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Application of p -regularity theory to nonlinear boundary value problems

Wiesław Grzegorzczak^{1†}, Beata Medak^{1*†} and Alexey A Tret'yakov^{1,2,3†}

*Correspondence:

bmedak@uph.edu.pl

¹Siedlce University of Natural Sciences and Humanities, 3-go Maja 54, Siedlce, 08-110, Poland

[†]Equal contributors

Full list of author information is available at the end of the article

Abstract

The paper studies the question of solution existence to a nonlinear equation in the degenerate case. This question is studied for three particular boundary value problems for ordinary and partial second-order differential equations. The so-called p -regularity theory is applied to these purposes as an effective apparatus to investigate many nonlinear mathematical, physical and numerical problems. All results obtained in the paper are based on the constructions of this theory whose basic concepts were described by Tret'yakov. We recall the main definitions and theorems of p -regularity theory and illustrate the results by examples including singular boundary value problems. In the first and second ones, the description of solutions by a tangent cone at an initial point are given. In the third example, we formulate a sufficient condition for p -regularity ($p = 2$), which can be tested using the notion of resultant.

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Introduction

In the paper, we study the question of solution existence to a nonlinear equation

$$F(x) = 0, \quad (1)$$

where X, Y are Banach spaces and $F \in \mathcal{C}^p(X, Y)$ ($p \geq 2$). Let x_* be a solution to this equation, i.e., $F(x_*) = 0$. The above problem is called regular at the point x_* if $\text{Im } F'(x_*) = Y$. Otherwise, problem (1) is called irregular, degenerate or singular at x_* .

The construction of p -regularity [1–4] gives new possibilities for solving or describing degenerate problems (see, for instance, [5–7]). We are going to use it to certain questions that appear in many numerical applications. Namely, we consider the equation of rod bending and the nonlinear Laplace equation that describe many mathematical physics problems like string oscillation, membrane oscillation and so on.

The following degenerate nonlinear boundary value problem:

$$F(x, \varepsilon) = x'' + (1 + \varepsilon)(x + x^2) = 0, \quad (2)$$

where $F : \mathcal{C}_0^2[0, \pi] \times \mathbb{R} \rightarrow \mathcal{C}[0, \pi]$ and $\mathcal{C}_0^2[0, \pi] = \{x \in \mathcal{C}^2[0, \pi] : x(0) = x(\pi) = 0\}$, is the first of them.

The second one is a modification of equation (2), i.e.,

$$F(x, \varepsilon) = x'' + x + \varepsilon x^{p-1} + x^p = 0, \quad (3)$$

where $F : C_0^2[0, \pi] \times \mathbb{R} \rightarrow C[0, \pi]$, $p \geq 2$.

The third one is the partial differential equation of the form

$$F(u, \varepsilon) = \Delta u - (\varepsilon - 10)g(u) = 0, \quad (4)$$

where $F : C_0^2[\Omega] \times \mathbb{R} \rightarrow C[\Omega]$, $\Omega = [0, \pi] \times [0, \pi]$, Δ is the Laplacian and $C_0^2[\Omega] = \{u \in C^2[\Omega] : u|_{\partial\Omega} = 0\}$; moreover, $g : \Omega \rightarrow \mathbb{R}$ is a certain function such that $g(0) = 0$, $g'(0) = 1$, $g''(0) \neq 0$.

Note that the numbers -1 and -10 are the eigenvalues of the operators $(\cdot)''$ and Δ in equations (2) and (4), respectively. Moreover, in the first example, we could take $g(x)$ instead of $x + x^2$ such that $g(0) = 0$, $g'(0) = 1$, $g''(0) \neq 0$. It is important that the 2-regularity condition is fulfilled for the mapping $F(x, \varepsilon)$ at the point $(0, 0)$. We chose equation (3) in order to expose our results not only for $p = 2$. In equation (4) we have taken $\lambda = -10$ as an arbitrary representative element of eigenvalues of the Laplacian of the form $-(k^2 + n^2)$, $k, n \in \mathbb{N}$. Of course it is possible to take for example $\lambda = -2$.

We begin with some notation. Suppose X and Y are Banach spaces and denote the space of all continuous linear operators from X to Y by $\mathcal{L}(X, Y)$. Let p be a natural number, and let $B : X \times X \times \cdots \times X$ (p -copies of X) $\rightarrow Y$ be a continuous symmetric p -multilinear mapping. The p form associated to B is the map $B[\cdot]^p : X \rightarrow Y$ defined by $B[x]^p = B(x, x, \dots, x)$ for $x \in X$. Alternatively, we may simply view $B[\cdot]^p$ as a homogeneous polynomial $Q : X \rightarrow Y$ of degree p , and hence $Q(\alpha x) = \alpha^p Q(x)$. Throughout this paper, we assume that the mapping $F : X \rightarrow Y$ is p -times continuously Fréchet differentiable on X and its p th order derivative at $x \in X$ will be denoted as $F^{(p)}(x)$ (a symmetric multilinear map of p copies of X to Y) and the associated p -form, also called a p th-order mapping, is

$$F^{(p)}(x)[h]^p = F^{(p)}(x)[h, h, \dots, h].$$

We also use the notation

$$\text{Ker}^p F^{(p)}(x) = \{h \in X : F^{(p)}(x)[h]^p = 0\}$$

and refer to it as the p -kernel of the p th-order mapping.

The set $M(x_*) = \{x \in X : F(x) = F(x_*) = 0\}$ is called *the solution set* for the mapping F . We call h a *tangent vector* to the set $M \subseteq X$ at $x_* \in M$ if there exist $\varepsilon > 0$ and a function $r : [0, \varepsilon] \rightarrow X$ with the property that for $t \in [0, \varepsilon]$ we have $x_* + th + r(t) \in M$ and $\|r(t)\| = o(t)$ as $t \rightarrow 0$ (for the sake of simplicity, the record $t \rightarrow 0$ will be omitted in the following text of paper). The set of all tangent vectors at x_* is called *the tangent cone* to M at x_* and is denoted by $T_1 M(x_*)$. A map $F : X \rightarrow Y$ is regular at $x_* \in X$ if $\text{Im } F'(x_*) = Y$. In the regular case, the tangent cone to the solution set coincides with the kernel of the first derivative of the map F . Recall the following theorem.

Theorem 1 (Classical Lyusternic theorem) *Let X and Y be Banach spaces and a map $F : X \rightarrow Y$ be regular at $x_* \in X$. Then*

$$T_1 M(x_*) = \text{Ker } F'(x_*).$$

The notion of regularity is generalized to the notion of the so-called p -regularity which will be described in the next section.

Elements of p -regularity theory

Assume that $x_* \in U \subseteq X$, U is a neighborhood of an element x_* . Let a mapping $F : U \rightarrow Y$ be p -times Fréchet differentiable in U and $\text{Im } F'(x_*) \neq Y$ (the regularity condition does not hold). In order to define the notion of p -regularity, let us define the so-called p -factor operator (see [1]). Assume that the space Y is decomposed into a direct sum

$$Y = Y_1 \oplus \cdots \oplus Y_p, \quad (5)$$

where $Y_1 = \text{cl}(\text{Im } F'(x_*))$ (the closure of the image of the first derivative of F evaluated at x_*) and the next spaces are defined as follows. Let Z_2 be a closed complementary subspace to Y_1 , that is, $Y = Y_1 \oplus Z_2$ (we assume that such a closed complement exists), and let $P_{Z_2} : Y \rightarrow Z_2$ be a projection operator onto Z_2 along Y_1 . Let $Y_2 = \text{cl}(\text{span Im } P_{Z_2} F''(x_*)[\cdot]^2) \subseteq Z_2$ (the closure of the linear span of the image of the quadratic map $P_{Z_2} F''(x_*)[\cdot]^2$). More generally, define

$$Y_i = \text{cl}(\text{span Im } P_{Z_i} F^{(i)}(x_*)[\cdot]^i) \subseteq Z_i, \quad i = 2, \dots, p-1,$$

where Z_i is a closed complementary subspace to $Y_1 \oplus \cdots \oplus Y_{i-1}$, $i = 2, \dots, p$, with respect to Y , and $P_{Z_i} : Y \rightarrow Z_i$ is a projection operator onto Z_i along $Y_1 \oplus \cdots \oplus Y_{i-1}$, $i = 2, \dots, p$. Finally, $Y_p = Z_p$. The order p is chosen as the minimal number (if it exists) for which the above decomposition (5) holds.

Now, let us define the following mappings:

$$f_i(x) : U \rightarrow Y_i, \quad f_i(x) = P_{Y_i} F(x), \quad i = 1, \dots, p,$$

where $P_{Y_i} : Y \rightarrow Y_i$ is a projection operator from Y along $Y_1 \oplus \cdots \oplus Y_{i-1} \oplus Y_{i+1} \oplus \cdots \oplus Y_p$. Below we recall some important definitions for further considerations.

We introduce the following operator:

$$\Psi_p(x_*, \cdot) = f_1'(x_*) + f_2''(x_*)[\cdot] + \cdots + f_p^{(p)}(x_*)[\cdot]^{p-1}.$$

This means that

$$\Psi_p(x_*, h) = f_1'(x_*) + f_2''(x_*)[h] + \cdots + f_p^{(p)}(x_*)[h]^{p-1}.$$

Definition 2 Let $h \in X$. The linear operator $\Psi_p(x_*, h) \in \mathcal{L}(X, Y)$ is called the p -factor operator.

Sometimes it is convenient to use the following representation of the p -factor operator:

$$\begin{aligned} \Psi_p(x_*, h) &= (f_1'(x_*), f_2''(x_*)[h], \dots, f_p^{(p)}(x_*)[h]^{p-1}) \\ &= (P_{Y_1} F'(x_*), P_{Y_2} F''(x_*)[h], \dots, P_{Y_p} F^{(p)}(x_*)[h]^{p-1}) \end{aligned}$$

for $h \in X$. In this representation, we assume that $Y = Y_1 \times \cdots \times Y_p$.

We say that F is *completely degenerate* at x_* up to order p if $F^{(i)}(x_*) = 0$, $i = 1, \dots, p-1$. In the completely degenerate case, the p -factor operator is equal to $F^{(p)}(x_*)[h]^{p-1}$.

Definition 3 The p -kernel of the operator $\Psi_p(x_*, \cdot)$ is a set

$$\begin{aligned} H_p(x_*) &= \text{Ker}^p \Psi_p(x_*, \cdot) = \{h \in X : \Psi_p(x_*, h)[h] = 0\} \\ &= \{h \in X : f'_1(x_*)[h] + f''_2(x_*)[h]^2 + \dots + f_p^{(p)}(x_*)[h]^p = 0\}. \end{aligned}$$

Note that

$$\text{Ker}^p \Psi_p(x_*, \cdot) = \left\{ \bigcap_{i=1}^p \text{Ker}^i f_i^{(i)}(x_*) \right\}.$$

In the completely degenerate case, $H_p(x_*)$ is equal to

$$\text{Ker}^p F^{(p)}(x_*) = \{h \in X : F^{(p)}(x_*)[h]^p = 0\}.$$

Definition 4 A mapping F is called p -regular at x_* along h , $p > 1$, if $\text{Im } \Psi_p(x_*, h) = Y$ (i.e., the operator $\Psi_p(x_*, h)$ is surjection).

Definition 5 A mapping F is called p -regular at x_* , $p > 1$, if it is p -regular along every h belonging to the set $H_p(x_*) \setminus \{0\}$.

Note that if $H_p(x_*) = \{0\}$, then F is automatically p -regular.

The following theorem gives a description of the tangent cone in the degenerate case.

Theorem 6 (Generalized Lyusternik theorem [1]) *Let X and Y be Banach spaces, $F \in C^p(X, Y)$ be p -regular at $x_* \in U \subset X$. Then*

$$T_1 M(x_*) = H_p(x_*).$$

Applications

The following lemma will be important in the study of surjectivity of p -factor operators in the mentioned examples.

Lemma 7 *Suppose that $Y = Y_1 \oplus Y_2$, where Y_1, Y_2 are closed subspaces in Y , $A, B \in \mathcal{L}(X, Y)$, $\text{Im } A = Y_1$. Let also P_2 be a projection onto Y_2 along Y_1 . Then $(A + P_2 B)X = Y \Leftrightarrow (P_2 B) \text{Ker } A = Y_2$.*

This lemma is a straightforward consequence of the following simple lemma.

Lemma 8 *Suppose that $Y = Y_1 \oplus Y_2$, where Y_1, Y_2 are closed subspaces in Y , $A_1, A_2 \in \mathcal{L}(X, Y)$, $A_1 X \subset Y_1$, $A_2 X \subset Y_2$. Then $(A_1 + A_2)X = Y$ iff $A_1 \text{Ker } A_2 = Y_1$ and $A_2 \text{Ker } A_1 = Y_2$.*

The proof is obvious. Lemma 7 follows from Lemma 8 if we put $A_1 = A$ and $A_2 = P_2 B$.

Example 9 Now we apply the above theory to differential equation (2)

$$F(x, \varepsilon) = x'' + (1 + \varepsilon)(x + x^2) = 0,$$

where $F : C_0^2[0, \pi] \times \mathbb{R} \rightarrow C[0, \pi]$ and $C_0^2[0, \pi] = \{x \in C^2[0, \pi] : x(0) = x(\pi) = 0\}$. Observe that $x_* = (0, 0)$ is a trivial solution of this equation.

The first derivative of the mapping F is

$$F'(x, \varepsilon) = [F'_x(x, \varepsilon), F'_\varepsilon(x, \varepsilon)] = \left[\frac{d^2}{dt^2} + (1 + \varepsilon)(1 + 2x), x + x^2 \right].$$

In our case, $F'_x(0, 0) = (\cdot)'' + (\cdot)$ and $F'_\varepsilon(0, 0) = 0$.

Note that $\text{Ker } F'_x(0, 0) = \{x \in C^2[0, \pi] : \frac{d^2 x}{dt^2} + x = 0\}$. The general solution of the equation $x'' + x = 0$ is $x(t) = c_1 \cos t + c_2 \sin t$. Taking into account the boundary condition, we obtain $c_1 = 0$, $x(t) = c_2 \sin t$ and $\text{Ker } F'_x(0, 0) = \text{span}\{\sin t\}$.

The image of $F'_x(0, 0)$ is defined as follows:

$$\begin{aligned} \text{Im } F'_x(0, 0) &= \{y \in C[0, \pi] : \exists x \in C^2[0, \pi] \text{ } F'_x(0, 0)x = y, x(0) = x(\pi) = 0\} \\ &= \{y \in C[0, \pi] : \exists x \in C^2[0, \pi] \text{ such that } x'' + x = y, x(0) = x(\pi) = 0\}. \end{aligned}$$

The general solution of the above equation has the form

$$x(t) = c_1 \cos t + c_2 \sin t - \cos t \int_0^t y(\tau) \sin \tau \, d\tau + \sin t \int_0^t y(\tau) \cos \tau \, d\tau.$$

In view of the boundary conditions $x(0) = x(\pi) = 0$, we obtain

$$\text{Im } F'_x(0, 0) = \left\{ y \in C[0, \pi] : \int_0^\pi y(\tau) \sin \tau \, d\tau = 0 \right\}.$$

One can easily show that the boundary value problem $x'' + x = \sin t$, $x(0) = x(\pi) = 0$ does not have a solution. Moreover, since $\int_0^\pi \sin^2(t) \, dt \neq 0$, it follows that $y(t) = \sin t \notin \text{Im } F'_x(0, 0)$. This implies that the operator $F'_x(0, 0)$ is not surjective, i.e., $\text{Im } F'_x(0, 0) \neq C[0, \pi] = Y$. Then $Y = Y_1 \oplus Z_2$, where $Y_1 = \text{Im } F'_x(0, 0)$ and $Z_2 = \text{Ker } F'_x(0, 0)$.

The projector $P_{Z_2} : Y \rightarrow Z_2$ along Y_1 can be described as

$$P_{Z_2} y = \frac{2}{\pi} \sin t \int_0^\pi y(\tau) \sin \tau \, d\tau, \quad y \in Y.$$

Since

$$\begin{aligned} F''(x, \varepsilon) &= \begin{bmatrix} 2(1 + \varepsilon) & 1 + 2x \\ 1 + 2x & 0 \end{bmatrix}, & F''(0, 0) &= \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}, \\ F''(0, 0)[h_z, h_\varepsilon] &= [2h_z + h_\varepsilon, h_z], & F''(0, 0)[h_z, h_\varepsilon]^2 &= 2h_z^2 + 2h_z h_\varepsilon, \end{aligned}$$

then

$$\begin{aligned} Y_2 &= \text{span}(\text{Im } P_{Z_2} F''(0, 0)[\cdot]^2) \\ &= \text{span} \left\{ y(t) \in Y : y(t) = \frac{2}{\pi} \sin t \int_0^\pi F''(0, 0)[\cdot]^2 \sin \tau \, d\tau \right\} = \text{span}\{\sin t\} = Z_2 \end{aligned}$$

and

$$Y = \mathcal{C}[0, \pi] = Y_1 \oplus Y_2 = \operatorname{Im} F'_x(0, 0) \oplus \operatorname{Ker} F'_x(0, 0).$$

Consequently, $[\operatorname{Im} F'_x(0, 0)]^\perp = \operatorname{Ker} F'_x(0, 0)$.

Using p -regularity theory and the generalized Lyusternik theorem, we obtain the following assertion. If the mapping F is p -regular ($p = 2$) at the point $x_* = (0, 0)$ with respect to the element $h = [h_z, h_\varepsilon]$, where $h_z = z \sin t$, $z \in \mathbb{R}$, $h_\varepsilon = \bar{\varepsilon}$, then there exist solutions $x = x(t, \varepsilon) = \varepsilon z \sin t + r(\varepsilon)$, $\|r(\varepsilon)\| = o(\varepsilon)$ of equation (2) for $\varepsilon \in (-\sigma, \sigma)$, where $\sigma > 0$ is sufficiently small. Below we will describe a 2-factor operator and show its surjectivity.

Let $P_1 = P_{Y_1}$ be a projection onto $Y_1 = \operatorname{Im} F'_x(0, 0)$ and $P_2 = P_{Y_2}$ be a projection onto $Y_2 = \operatorname{Ker} F'_x(0, 0)$ along Y_1 . The 2-factor operator has the form

$$\begin{aligned} \Psi_2((0, 0), [h_z, h_\varepsilon]) &= f'_1(0, 0) + f'_2(0, 0)[h_z, h_\varepsilon] \\ &= P_1 F'_x(0, 0) + P_2 F''(0, 0)[h_z, h_\varepsilon] \\ &= F'_x(0, 0) + P_2 F''(0, 0)[h_z, h_\varepsilon] \end{aligned}$$

because $P_1 F'_x = F'_x$.

Using the second derivative of F , we obtain, for $[h_u, h_\lambda] \in \mathcal{C}^2[0, \pi] \times \mathbb{R}$,

$$\begin{aligned} \Psi_2((0, 0), [h_z, h_\varepsilon])[h_u, h_\lambda] &= \frac{d^2 h_u}{dt^2} + h_u + P_2[2h_z h_u + h_\varepsilon h_u + h_z h_\lambda] \\ &= \frac{d^2 h_u}{dt^2} + h_u + \frac{2}{\pi} \langle 2h_z h_u + h_\varepsilon h_u + h_z h_\lambda, \sin t \rangle \sin t \\ &= \frac{d^2 h_u}{dt^2} + h_u + \frac{2}{\pi} \sin t \int_0^\pi (2h_z h_u + h_\varepsilon h_u + h_z h_\lambda) \sin t \, dt. \end{aligned}$$

Now we describe the 2-kernel of the operator $\Psi_2((0, 0), \cdot)$

$$H_2(0, 0) = \{h = [h_z, h_\varepsilon] \in \mathcal{C}^2[0, \pi] \times \mathbb{R} : \Psi_2((0, 0), h)[h] = 0\}.$$

Assume that $h = [h_z, h_\varepsilon] \in \operatorname{Ker} F'_x(0, 0)$, i.e., $F'_x(0, 0)[h_z, h_\varepsilon] = 0$. Then $h_z = z \sin t$, $z \in \mathbb{R}$, $h_\varepsilon \in \mathbb{R}$ and

$$\begin{aligned} \Psi_2((0, 0), [h_z, h_\varepsilon])[h_z, h_\varepsilon] &= P_2 F''(0, 0)[h_z, h_\varepsilon]^2 = \frac{2}{\pi} \sin t \int_0^\pi (2h_z^2 + 2h_z h_\varepsilon) \sin t \, dt = 0 \\ \Leftrightarrow \int_0^\pi (2h_z^2 + 2h_z h_\varepsilon) \sin t \, dt &= 0 \\ \Leftrightarrow z^2 \int_0^\pi \sin^3 t \, dt + zh_\varepsilon \int_0^\pi \sin^2 t \, dt &= 0 \\ \Leftrightarrow \frac{4}{3} z^2 + \frac{\pi}{2} zh_\varepsilon = 0 \quad \Leftrightarrow \quad z = 0 \vee z = -\frac{3}{8} \pi h_\varepsilon \\ \Leftrightarrow h_z = 0 \vee h_z = -\frac{3}{8} \pi h_\varepsilon \sin t. \end{aligned}$$

This implies that $H_2(0, 0) = \{[0, \bar{\varepsilon}]\} \cup \{[-\frac{3}{8} \pi \bar{\varepsilon} \sin t, \bar{\varepsilon}]\}$, where $h_\varepsilon = \bar{\varepsilon}$.

Next we verify that the mapping $\Psi_2((0, 0), [h_z, h_\varepsilon])$ is surjective onto $\mathcal{C}[0, \pi]$ for elements belonging to the set $H_2(0, 0)$, that is,

$$\forall y \in \mathcal{C}[0, \pi] \exists [h_u, h_\lambda] \in \mathcal{C}^2[0, \pi] \times \mathbb{R} \text{ such that } \Psi_2((0, 0), [h_z, h_\varepsilon])[h_u, h_\lambda] = y.$$

Consider the first case $[h_z, h_\varepsilon] = [0, \bar{\varepsilon}]$, $\bar{\varepsilon} \neq 0$.

Assume that

$$\frac{d^2 h_u(t)}{dt^2} + h_u(t) + \frac{2}{\pi} \sin t \int_0^\pi \bar{\varepsilon} h_u(t) \sin t \, dt = y = y_1 + y_2, \quad (6)$$

where

$$y_1 = \frac{d^2 h_u(t)}{dt^2} + h_u(t) = P_1 y \in Y_1 = \text{Im } F'_x(0, 0),$$

$$y_2 = \frac{2}{\pi} \sin t \int_0^\pi \bar{\varepsilon} h_u(t) \sin t \, dt = P_2 y \in Y_2 = \text{Ker } F'_x(0, 0).$$

Putting $y_2 = a \sin t$ and using Lemma 7, it suffices to take $h_u = b \sin t \in \text{Ker } F'_x(0, 0)$. Then from (6) we obtain $y_1 = 0$ and $\frac{2}{\pi} \bar{\varepsilon} \sin t \int_0^\pi b \sin^2 t \, dt = a \sin t$ from where $b = \frac{a}{\bar{\varepsilon}}$, $h_u = \frac{a}{\bar{\varepsilon}} \sin t$. The solutions of equation (6) exist, and hence $\Psi_2((0, 0), [h_z, h_\varepsilon])$ is surjective.

Verify the second case $[h_z, h_\varepsilon] = [-\frac{3}{8}\pi \bar{\varepsilon} \sin t, \bar{\varepsilon}]$, $\bar{\varepsilon} \neq 0$. Consider the equation

$$\begin{aligned} \frac{d^2 h_u(t)}{dt^2} + h_u(t) + \frac{2}{\pi} \sin t \int_0^\pi \left[2\bar{\varepsilon} \left(-\frac{3}{8}\pi \sin t \right) h_u(t) + \bar{\varepsilon} h_u(t) \right. \\ \left. + \bar{\varepsilon} \left(-\frac{3}{8}\pi \sin t \right) h_\lambda \right] \sin t \, dt = y = y_1 + y_2. \end{aligned} \quad (7)$$

Putting $y_2 = a \sin t$, using Lemma 7 and taking $h_u = b \sin t \in \text{Ker } F'_x(0, 0)$, we have $y_1 = 0$ and

$$-\frac{3\pi}{8} h_\lambda \bar{\varepsilon} \sin t + \frac{2}{\pi} \bar{\varepsilon} \sin t \int_0^\pi \left(-\frac{3}{4}\pi \sin^2 t + \sin t \right) b \sin t \, dt = a \sin t.$$

Hence $b = -\frac{3}{8}\pi h_\lambda - \frac{a}{\bar{\varepsilon}}$, and $h_u = (-\frac{3}{8}\pi h_\lambda - \frac{a}{\bar{\varepsilon}}) \sin t$. The solutions of equation (7) exist, hence $\Psi_2((0, 0), [h_z, h_\varepsilon])$ is surjective in this case too. Therefore the mapping F is 2-regular at the point $x_* = (0, 0)$ with respect to the element $h = [h_z, h_\varepsilon]$.

Using the generalized Lyusternik theorem, we can describe the solutions belonging to the tangent cone $T_1 M(0, 0)$. For $[h_z, h_\varepsilon] = [0, \bar{\varepsilon}]$ we have $x_1(t, \varepsilon) = r_1(\varepsilon)$ and for $[h_z, h_\varepsilon] = [-\frac{3}{8}\pi \bar{\varepsilon} \sin t, \bar{\varepsilon}]$ we have $x_2(t, \varepsilon) = -\frac{3}{8}\pi \varepsilon \sin t + r_2(\varepsilon)$, where $\|r_i(\varepsilon)\| = o(\varepsilon)$ for $i = 1, 2$ and $\varepsilon \in (-\sigma, \sigma)$, $\sigma > 0$ is sufficiently small. Thus we obtain the following theorem.

Theorem 10 Equation (2) has two solutions $x_i(t, \varepsilon)$, $i = 1, 2$, such that

$$x_1(t, \varepsilon) = r_1(\varepsilon),$$

$$x_2(t, \varepsilon) = -\frac{3}{8}\pi \varepsilon \sin t + r_2(\varepsilon),$$

where $\|r_i(\varepsilon)\| = o(\varepsilon)$ for $i = 1, 2$ and $\varepsilon \in (-\sigma, \sigma)$, $\sigma > 0$ is sufficiently small.

Example 11 Equation (3), i.e.,

$$F(x, \varepsilon) = x'' + x + \varepsilon x^{p-1} + x^p = 0,$$

where $F : C_0^2[0, \pi] \times \mathbb{R} \rightarrow C[0, \pi]$, $p \geq 2$, can be investigated analogously. Similarly, we obtain the following result.

Theorem 12 Equation (3) has a nonzero solution $x(t, \varepsilon)$ such that

$$x(t, \varepsilon) = -\frac{\varepsilon}{p} c_p \sin t + r(\varepsilon),$$

where

$$c_p = \frac{\int_0^\pi \sin^p t \, dt}{\int_0^\pi \sin^{p+1} t \, dt}, \quad \|r(\varepsilon)\| = o(\varepsilon)$$

and $\varepsilon \in (-\sigma, \sigma)$, $\sigma > 0$ is sufficiently small.

Example 13 Consider now equation (4)

$$F(u, \varepsilon) = \Delta u - (\varepsilon - 10)g(u) = 0,$$

where $F : C_0^2[\Omega] \times \mathbb{R} \rightarrow C[\Omega]$, $\Omega = [0, \pi] \times [0, \pi]$, Δ is the Laplacian, with the assumption $g(0) = 0$, $g'(0) = 1$, $g''(0) \neq 0$. The point $x_* = (0, 0)$ is the trivial solution of the above equation.

We put $u = u(y) = u(y_1, y_2)$.

The first derivative of the mapping F is the following:

$$F'(u, \varepsilon) = [F'_u(u, \varepsilon), F'_\varepsilon(u, \varepsilon)] = [\Delta - (\varepsilon - 10)g'(u), -g(u)].$$

In our case $F'_u(0, 0) = \Delta + 10$.

Note that $\text{Ker } F'_u(0, 0) = \{u \in C_0^2[\Omega] : \Delta u + 10u = 0\}$. This space is spanned by the L^2 orthonormal functions (see [8], p.424)

$$u_1 = \frac{2}{\pi} \sin 3y_1 \sin y_2, \quad u_2 = \frac{2}{\pi} \sin y_1 \sin 3y_2$$

(that is, $\langle u_i, u_j \rangle = \int_\Omega u_i u_j \, dy_1 \, dy_2$ and $\int_\Omega u_i^2 \, dy_1 \, dy_2 = 1$, $\int_\Omega u_i u_j \, dy_1 \, dy_2 = 0$ for $i \neq j$). Moreover,

$$C[\Omega] = \text{Im } F'_u(0, 0) \oplus \text{Ker } F'_u(0, 0)$$

and

$$[\text{Im } F'_u(0, 0)]^\perp = \text{Ker } F'_u(0, 0).$$

From p -regularity theory and the generalized Lyusternik theorem, we obtain the following assertion. If the mapping F is p -regular at the point $x_* = (0, 0)$ with respect to the element $h = [h_z, h_\varepsilon]$, where $z = (z_1, z_2)$, $h_z = z_1 u_1 + z_2 u_2$, $h_\varepsilon = \bar{\varepsilon}$, then there exist solutions $u = u(y, \varepsilon) =$

$\varepsilon(z_1 u_1 + z_2 u_2) + r(\varepsilon)$, $\|r(\varepsilon)\| = o(\varepsilon)$ of equation (4) for $\varepsilon \in (-\sigma, \sigma)$, where $\sigma > 0$ is sufficiently small.

Let $P_1 = P_{Y_1}$ be the projection of Y onto $Y_1 = \text{Im } F'_u(0, 0)$, and let $P_2 = P_{Y_2}$ be the projection of Y onto $Y_2 = \text{Ker } F'_u(0, 0)$ along Y_1 . We define the 2-factor operator:

$$\begin{aligned}\Psi_2((0, 0), h) &= \Psi_2((0, 0), [h_z, h_\varepsilon]) = f'_1(0, 0) + f'_2(0, 0)[h_z, h_\varepsilon] \\ &= P_1 F'_u(0, 0) + P_2 F''(0, 0)[h_z, h_\varepsilon] \\ &= F'_u(0, 0) + P_2 F''(0, 0)[h_z, h_\varepsilon] \quad (\text{because } P_1 F'_u = F'_u).\end{aligned}$$

Using the second derivative of F , that is,

$$\begin{aligned}F''(u, \varepsilon) &= \begin{bmatrix} -(\varepsilon - 10)g''(u) & -g'(u) \\ -g'(u) & 0 \end{bmatrix}, \quad F''(0, 0) = \begin{bmatrix} 10g''(0) & -1 \\ -1 & 0 \end{bmatrix}, \\ F''(0, 0)[h_z, h_\varepsilon] &= \begin{bmatrix} 10g''(0) & -1 \\ -1 & 0 \end{bmatrix} [h_z, h_\varepsilon] = [10g''(0)h_z - h_\varepsilon, -h_z],\end{aligned}$$

we obtain, for $[h_u, h_\lambda] \in C_0^2[\Omega] \times \mathbb{R}$,

$$F''(0, 0)[h_z, h_\varepsilon][h_u, h_\lambda] = [10g''(0)h_z - h_\varepsilon, -h_z][h_u, h_\lambda] = 10g''(0)h_z h_u - h_\varepsilon h_u - h_z h_\lambda,$$

and hence

$$F''(0, 0)[h_z, h_\varepsilon]^2 = 10g''(0)h_z^2 - 2h_z h_\varepsilon.$$

From the above calculations it follows that the value of the 2-factor operator $\Psi_2((0, 0), [h_z, h_\varepsilon])$ on an element $x = [h_u, h_\lambda] \in C_0^2[\Omega] \times \mathbb{R}$ is equal to

$$\begin{aligned}\Psi_2((0, 0), h)[x] &= \Psi_2((0, 0), [h_z, h_\varepsilon])[h_u, h_\lambda] \\ &= \Delta h_u + 10h_u + P_2[10g''(0)h_z h_u - h_\varepsilon h_u - h_z h_\lambda] \\ &= \Delta h_u + 10h_u + \sum_{i=1}^2 \langle 10g''(0)h_z h_u - h_\varepsilon h_u - h_z h_\lambda, u_i \rangle u_i.\end{aligned}$$

Now we describe the 2-kernel of the operator $\Psi_2((0, 0), \cdot)$, which is the set

$$H_2(0, 0) = \{h = [h_z, h_\varepsilon] \in C_0^2[\Omega] \times \mathbb{R} : \Psi_2((0, 0), h)[h] = 0\}.$$

Assuming that $[h_z, h_\varepsilon] = [z_1 u_1 + z_2 u_2, \bar{\varepsilon}] \in \text{Ker } F'_u(0, 0)$, we have

$$F'_u(0, 0)[h_z, h_\varepsilon] = 0$$

and

$$\begin{aligned}\Psi_2((0, 0), [h_z, h_\varepsilon])[h_z, h_\varepsilon] \\ = P_2 F''(0, 0)[h_z, h_\varepsilon]^2 = \sum_{i=1}^2 \langle F''(0, 0)[h_z, h_\varepsilon]^2, u_i \rangle u_i\end{aligned}$$

$$\begin{aligned}
 &= \sum_{i=1}^2 u_i \int_{\Omega} (10g''(0)h_z^2 - 2h_z h_{\varepsilon}) u_i dy_1 dy_2 \\
 &= \sum_{i=1}^2 u_i \int_{\Omega} [10g''(0)(z_1 u_1 + z_2 u_2)^2 - 2(z_1 u_1 + z_2 u_2)\bar{\varepsilon}] u_i dy_1 dy_2 \\
 &= \sum_{i=1}^2 u_i \left[10g''(0) \int_{\Omega} u_i (z_1 u_1 + z_2 u_2)^2 dy_1 dy_2 - 2\bar{\varepsilon} \int_{\Omega} u_i (z_1 u_1 + z_2 u_2) dy_1 dy_2 \right] \\
 &= \sum_{i=1}^2 u_i \left[10g''(0) \int_{\Omega} u_i (z_1 u_1 + z_2 u_2)^2 dy_1 dy_2 - 2\bar{\varepsilon} z_i \right].
 \end{aligned}$$

Next, introducing the quadratic forms

$$Q_i(h_z, \bar{\varepsilon}) = Q_i(z_1, z_2, \bar{\varepsilon}) = 10g''(0) \int_{\Omega} u_i (z_1 u_1 + z_2 u_2)^2 dy_1 dy_2 - 2\bar{\varepsilon} z_i, \quad i = 1, 2,$$

we obtain the following system of equations for elements belonging to the set $H_2(0, 0)$:

$$\begin{cases} Q_1(h_z, \bar{\varepsilon}) = Q_1(z_1, z_2, \bar{\varepsilon}) = 10g''(0) \int_{\Omega} u_1 (z_1 u_1 + z_2 u_2)^2 dy_1 dy_2 - 2\bar{\varepsilon} z_1 = 0, \\ Q_2(h_z, \bar{\varepsilon}) = Q_2(z_1, z_2, \bar{\varepsilon}) = 10g''(0) \int_{\Omega} u_2 (z_1 u_1 + z_2 u_2)^2 dy_1 dy_2 - 2\bar{\varepsilon} z_2 = 0 \end{cases} \quad (8)$$

or, equivalently,

$$\begin{cases} 10g''(0) \int_{\Omega} (u_1^3 z_1^2 + 2z_1 z_2 u_1^2 u_2 + z_2^2 u_1 u_2^2) dy_1 dy_2 - 2\bar{\varepsilon} z_1 = 0, \\ 10g''(0) \int_{\Omega} (u_1 u_2^2 z_1^2 + 2z_1 z_2 u_1 u_2^2 + z_2^2 u_2^3) dy_1 dy_2 - 2\bar{\varepsilon} z_2 = 0. \end{cases}$$

Simple calculations give us

$$\begin{aligned}
 a &= \int_{\Omega} u_1^3 dy_1 dy_2 = \int_{\Omega} u_2^3 dy_1 dy_2 = \frac{16}{27} \cdot \frac{8}{\pi^3}, \\
 b &= \int_{\Omega} u_1^2 u_2 dy_1 dy_2 = \int_{\Omega} u_1 u_2^2 dy_1 dy_2 = -\frac{48}{175} \cdot \frac{8}{\pi^3},
 \end{aligned}$$

and we obtain the following form of system (8):

$$\begin{cases} 10g''(0)(a z_1^2 + 2b z_1 z_2 + b z_2^2) - 2\bar{\varepsilon} z_1 = 0, \\ 10g''(0)(b z_1^2 + 2b z_1 z_2 + a z_2^2) - 2\bar{\varepsilon} z_2 = 0, \end{cases} \quad (9)$$

or putting $C_i^{jk} = 10g''(0) \int_{\Omega} u_i u_j u_k dy_1 dy_2$, we get

$$\sum_{j,k=1}^2 C_i^{jk} z_j z_k - 2\bar{\varepsilon} z_i = 0, \quad i = 1, 2.$$

Note that if $M(z_1, z_2)$ is the 2×2 symmetric matrix whose (i, j) entry is

$$a_{ij} = \sum_{m=1}^2 \left(\int_{\Omega} u_i u_j u_m dy_1 dy_2 \right) z_m,$$

that is,

$$M(z_1, z_2) = \begin{bmatrix} az_1 + bz_2 & bz_1 + bz_2 \\ bz_1 + bz_2 & bz_1 + az_2 \end{bmatrix},$$

then system (9) is equivalent to the following equation:

$$[10g''(0)M(z) - 2\bar{\varepsilon}I] \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = 0, \quad \text{where } I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus we get the following condition: elements $[\bar{z}_1 u_1 + \bar{z}_2 u_2, \bar{\varepsilon}] \in H_2(0, 0)$ (that is, elements $(\bar{z}_1, \bar{z}_2, \bar{\varepsilon})$ are the solutions of system (9)) if and only if either (\bar{z}_1, \bar{z}_2) is an eigenvector of $10g''(0)M(z)$ corresponding to the eigenvalue $2\bar{\varepsilon}$ or $(\bar{z}_1, \bar{z}_2) = (0, 0)$. We will prove that for $[0u_1 + 0u_2, \bar{\varepsilon}] \in H_2(0, 0)$, the mapping F is always 2-regular and for any other $[\bar{z}_1 u_1 + \bar{z}_2 u_2, \bar{\varepsilon}] \in H_2(0, 0)$, F is 2-regular if $\bar{\varepsilon}$ is not an eigenvalue of $10g''(0)M(z)$.

Note that $Q(z_1, z_2, \bar{\varepsilon}) = (Q_1(z_1, z_2, \bar{\varepsilon}), Q_2(z_1, z_2, \bar{\varepsilon}))$ is a quadratic mapping associated to the 2-factor operator $\Psi_2((0, 0), h)$. If $[\bar{z}_1 u_1 + \bar{z}_2 u_2, \bar{\varepsilon}] \in H_2(0, 0)$, then the first differentiation of Q gives us the matrix form of the operator $\Psi_2((0, 0), h)$. Consider the calculations:

$$Q'(z_1, z_2, \bar{\varepsilon}) = \begin{bmatrix} 10g''(0)(2z_1 a + 2z_2 b) - 2\bar{\varepsilon} & 10g''(0)(2z_1 b + 2z_2 b) & -2z_1 \\ 10g''(0)(2z_1 b + 2z_2 b) & 10g''(0)(2z_1 b + 2z_2 a) - 2\bar{\varepsilon} & -2z_2 \end{bmatrix},$$

$$Q''(z_1, z_2, \bar{\varepsilon}) = \begin{bmatrix} \begin{bmatrix} 10g''(0)2a & 10g''(0)2b & -2 \\ 10g''(0)2b & 10g''(0)2b & 0 \\ -2 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 10g''(0)2b & 10g''(0)2b & 0 \\ 10g''(0)2b & 10g''(0)2a & -2 \\ 0 & -2 & 0 \end{bmatrix} \end{bmatrix},$$

$$\text{and } \Psi_2((0, 0), h) = Q''(z_1, z_2, \bar{\varepsilon}) \begin{bmatrix} z_1 \\ z_2 \\ \bar{\varepsilon} \end{bmatrix}.$$

It follows that

$$\Psi_2((0, 0), h) = \begin{bmatrix} 10g''(0)(2z_1 a + 2z_2 b) - 2\bar{\varepsilon} & 10g''(0)(2z_1 b + 2z_2 b) & -2z_1 \\ 10g''(0)(2z_1 b + 2z_2 b) & 10g''(0)(2z_1 b + 2z_2 a) - 2\bar{\varepsilon} & -2z_2 \end{bmatrix}, \quad (10)$$

that is, $\Psi_2((0, 0), h) = Q'$.

For an arbitrary element $h = (z_1, z_2, \bar{\varepsilon}) \in H_2(0, 0)$, we examine the surjectivity of the mapping $\Psi_2((0, 0), h)$, that is, the following condition:

$$\forall \eta \in C[\Omega] \exists \xi \in C_0^2[\Omega] \times \mathbb{R} \text{ such that } \Psi_2((0, 0), h)[\xi] = \eta.$$

For $\xi = (\xi_1, \xi_2, \lambda)$, $\eta = (\eta_1, \eta_2)$, we have

$$\Psi_2((0, 0), h) \begin{bmatrix} \xi_1 \\ \xi_2 \\ \lambda \end{bmatrix} = \begin{bmatrix} [10g''(0)(2\bar{z}_1 a + 2\bar{z}_2 b) - 2\bar{\varepsilon}]\xi_1 + [10g''(0)(2\bar{z}_1 b + 2\bar{z}_2 b)]\xi_2 - 2\bar{z}_1 \lambda \\ 10g''(0)(2\bar{z}_1 b + 2\bar{z}_2 b)\xi_1 + [10g''(0)(2\bar{z}_1 b + 2\bar{z}_2 a) - 2\bar{\varepsilon}]\xi_2 - 2\bar{z}_2 \lambda \end{bmatrix} = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix},$$

or

$$\begin{bmatrix} 10g''(0) \begin{bmatrix} 2a\bar{z}_1 + 2b\bar{z}_2 & 2b\bar{z}_1 + 2b\bar{z}_2 \\ 2b\bar{z}_1 + 2b\bar{z}_2 & 2b\bar{z}_1 + 2a\bar{z}_2 \end{bmatrix} - 2\bar{\varepsilon} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} - 2\lambda \begin{bmatrix} \bar{z}_1 \\ \bar{z}_2 \end{bmatrix} = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}$$

and

$$2[10g''(0)M(z) - \bar{\varepsilon}I] \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = 2\lambda \begin{bmatrix} \bar{z}_1 \\ \bar{z}_2 \end{bmatrix} + \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}. \quad (11)$$

Observe that for $[0u_1 + 0u_2, \bar{\varepsilon}]$, $\bar{\varepsilon} \neq 0$, F is always 2-regular (since the matrix (10) has the maximal rank). Now put $\bar{\varepsilon} = 0$ in (9). Then we get $\bar{z}_1 = \bar{z}_2 = 0$. This means that in this case $H_2(0, 0) = \{0\}$ and then F is 2-regular (in this trivial case, the equation $F(u, \varepsilon) = 0$ has the unique solution $u = 0$). Consider the other $[\bar{z}_1u_1 + \bar{z}_2u_2, \bar{\varepsilon}] \in H_2(0, 0)$. Then equation (11) has the solution if $\det[10g''(0)M(z) - \bar{\varepsilon}I] \neq 0$. This means that $\bar{\varepsilon}$ is not an eigenvalue of $10g''(0)M(z)$.

From the above consideration, we obtain the following sufficient condition for 2-regularity: the matrix $10g''(0)M(z)$ has no eigenvalues $\bar{\varepsilon}$ and $2\bar{\varepsilon}$. This condition can be tested using the notion of resultant. We evidently have $10g''(0) \neq 0$. Let λ and 2λ be the eigenvalues, and let $\lambda \neq 0$. Then we obtain

$$\begin{aligned} \det[M(z_1, z_2) - \lambda I] &= \det \left[\begin{bmatrix} az_1 + bz_2 & bz_1 + bz_2 \\ bz_1 + bz_2 & bz_1 + az_2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right] \\ &= \lambda^2 + A_1\lambda + A_2 = 0, \end{aligned} \quad (12)$$

where $A_1 = -(a + b)(z_1 + z_2)$ and $A_2 = (ab - b^2)z_1^2 + (a^2 - b^2)z_1z_2 + (ab - b^2)z_2^2$.

Likewise,

$$\det[M(z_1, z_2) - 2\lambda I] = 4\lambda^2 + 2A_1\lambda + A_2 = 0. \quad (13)$$

Subtracting equation (12) from (13) gives $3\lambda + A_1 = 0$ (because $\lambda \neq 0$). The condition that an eigenvalue multiplied by two is not an eigenvalue means that the polynomials $f = \lambda^2 + A_1\lambda + A_2$ and $g = 3\lambda + A_1$ do not have a common root. This is equivalent to the nonvanishing of the resultant:

$$R(f, g) = \begin{vmatrix} 1 & A_1 & A_2 \\ 3 & A_1 & 0 \\ 0 & 3 & A_1 \end{vmatrix} = Az_1^2 + 2Bz_1z_2 + Az_2^2.$$

This resultant is a quadratic form with $A = 9(ab - b^2) - 2(a + b)^2$, $B = \frac{1}{2}[9(a^2 - b^2) - 4(a + b)^2]$. Thus a sufficient condition of 2-regularity is that this form is positively or negatively defined. Therefore one obtains $A^2 - B^2 > 0$. In our case, one finds $A^2 = [2a^2 - 5ab + 11b^2]^2 \approx 5.49 \cdot (\frac{8}{\pi^3})^4$, $B^2 = \frac{1}{4}[13b^2 - 5a^2 + 8ba]^2 \approx 0.07 \cdot (\frac{8}{\pi^3})^4$ and $A^2 > B^2$, so the test with resultants is effective.

Summing up we conclude from the generalized Lyusternic theorem: the tangent cone coincides with the 2-kernel $H_2(0, 0)$ and this gives the solutions.

Theorem 14 Equation (4) has a nonzero solution $u(\varepsilon)$ such that

$$u(\varepsilon) = \varepsilon(\bar{z}_1 u_1 + \bar{z}_2 u_2) + r(\varepsilon), \quad \|r(\varepsilon)\| = o(\varepsilon)$$

for $\varepsilon \in (-\sigma, \sigma)$, where $\sigma > 0$ is sufficiently small.

Conclusion

The paper was inspired by Buchner, Marsden and Schecter's article [8]. The authors consider the bifurcation problem $F(x, \lambda) = Lx + (\lambda - \lambda_0)x + R(x) = 0$, where L is the elliptic self-adjoint operator on a suitable Banach space Y of functions, with another suitable Banach space of function X - the domain of $L \subset Y$, $R: X \rightarrow Y$ is a smooth map with $R(0) = 0$ and $R'(0) = 0$, λ_0 - is an eigenvalue of L of multiplicity n , $x \in X$ and $\lambda \in \mathbb{R}$. They use Lyapunov-Schmidt procedure to examine the above equation and show that $(0, 0)$ is a bifurcation point and that there exist solutions different from $(0, \lambda)$. Our examples are special cases of the above problem, and we use p -regularity theory to prove the existence of solutions and give an approximative description of the solution set. The structure of the solution set is reduced to a study of the system of homogeneous algebraic equations.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors read and approved the final manuscript.

Author details

¹Siedlce University of Natural Sciences and Humanities, 3-go Maja 54, Siedlce, 08-110, Poland. ²System Research Institute, Polish Academy of Sciences, Newelska 6, Warsaw, 01-447, Poland. ³Dorodnicyn Computer Center, Russian Academy of Sciences, Vavilova 40, Moscow, 119991, Russia.

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