k, q)^{q+1} \|u\|_{k+1}^{\alpha_1}] \leq 0$$

by (6) and the inequality above, where $\alpha_1 = \frac{(q+1)(1-\theta_3)(m+1)}{m+1-(q+1)\theta_3} - (k+1) = \frac{2(q-k)(k+1)}{2(k+1)+N(m-q)} \geq 0$. We can choose η_3 small enough so that $C_5 = \frac{4md}{(m+1)^2} - \eta_3 \lambda |\Omega| C(N, k, q)^{q+1} > 0$. Once η_3 is fixed, we may choose β large enough so that

$$\beta - C(\eta_3) \lambda |\Omega| C(N, k, q)^{q+1} \|u\|_{k+1}^{\alpha_1} \geq 0.$$

Hence, we have the inequality

$$\frac{d}{dt} \|u\|_2 + C_5 \|u\|_2^m \leq 0,$$

from which the following decay estimates can be obtained by a similar argument as the one used in the proof of Theorem 1:

$$\|u\|_2 \leq [\|u_0\|_2^{1-m} - C_5(1-m)t]^{-\frac{1}{m-1}}, \quad t \in [0, T_5),$$

$$\|u\|_2 = 0, \quad t \in [T_5, +\infty),$$

where $T_5 = \frac{\|u_0\|_2^{1-m}}{C_5(1-m)}$.

Secondly, we consider the case that $N > 2$. If $\frac{N-2}{N+2} \leq m < 1$, multiplying both sides of (1) by u^s ($s > m$) and integrating the result over Ω , and then using Hölder's inequality, we get

$$\frac{1}{s+1} \frac{d}{dt} \|u\|_{s+1}^{s+1} + \frac{4msd}{(m+s)^2} \|\nabla u^{\frac{m+s}{2}}\|_2^2 + \beta \|u\|_{k+s}^{k+s} = \lambda \int_{\Omega} u^q dx \int_{\Omega} u^s dx \leq \lambda |\Omega| \|u\|_{q+s}^{q+s}.$$

By Lemma 2, it can be shown that

$$\|u\|_{q+s} \leq C(N, k, q, s) \|u\|_{k+s}^{1-\theta_4} \|\nabla u^{\frac{m+s}{2}}\|_2^{\frac{2\theta_4}{m+s}}, \tag{22}$$

where $\theta_4 = \frac{m+s}{2} (\frac{1}{k+s} - \frac{1}{q+s}) (\frac{1}{N} - \frac{1}{2} + \frac{m+s}{2} \frac{1}{k+s})^{-1} = \frac{N(q-k)(m+s)}{(q+s)[2(k+s)+N(m-k)]} \in [0, 1)$. Since $q < m$, we have $m+s - (q+s)\theta_4 > 0$. It then follows from (22) and Young's inequality that

$$\lambda |\Omega| \|u\|_{q+s}^{q+s} \leq \lambda |\Omega| C(N, k, q, s)^{q+s} \|u\|_{k+s}^{(q+s)(1-\theta_4)} \|\nabla u^{\frac{m+s}{2}}\|_2^{\frac{2(q+s)\theta_4}{m+s}} \leq \lambda |\Omega| C(N, k, q, s)^{q+s} (\eta_4 \|\nabla u^{\frac{m+s}{2}}\|_2^2 + C(\eta_4) \|u\|_{k+s}^{\frac{(q+s)(1-\theta_4)(m+s)}{m+s-(q+s)\theta_4}}), \tag{23}$$

where η_4 will be determined later. From (19) and (23), one can see that

$$\frac{1}{s+1} \frac{d}{dt} \|u\|_{s+1}^{s+1} + \left[\frac{4msd}{(m+s)^2} - \eta_4 \lambda |\Omega| C(N, k, q, s)^{q+s} \right] \|\nabla u^{\frac{m+s}{2}}\|_2^2 + \beta \|u\|_{k+s}^{k+s} \leq C(\eta_4) \lambda |\Omega| C(N, k, q, s)^{q+s} \|u\|_{k+s}^{\frac{(q+s)(1-\theta_4)(m+s)}{m+s-(q+s)\theta_4}}.$$

By (12), (13), and the inequality above, we can obtain the inequality

$$\frac{1}{s+1} \frac{d}{dt} \|u\|_{s+1}^{s+1} + \left[\frac{4msd}{(m+s)^2} - \eta_4 \lambda |\Omega| C(N, k, q, s)^{q+s} \right] C_0^{-2} |\Omega|^{\frac{N-2}{N} - \frac{m+s}{s+1}} \|u\|_{s+1}^{m+s} + \|u\|_{k+s}^{k+s} [\beta - C(\eta_4) \lambda |\Omega| C(N, k, q, s)^{q+s} \|u\|_{k+s}^{\alpha_2}] \leq 0,$$

where $\alpha_2 = \frac{(q+s)(1-\theta_4)(m+s)}{m+s-(q+s)\theta_4} - (k+s) = \frac{2(q-k)(k+s)}{2(k+s)+N(m-q)} \geq 0$. We can choose η_4 small enough so that $C_6 = [\frac{4msd}{(m+s)^2} - \eta_4 \lambda |\Omega| C(N, k, q, s)^{q+s}] C_0^{-2} |\Omega|^{\frac{N-2}{N} - \frac{m+s}{s+1}} > 0$. Once η_4 is fixed, we can choose β large enough so that

$$\beta - C(\eta_4) \lambda |\Omega| C(N, k, q, s)^{q+s} \|u\|_{k+s}^{\alpha_2} \geq 0.$$

Hence, we can obtain the inequality

$$\frac{d}{dt} \|u\|_{s+1} + C_6 \|u\|_{s+1}^m \leq 0,$$

from which the following decay estimates can be obtained:

$$\|u\|_{s+1} \leq [\|u_0\|_{s+1}^{1-m} - C_6(1-m)t]^{\frac{1}{1-m}}, \quad t \in [0, T_6),$$

$$\|u\|_{s+1} = 0, \quad t \in [T_6, +\infty),$$

where $T_6 = \frac{\|u_0\|_{s+1}^{1-m}}{C_6(1-m)}$.

Similarly, one can obtain the following decay estimates for $0 < m < \frac{N-2}{N+2}$:

$$\|u\|_{s+1} \leq \left\{ \|u_0\|_{s+1}^{1-m} - \left[\frac{4msd}{(m+s)^2} - \eta_4 \lambda |\Omega| C(N, k, q, s)^{q+s} \right] \times C_0^{-2} |\Omega|^{1 - \frac{(s+1)(N+2)}{N(m+s)}} (1-m)t \right\}^{\frac{1}{1-m}}, \quad t \in [0, T_6^*),$$

$$\|u\|_{s+1} = 0, \quad t \in [T_6^*, +\infty),$$

where $T_6^* = \frac{\|u_0\|_{s+1}^{1-m}}{[\frac{4msd}{(m+s)^2} - \eta_4 \lambda |\Omega| C(N, k, q, s)^{q+s}] C_0^{-2} |\Omega|^{1 - \frac{(s+1)(N+2)}{N(m+s)}} (1-m)}$. □

Remark 3 One can see from Theorems 1-3 that the extinction of nonnegative nontrivial weak solutions to problem (1)-(3) occurs when $0 < k \leq q < 1$.

Remark 4 Theorems 1-3 all require $|\Omega|$, λ , or u_0 to be sufficiently small or β to be sufficiently large.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the manuscript and read and approved the final manuscript.

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Acknowledgements

The second and third authors were supported by the National Science Foundation of Shandong Province of China (ZR2012AM018) and Changwon National University in 2012, respectively. The authors would like to express their sincere gratitude to the anonymous reviewers for their insightful and constructive comments.

Received: 25 September 2012 Accepted: 21 December 2012 Published: 5 March 2013

References

1. Bebernes, J, Eberly, D: *Mathematical Problems from Combustion Theory*. Springer, New York (1989)
2. Pao, CV: *Nonlinear Parabolic and Elliptic Equations*. Plenum, New York (1992)
3. Furter, J, Grinfeld, M: Local vs. nonlocal interactions in populations dynamics. *J. Math. Biol.* **27**, 65-80 (1989)
4. Chen, YP: Blow-up for a system of heat equations with nonlocal sources and absorptions. *Comput. Math. Appl.* **48**, 361-372 (2004)
5. Souplet, P: Blow-up in nonlocal reaction-diffusion equations. *SIAM J. Math. Anal.* **29**, 1301-1334 (1998)
6. Souplet, P: Uniform blow-up profiles and boundary behavior for diffusion equations with nonlocal nonlinear source. *J. Differ. Equ.* **153**, 374-406 (1999)
7. Wang, MX, Wang, YM: Properties of positive solutions for non-local reaction-diffusion problems. *Math. Methods Appl. Sci.* **19**, 1141-1156 (1996)
8. Galaktionov, VA, Levine, HA: A general approach to critical Fujita exponents in nonlinear parabolic problems. *Nonlinear Anal.* **34**, 1005-1027 (1998)
9. Afanas'eva, NV, Tedeev, AF: Theorems on the existence and nonexistence of solutions of the Cauchy problem for degenerate parabolic equations with nonlocal source. *Ukr. Math. J.* **57**, 1687-1711 (2005)
10. Evans, LC, Knerr, BF: Instantaneous shrinking of the support of nonnegative solutions to certain nonlinear parabolic equations and variational inequalities. III. *J. Math.* **23**, 153-166 (1979)
11. Ferreira, R, Vazquez, JL: Extinction behavior for fast diffusion equations with absorption. *Nonlinear Anal.* **43**, 353-376 (2001)
12. Li, YX, Wu, JC: Extinction for fast diffusion equations with nonlinear sources. *Electron. J. Differ. Equ.* **2005**, 23 (2005)
13. Yin, JX, Jin, CH: Critical extinction and blow-up exponents for fast diffusive Laplacian with sources. *Math. Methods Appl. Sci.* **30**, 1147-1167 (2007)
14. Zhou, J, Mu, CL: Critical blow-up and extinction exponents for non-Newton polytropic filtration equation with source. *Bull. Korean Math. Soc.* **46**, 1159-1173 (2009)
15. Liu, WJ, Wang, MX, Wu, B: Extinction and decay estimates of solutions for a class of porous medium equations. *J. Inequal. Appl.* **2007**, Article ID 087650 (2007)
16. Fang, ZB, Li, G: Extinction and decay estimates of solutions for a class of doubly degenerate equations. *Appl. Math. Lett.* **25**, 1795-1802 (2012)
17. Han, YZ, Gao, WJ: Extinction for a fast diffusion equation with a nonlinear nonlocal source. *Arch. Math.* **97**, 353-363 (2011)
18. Liu, WJ: Extinction and non-extinction of solutions for a nonlocal reaction-diffusion problem. *Electron. J. Qual. Theory Differ. Equ.* **2010**, 15 (2010)
19. Fang, ZB, Xu, XH: Extinction behavior of solutions for the p -Laplacian equations with nonlocal sources. *Nonlinear Anal., Real World Appl.* **13**, 1780-1789 (2012)
20. Antontsev, S, Diaz, JI, Shmarev, S: *Energy Methods for Free Boundary Problems: Applications to Nonlinear PDEs and Fluid Mechanics*. Progress in Nonlinear Differential Equations and Their Applications. Birkhäuser, Boston (2002)
21. Sacks, PE: Continuity of solutions of a singular parabolic equation. *Nonlinear Anal.* **7**, 387-409 (1983)
22. Wu, ZQ, Zhao, JN, Yin, JX, Li, HL: *Nonlinear Diffusion Equations*. World Scientific, River Edge (2001)
23. Ladyzhenskaya, OA, Solonnikov, VA, Ural'tseva, NN: *Linear and Quasilinear Equations of Parabolic Type*. Am. Math. Soc., Providence (1968)

doi:10.1186/1687-2770-2013-24

Cite this article as: Xu et al.: Extinction and decay estimates of solutions for a porous medium equation with nonlocal source and strong absorption. *Boundary Value Problems* 2013 **2013**:24.