Cheng *Boundary Value Problems* 2013, 2013:250
http://www.boundaryvalueproblems.com/content/2013/1/250

Boundary Value Problems
a SpringerOpen Journal

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Nontrivial solutions for Schrödinger-Kirchhoff-type problem in R^N

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Abstract

In the present paper, we use variational methods to prove two existence results of nontrivial solutions for the Schrödinger-Kirchhoff-type problem

$$\begin{cases} -(a+b\int_{R^N}|\nabla u|^2\,dx)\Delta u + V(x)u = f(x,u), & \text{for } x\in R^N,\\ u(x)\to 0, & \text{as } |x|\to\infty. \end{cases}$$

One deals with the asymptotic behaviors of f near zero and infinity and the other deals with 4-superlinearity of F at infinity.

Keywords: Schrödinger-Kirchhoff-type problem; Sobolev's embedding theorem; critical point; variational methods

1 Introduction and main results

In this paper, we consider the Schrödinger-Kirchhoff-type problem

$$\begin{cases} -(a+b\int_{R^N}|\nabla u|^2\,dx)\Delta u + V(x)u = f(x,u), & \text{for } x\in R^N,\\ u(x)\to 0, & \text{as } |x|\to\infty, \end{cases} \tag{1.1}$$

where constants $a > 0$, $b \geq 0$, $N = 1, 2$ or 3, $V \in C(R^N, R)$ and $f \in C(R^N \times R, R)$. We are concerned with the existence of nontrivial solutions of (1.1), corresponding to the critical points of the energy functional

$$I(u) = \frac{a}{2}\int_{R^N}|\nabla u|^2\,dx + \frac{b}{4}\left(\int_{R^N}|\nabla u|^2\,dx\right)^2 + \frac{1}{2}\int_{R^N}V(x)u^2\,dx - \int_{R^N}F(x,u)\,dx, \tag{1.2}$$

where $F(x,t) = \int_0^t f(x,s)\,ds$.

When $a = 1$, $b = 0$, problem (1.1) reduces to the following semilinear Schrödinger equation:

$$\begin{cases} -\Delta u + V(x)u = f(x,u), & \text{for } x\in R^N,\\ u(x)\to 0, & \text{as } |x|\to\infty. \end{cases} \tag{1.3}$$

Equation (1.3) has been studied extensively by many authors, and there is a large body of literature on the existence and multiplicity of results of solutions for equation (1.3); for example, we refer the readers to [1–3] and references therein.

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On the other hand, the Kirchhoff-type problem on a bounded domain $\Omega \subset R^N$,

$$\begin{cases} -(a + b\int_\Omega |\nabla u|^2\,dx)\triangle u = f(x,u), & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

is related to the stationary analogue of the Kirchhoff equation

$$u_{tt} - \left(a + b\int_{R^N} |\nabla u|^2\,dx\right)\triangle u = g(x,t),$$

which was proposed by Kirchhoff [4] as an extension of the classical D'Alembert wave equation for free vibrations of elastic strings. It is pointed out in [5] that Kirchhoff-type problem (1.4) models several physical and biological systems, where u describes the process which depends on the average of itself (for example, population density). For the case of bounded domain, some interesting studies by variational methods can be found in [6–14] for equation (1.4) with several growth conditions on f. Very recently, Kirchhoff-type equations on the unbounded domain or the whole space R^N have also attracted a lot of attention. Many solvability conditions on the nonlinearity have been given to obtain the existence and multiplicity of solutions for problem (1.1). In [15, 16], the authors studied the case of superlinear nonlinearity. In [17, 18], the authors considered the case of radial potentials. In [19], the authors studied the case of nonhomogeneous nonlinearity.

Equation (1.1) can be viewed as the combination of (1.3) and (1.4) in R^N. So we call it the Schrödinger-Kirchhoff-type problem. Compared with (1.3) or (1.4), problem (1.1) is much more complicated and R^N is in place of the bounded domain $\Omega \subset R^N$ in (1.4). This makes the study of equation (1.1) more difficult and interesting. In the present paper, the goal is to study the existence of nontrivial solutions for equation (1.1) for the case of asymptotical nonlinearity and weaker superlinear conditions compared with [15, 16].

For the potential V, we assume

(V) $V \in C(R^N, R)$ satisfies $\inf V(x) \geq \alpha > 0$ and for each $M > 0$, $\operatorname{meas}\{x \in R^N : V(x) \leq M\} < +\infty$, where α is a constant and meas denotes the Lebesgue measure in R^N.

Set

$$H^1(R^N) = \{u \in L^2(R^N) : \nabla u \in L^2(R^N)\}$$

with the norm

$$\|u\|_{H^1} = \left(\int_{R^N} (|\nabla u|^2 + u^2)\,dx\right)^{\frac{1}{2}}.$$

Denote

$$E = \left\{u \in H^1(R^N) : \int_{R^N} (|\nabla u|^2 + V(x)u^2)\,dx < +\infty\right\}$$

with the inner product and the norm

$$\langle u,v\rangle_E = \int_{R^N} (\nabla u \cdot \nabla v + V(x)uv)\,dx, \qquad \|u\|_E = \langle u,u\rangle_E^{\frac{1}{2}}.$$

Since $\inf V(x) \geq \alpha > 0$, it is easy to see that the Hilbert space E is continuously embedded in $H^1(R^N)$. By the Sobolev embedding theorem, we know the embedding

$$E \hookrightarrow L^s(R^N), \quad 2 \leq s \leq 2^*,$$

where $2^* = \frac{2N}{N-2}$, if $N \geq 3$, and $2^* = +\infty$, if $N = 1, 2$, is also continuous, and there is a constant $\gamma_s > 0$, $2 \leq s \leq 2^*$, such that

$$\|u\|_s \leq \gamma_s \|u\|_E, \quad \forall u \in E, \tag{1.5}$$

where

$$\|u\|_s = \left(\int_{R^N} |u|^s \, dx \right)^{\frac{1}{s}}.$$

Moreover, the embedding $E \hookrightarrow L^s(R^N)$ is compact for each $2 \leq s < 2^*$ due to the assumption (V). In fact, if $N = 3$, it follows from Lemma 3.4 in [20]. If $N = 1, 2$, we also claim that the compactness of the embedding is valid for $2^* = +\infty$. Indeed, let $\{u_n\} \subset E$ be a sequence of E such that $u_n \rightharpoonup u$ weakly in E. Similarly to the proof of Lemma 3.4 in [20], up to a subsequence, we can obtain $u_n \to u$ strongly in $L^2(R^N)$. Next, we shall prove $u_n \to u$ strongly in $L^s(R^N)$ for $2 < s < +\infty$. In fact, $\forall s \in (2, +\infty)$, there are $t \in (s, +\infty)$ and $\theta \in (0, 1)$ such that $s = 2\theta + (1 - \theta)t$. Then, by the Hölder inequality,

$$\|u_n - u\|_s^s = \int_{R^N} |u_n - u|^{2\theta} |u_n - u|^{(1-\theta)t} \, dx \leq \|u_n - u\|_2^{2\theta} \|u_n - u\|_t^{(1-\theta)t}.$$

Since the embedding

$$E \hookrightarrow L^s(R^N), \quad 2 \leq s < +\infty, N = 1, 2,$$

is continuous, $\{u_n\}$ is bounded in $L^t(R^N)$. This, together with $\|u_n - u\|_2 \to 0$, shows $u_n \to u$ strongly in $L^s(R^N)$, $2 \leq s < +\infty$, $N = 1, 2$. Therefore, the compactness result holds for $N = 1, 2, 3$.

Throughout this paper, we shall always assume $f(x,t)t \geq 0$, $\forall (x,t) \in R^N \times R$. To establish the existence of nontrivial solutions for Schrödinger-Kirchhoff-type problem (1.1) in R^N, we make the following assumptions:

(f_1) $f \in C(R^N \times R, R)$ and $|f(x,t)| \leq c(1 + |t|^{p-1})$ for some $2 \leq p < 2^*$, where c is a positive constant.
(f_2) $f(x,t) = o(|t|)$ as $|t| \to 0$ uniformly in $x \in R^N$.
(f_3) $\lim_{|t| \to +\infty} \frac{f(x,t)}{t^3} = l$ uniformly in $x \in R^N$.
(f_4) $4F(x,t) \leq f(x,t)t + \alpha t^2$ for all $(x,t) \in R^N \times R$, where $0 < \alpha < \frac{\min\{a,1\}}{\gamma_2^2}$ (γ_2 appears in (1.5)).

We consider the subcritical case in the present paper, (f_2) and (f_3) with $l < +\infty$ characterize the asymptotic behavior of f at zero and infinity. The condition (f_3) with $l = +\infty$ implies $\lim_{|t| \to +\infty} \frac{F(x,t)}{t^4} = +\infty$, that is, 4-superlinearity of F at infinity.

As usual, the Ambrosetti-Rabinowitz [21] type condition

$$\text{(AR)} \quad \exists \nu > 4 : \nu F(x,t) \leq tf(x,t), \quad |t| \text{ large},$$

is assumed to ensure the boundedness of a Palais-Smale sequence. It implies that there exists a constant $c > 0$ such that $F(x,t) \geq c(|t|^\nu - 1)$. By a simple calculation, it is easy to see that (AR) implies that $\lim_{|t|\to+\infty} \frac{F(x,t)}{t^4} = +\infty$, and hence ($f_3$) with $l < +\infty$ does not ensure the condition (AR). Furthermore, our condition (f_3) with $l \equiv +\infty$ is much weaker than the condition (AR). It is important to show that there are many functions satisfying the conditions (f_1)-(f_4) with $l \in (b\Lambda^2, +\infty)$ or $l = +\infty$, where

$$\Lambda = \inf\left\{ \int_{R^N} \left(|\nabla u|^2 + V(x)u^2\right) dx : u \in E, \int_{R^N} u^4\, dx = 1 \right\}. \tag{1.6}$$

But the condition (AR) is not satisfied.

Example 1 For any given $M \in (b\Lambda^2, +\infty)$, set

$$f(x,t) = \begin{cases} |t|^3 t, & |t| \leq M; \\ Mt^3, & |t| > M. \end{cases}$$

Then it is easy to verify that $f(x,t)$ satisfies (f_1)-(f_4) with $l \equiv M$, but does not satisfy the condition (AR).

Example 2 Set

$$f(x,t) = \begin{cases} t^3, & |t| \leq e; \\ t^3 \ln|t|, & |t| > e. \end{cases}$$

Then it is easy to verify that $f(x,t)$ satisfies (f_1)-(f_4) with $l = +\infty$, but does not satisfy the condition (AR).

Our main results are stated as the following theorems.

Theorem 1.1 *Let conditions* (V) *and* (f_1)-(f_4) *hold and* $l \in (b\Lambda^2, +\infty)$. *Then problem* (1.1) *has at least one nontrivial solution in E*.

Theorem 1.2 *Let conditions* (V), (f_1)-(f_4) *with* $l \equiv +\infty$ *hold*. *Then problem* (1.1) *has at least one nontrivial solution in E*.

Corollary 1.3 *If the following* (f_5) *or* (f_6) *is used in place of* (f_4):

(f_5) $4F(x,t) \leq f(x,t)t$, $\forall x \in R^N$, $\forall t \in R$.
(f_6) $\frac{f(x,t)}{|t|^3}$ *is nondecreasing with respect to* t.

Then the conclusions of Theorem 1.1 *and Theorem* 1.2 *hold*.

2 The preliminary lemmas

First, under assumptions (V), the embedding $E \hookrightarrow L^s(R^N)$ is compact for each $2 \leq s < 2^*$. Then the condition (f_1) implies $I \in C^1(E,R)$,

$$\langle I'(u), v\rangle = \left(a + b\int_{R^N} |\nabla u|^2\, dx\right) \int_{R^N} \nabla u \nabla v\, dx + \int_{R^N} V(x)uv\, dx - \int_{R^N} f(x,u)v\, dx \tag{2.1}$$

for all $u, v \in E$, and the weak solutions of problem (1.1) correspond to the critical points of energy functional I.

Recall that we say that I satisfies the (PS) condition at the level $c \in R$ ($(PS)_c$ condition for short) if any sequence $\{u_n\} \subset E$ along with $I(u_n) \to c$ and $I'(u_n) \to 0$ as $n \to \infty$ possesses a convergent subsequence. If I satisfies the $(PS)_c$ condition for each $c \in R$, then we say that I satisfies the (PS) condition.

For the proof of our main results, we will make use of the following lemmas.

Lemma 2.1 *$\Lambda > 0$ defined by (1.6) achieves by some $\varphi_\Lambda \in E$ with $\int_{R^N} \varphi_\Lambda^4\,dx = 1$ and $\varphi_\Lambda > 0$ a.e. in R^N.*

Proof By the Sobolev embedding theorem, one has $\Lambda > 0$. In order to prove that the infimum is achieved, we consider a minimizing sequence $\{u_n\} \subset E$ such that

$$\|u_n\|_4 = 1, \qquad \|u_n\|_E^2 \to \Lambda, \quad n \to \infty.$$

By the embedding $E \hookrightarrow L^s(R^N)$ is compact for each $2 \leq s < 2^*$, up to a subsequence, we may assume that there is $\varphi_\Lambda \in E$ such that

$$u_n \rightharpoonup \varphi_\Lambda \quad \text{in } E, \qquad u_n \to \varphi_\Lambda \quad \text{in } L^4\bigl(R^N\bigr),$$

so that

$$\|\varphi_\Lambda\|_E^2 \leq \liminf_{n\to\infty} \|u_n\|_E^2 = \Lambda, \qquad \|\varphi_\Lambda\|_4 = 1,$$

because of the weak lower semicontinuity of $\|\cdot\|_E^2$. Furthermore, we may assume that $\varphi_\Lambda(x) > 0$ a.e. in R^N. Otherwise, we can replace φ_Λ by $|\varphi_\Lambda|$. □

Lemma 2.2 *Set $Q(u) = \int_{R^N} |\nabla u|^2\,dx$, $u \in E$. Then Q is weakly lower semicontinuous on E.*

Proof The proof has been given by Lemma 2 in [16]. Next we give another direct method to prove it, which is much easier than Lemma 2 in [16]. Let $\{u_n\} \subset E$ and $u_n \rightharpoonup u$ in E. By the embedding $E \hookrightarrow L^2(R^N)$ is compact and the weak lower semicontinuity of $\|\cdot\|_E^2$, then

$$\begin{aligned}
\liminf_{n\to\infty} Q(u_n) &= \left[\liminf_{n\to\infty} Q(u_n) + \int_{R^N} u^2\,dx\right] - \int_{R^N} u^2\,dx \\
&= \liminf_{n\to\infty}\left[\int_{R^N} \bigl(|\nabla u_n|^2 + u_n^2\bigr)\,dx\right] - \int_{R^N} u^2\,dx \\
&\geq \int_{R^N} \bigl(|\nabla u|^2 + u^2\bigr)\,dx - \int_{R^N} u^2\,dx \\
&= Q(u).
\end{aligned}$$

This shows that Q is weakly lower semicontinuous on E. □

Lemma 2.3 *Let* (f_4) *hold. Then any* (PS) *sequence of* I *is bounded in* E.

Proof Let $\{u_n\} \subset E$ be a (PS) sequence with

$$I(u_n) \to c, \qquad I'(u_n) \to 0, \quad n \to \infty.$$

Hence, for large n, the combination of (1.5) with (f_4) implies that

$$\begin{aligned} c + 1 + \|u_n\|_E &\geq I(u_n) - \frac{1}{4}\langle I'(u_n), u_n\rangle \\ &\geq \frac{1}{4}\min\{a,1\}\|u_n\|_E^2 + \int_{R^N}\left[\frac{1}{4}f(x,u_n)u_n - F(x,u_n)\right]dx \\ &\geq \frac{1}{4}\min\{a,1\}\|u_n\|_E^2 - \frac{1}{4}\alpha\int_{R^N} u_n^2\,dx \\ &\geq \frac{1}{4}\left[\min\{a,1\} - \alpha\gamma_2^2\right]\|u_n\|_E^2. \end{aligned} \tag{2.2}$$

Therefore, the conclusion follows from (2.2) and $\alpha\gamma_2^2 < \min\{a,1\}$. □

3 Proof of main results

Proof of Theorem 1.1 To begin with, we prove that there exist $\rho, \beta > 0$ such that $I(u) \geq \beta$ for all $u \in E$ with $\|u\|_E = \rho$, and $I(t\varphi_\Lambda) \to -\infty$ as $t \to +\infty$. Indeed, for any $\epsilon > 0$, by (f_1), (f_2) and (f_3), there exists $C(\varepsilon) > 0$ such that

$$F(x,t) \leq \frac{1}{2}\varepsilon|t|^2 + \frac{C(\varepsilon)}{4}|t|^4, \quad \forall (x,t) \in R^N \times R. \tag{3.1}$$

Choosing $0 < \varepsilon < \frac{\min\{a,1\}}{\gamma_2^2}$ (γ_2 appears in (1.5)), by (1.5) and (3.1),

$$\begin{aligned} I(u) &\geq \frac{1}{2}\min\{a,1\}\|u\|_E^2 - \frac{\varepsilon}{2}\|u\|_2^2 - \frac{C(\varepsilon)}{4}\|u\|_4^4 \\ &\geq \frac{1}{2}\left(\min\{a,1\} - \gamma_2^2\varepsilon\right)\|u\|_E^2 - \frac{C(\varepsilon)}{p}\gamma_4^4\|u\|_E^4. \end{aligned}$$

Therefore, we can choose small $\rho > 0$ such that

$$I(u) \geq \frac{1}{4}\left(\min\{a,1\} - \gamma_2^2\varepsilon\right)\rho^2 := \beta > 0$$

whenever $u \in E$ with $\|u\|_E = \rho$.

Since $l > b\Lambda^2$, by Fatou's lemma and (f_3), we have

$$\begin{aligned} \limsup_{t\to+\infty}\frac{I(t\varphi_\Lambda(x))}{t^4} &\leq \limsup_{t\to+\infty}\left\{\frac{\max\{a,1\}\|\varphi_\Lambda\|_E^2}{2t^2} + \frac{b}{4}\|\varphi_\Lambda\|_E^4 - \int_{R^N}\frac{F(x,t\varphi_\Lambda(x))}{t^4}\,dx\right\} \\ &\leq \frac{b}{4}\|\varphi_\Lambda\|_E^4 - \liminf_{t\to+\infty}\int_{R^N}\frac{F(x,t\varphi_\Lambda(x))}{t^4}\,dx \\ &\leq \frac{b}{4}\|\varphi_\Lambda\|_E^4 - \int_{R^N}\lim_{t\to+\infty}\frac{F(x,t\varphi_\Lambda(x))}{t^4}\,dx \\ &= \frac{b}{4}\|\varphi_\Lambda\|_E^4 - \int_{R^N}\lim_{t\to+\infty}\frac{f(x,t\varphi_\Lambda(x))\varphi_\Lambda(x)}{4t^3}\,dx \end{aligned}$$

$$
\begin{aligned}
&= \frac{b}{4}\|\varphi_\Lambda\|_E^4 - \frac{1}{4}\int_{R^N} \lim_{t\to+\infty} \frac{f(x,t\varphi_\Lambda(x))}{(t\varphi_\Lambda(x))^3}\cdot \varphi_\Lambda^4(x)\,dx \\
&= \frac{b}{4}\|\varphi_\Lambda\|_E^4 - \frac{l}{4}\int_{R^N} \varphi_\Lambda^4(x)\,dx \\
&= \frac{b}{4}\Lambda^2 - \frac{l}{4} \\
&< 0.
\end{aligned}
$$

Hence, $I(t\varphi_\Lambda) \to -\infty$ as $t \to +\infty$. Therefore, we can find large $t_0 > 0$ such that $\|t_0\varphi_\Lambda\|_E > \rho$, $I(t_0\varphi_\Lambda) < 0$.

Now, we prove that I satisfies the (PS) condition. Indeed, if a sequence $\{u_n\} \subset E$ is such that

$$
I(u_n) \to c, \qquad I'(u_n) \to 0, \quad n \to \infty,
$$

then by Lemma 2.3, $\{u_n\}$ is bounded in E. Since the embedding $E \hookrightarrow L^s(R^N)$ is compact for each $s \in [2, 2^*)$, up to a subsequence, there is $u \in E$ such that

$$
u_n \rightharpoonup u \quad \text{in } E, \qquad u_n \to u \quad \text{in } L^s(R^N), \quad s \in [2, 2^*).
$$

By (2.1) and a simple computation, we conclude

$$
\begin{aligned}
&\langle I'(u_n) - I'(u), u_n - u\rangle \\
&\quad = \left(a + b\int_{R^N} |\nabla u_n|^2\,dx\right)\int_{R^N} \nabla u_n\cdot\nabla(u_n - u)\,dx + \int_{R^N} V(x)|u_n - u|^2\,dx \\
&\qquad - \left(a + b\int_{R^N} |\nabla u|^2\,dx\right)\int_{R^N} \nabla u\cdot\nabla(u_n - u)\,dx \\
&\qquad - \int_{R^N} \big[f(x,u_n) - f(x,u)\big](u_n - u)\,dx \\
&\quad = \left(a + b\int_{R^N} |\nabla u_n|^2\,dx\right)\int_{R^N} \big|\nabla(u_n - u)\big|^2\,dx + \int_{R^N} V(x)|u_n - u|^2\,dx \\
&\qquad - b\left(\int_{R^N} |\nabla u|^2\,dx - \int_{R^N} |\nabla u_n|^2\,dx\right)\int_{R^N} \nabla u\cdot\nabla(u_n - u)\,dx \\
&\qquad - \int_{R^N} \big[f(x,u_n) - f(x,u)\big](u_n - u)\,dx \\
&\quad \ge \min\{a,1\}\|u_n - u\|_E^2 \\
&\qquad - b\left(\int_{R^N} |\nabla u|^2\,dx - \int_{R^N} |\nabla u_n|^2\,dx\right)\int_{R^N} \nabla u\cdot\nabla(u_n - u)\,dx \\
&\qquad - \int_{R^N} \big[f(x,u_n) - f(x,u)\big](u_n - u)\,dx.
\end{aligned}
\tag{3.2}
$$

Then (3.2) implies that

$$
\begin{aligned}
&\min\{a,1\}\|u_n - u\|_E^2 \\
&\quad \le \langle I'(u_n) - I'(u_n), u_n - u\rangle
\end{aligned}
$$

$$\begin{aligned} &+ b\left(\int_{R^N} |\nabla u|^2\,dx - \int_{R^N} |\nabla u_n|^2\,dx\right)\int_{R^N} \nabla u\cdot\nabla(u_n-u)\,dx \\ &+ \int_{R^N} \bigl[f(x,u_n)-f(x,u)\bigr](u_n-u)\,dx. \end{aligned} \tag{3.3}$$

Define the functional $h_u: E\to R$ by

$$h_u(v) = \int_{R^N} \nabla u\cdot\nabla v\,dx, \quad \forall v\in E.$$

Obviously, h_u is a linear functional on E. Furthermore,

$$\bigl|h_u(v)\bigr| \le \int_{R^N} |\nabla u\cdot\nabla v|\,dx \le \sqrt{Q(u)}\|v\|_E, \quad \forall v\in E,$$

which implies that h_u is bounded on E, where $Q(u)$ is defined in Lemma 2.2. Hence $h_u\in E^*$. Since $u_n \rightharpoonup u$ in E, it has $\lim_{n\to\infty} h_u(u_n) = h_u(u)$, that is, $\int_{R^N} \nabla u\cdot\nabla(u_n-u)\,dx\to 0$ as $n\to\infty$. Consequently, by the boundedness of $\{u_n\}$, it has

$$b\left(\int_{R^N} |\nabla u|^2\,dx - \int_{R^N} |\nabla u_n|^2\,dx\right)\int_{R^N} \nabla u\cdot\nabla(u_n-u)\,dx \to 0, \quad n\to+\infty. \tag{3.4}$$

Moreover, for any $\varepsilon>0$, by (f_1), (f_2) and (f_3), there exists $C(\varepsilon)>0$ such that

$$\bigl|f(x,t)\bigr| \le \varepsilon|t| + C(\varepsilon)|t|^3, \quad \forall(x,t)\in R^N\times R. \tag{3.5}$$

Hence, (3.5) implies

$$\begin{aligned} &\left|\int_{R^N} \bigl[f(x,u_n)-f(x,u)\bigr](u_n-u)\,dx\right| \\ &\quad\le \int_{R^N} \bigl[\varepsilon\bigl(|u_n|+|u|\bigr) + C(\varepsilon)\bigl(|u_n|^3+|u|^3\bigr)\bigr]|u_n-u|\,dx \\ &\quad\le \varepsilon c_1 + C(\varepsilon)\|u_n-u\|_4\bigl(\|u_n\|_4^3+\|u\|_4^3\bigr) \\ &\quad\le \varepsilon c_1 + c_2C(\varepsilon)\|u_n-u\|_4, \end{aligned}$$

where c_1 and c_2 are independent of ε and n. Since $\|u_n-u\|_4\to 0$ as $n\to+\infty$, we conclude

$$\int_{R^N} \bigl[f(x,u_n)-f(x,u)\bigr](u_n-u)\,dx \to 0 \quad \text{as } n\to+\infty. \tag{3.6}$$

Since $I'(u_n)\to 0$, the combination of (3.3), (3.4) and (3.6) implies that $\|u_n-u\|_E\to 0$.

Note that $I(0)=0$ applying the mountain pass theorem (Theorem 2.2 in [21]), then I possesses a critical value $c\ge\beta$, *i.e.*, problem (1.1) has a nontrivial solution in E. This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2 Set $0<\varepsilon<\frac{\min\{a,1\}}{\gamma_2^2}$ (γ_2 appears in (1.5)). By (f_1) and (f_2), there exists $C(\varepsilon)>0$ such that

$$\bigl|f(x,t)\bigr| \le \varepsilon|t| + C(\varepsilon)|t|^{p-1}. \tag{3.7}$$

Then, by (3.7), one has

$$F(x,t) \leq \frac{1}{2}\varepsilon|t|^2 + \frac{C(\varepsilon)}{p}|t|^p, \quad \forall(x,t) \in R^N \times R.$$

Since $p > 2$, there exist constants $\rho, \beta > 0$ such that $I(u) \geq \beta$ for all $u \in E$ with $\|u\|_E = \rho$ (see the proof of Theorem 1.1).

Since $E \hookrightarrow L^2(R^N)$ and $L^2(R^N)$ is a separable Hilbert space, E has a countable orthogonal basis $\{e_i\}$. Set $E_k = \operatorname{span}\{e_1, \dots, e_k\}$ and $Z_k = E_k^{\perp}$. Since all norms are equivalent in a finite dimensional space, there is a constant $C_4 > 0$ such that

$$\|u\|_4 \geq C_4\|u\|_E, \quad \forall u \in E_k. \tag{3.8}$$

By (f_1)-(f_3) with $l \equiv +\infty$, for any $M > \frac{b}{4C_4^4}$, there is a constant $C(M) > 0$ such that

$$F(x,u) \geq M|u|^4 - C(M)|u|^2, \quad \forall(x,u) \in R^N \times R. \tag{3.9}$$

Hence, combining (3.8) and (3.9), we deduce

$$\begin{aligned} I(u) &\leq \frac{1}{2}\max\{a,1\}\|u\|_E^2 + \frac{b}{4}\|u\|_E^4 - M\|u\|_4^4 + C(M)\|u\|_2^2 \\ &\leq \frac{1}{2}\max\{a,1\}\|u\|_E^2 - \left(MC_4^4 - \frac{b}{4}\right)\|u\|_E^4 + C(M)\gamma_2^2\|u\|_E^2 \end{aligned}$$

for all $u \in E_k$. Consequently, there is a point $e \in E$ with $\|e\|_E > \rho$ such that $I(e) < 0$.

Next, we show that I satisfies the (PS) condition. In fact, let $\{u_n\} \subset E$ be a (PS) sequence with

$$I(u_n) \to c, \qquad I'(u_n) \to 0, \quad n \to \infty.$$

Then, by Lemma 2.3, $\{u_n\}$ is bounded in E. Since the embedding $E \hookrightarrow L^s(R^N)$ is compact for each $s \in [2, 2^*)$, up to a subsequence, there is $u \in E$ such that

$$u_n \rightharpoonup u \quad \text{in } E, \qquad u_n \to u \quad \text{in } L^s(R^N), \quad s \in [2, 2^*).$$

Thanks to (3.7), we have

$$\begin{aligned} &\left|\int_{R^N}\left[f(x,u_n) - f(x,u)\right](u_n - u)\,dx\right| \\ &\quad\leq \int_{R^N}\left[\varepsilon\left(|u_n| + |u|\right) + c(\varepsilon)\left(|u_n|^{p-1} + |u|^{p-1}\right)\right]|u_n - u|\,dx \\ &\quad\leq \varepsilon c_1 + c(\varepsilon)\|u_n - u\|_p\left(\|u_n\|_p^{p-1} + \|u\|_p^{p-1}\right) \\ &\quad\leq \varepsilon c_1 + c_2 c(\varepsilon)\|u_n - u\|_p, \end{aligned}$$

where c_1 and c_2 are independent of ε and n. Since $\|u_n - u\|_p \to 0$ as $n \to +\infty$, we conclude

$$\int_{R^N}\left[f(x,u_n) - f(x,u)\right](u_n - u)\,dx \to 0 \quad \text{as } n \to +\infty.$$

The rest of the proof is the same as the proof of Theorem 1.1, the desired conclusion follows from the mountain pass theorem (Theorem 2.2 in [21]). This completes the proof of Theorem 1.2. □

Proof of Corollary 1.3 In order to obtain the desired conclusions, by the proof of Theorem 1.1 and Theorem 1.2, it is sufficient to show that the condition (f_5) or (f_6) implies the condition (f_4). First, it is obvious that (f_5) implies (f_4). Next, we only show (f_6) implies (f_4). Indeed, by (f_6), whenever $u > 0$,

$$F(x,u) = \int_0^1 f(x,ut)u\,dt = \int_0^1 \frac{f(x,ut)}{(ut)^3}u^4t^3\,dt \le \int_0^1 \frac{f(x,u)}{u^3}u^4t^3\,dt = \frac{1}{4}f(x,u)u. \tag{3.10}$$

Whenever $u < 0$,

$$\begin{aligned} F(x,u) &= \int_0^1 f(x,ut)u\,dt = -\int_0^1 \frac{f(x,ut)}{(-ut)^3}u^4t^3\,dt \le \int_0^1 \frac{f(x,u)}{u^3}u^4t^3\,dt \\ &= \frac{1}{4}f(x,u)u. \end{aligned} \tag{3.11}$$

Thus, (3.10) and (3.11) imply that $4F(x,u) \le f(x,u)u$, $\forall x \in R^N$, $\forall u \in R$. Hence (f_6) implies (f_4). □

Remark In [16], Wu considered the superlinear nonlinearity case and obtained the existence of nontrivial solutions for problem (1.1) under the same conditions of our Corollary 1.3 with superlinear case (see Theorem 1 and Theorem 2 in [16]). Therefore, our Theorem 1.2 is a generalization of Theorem 1 and Theorem 2 in [16].

Competing interests
The author declares that he has no competing interests.

Acknowledgements
The author would like to thank the referees for several comments and valuable suggestions. This work is partly supported by the National Natural Science Foundation of China (11361048), the Foundation of Education of Commission of Yunnan Province (2012Y410, 2013Y015) and the Youth Program of Yunnan Provincial Science and Technology Department (2013FD046).

Received: 25 June 2013 Accepted: 28 October 2013 Published: 22 Nov 2013

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10.1186/1687-2770-2013-250
Cite this article as: Cheng: Nontrivial solutions for Schrödinger-Kirchhoff-type problem in R^N. *Boundary Value Problems* 2013, 2013:250