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Homogenization of an elastic double-porosity medium with imperfect interface via the periodic unfolding method

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Abstract

We study an ε-periodic model of a medium with double porosity which consists of two components, one of them being connected. We assume that the elasticity of the medium in the inclusion is of order ε^2 and also, on the interface between the two components, we consider a jump of the displacement vector condition, proportional to the stress tensor which is continuous. The aim of the paper is to prove the convergence of the homogenization process using the periodic unfolding method.

Keywords: homogenization; elasticity; double porosity; interface jumps

1 Introduction

This paper deals with the homogenization of a double porosity model in elasticity describing a medium occupying an open set Ω in $\mathbb{R}^N$ which consists of two components, one of them being connected and the second one disconnected.

More exactly, we suppose that Ω is the union of two open subsets Ω_1^ε and Ω_2^ε and their common boundary Γ^ε, and we consider the problem

$$\begin{cases} -\frac{\partial \sigma_{ij}^{\alpha\varepsilon}}{\partial x_j} = g_i & \text{in } \Omega_\alpha^\varepsilon, \alpha \in \{1, 2\}, \\ \sigma_{ij}^{1\varepsilon} n_j = \sigma_{ij}^{2\varepsilon} n_j & \text{on } \Gamma_\varepsilon, \\ \sigma_{ij}^{1\varepsilon} n_j = \varepsilon h_\varepsilon(u_i^{2\varepsilon} - u_i^{1\varepsilon}) & \text{on } \Gamma_\varepsilon, \\ u^{1\varepsilon} = 0 & \text{on } \partial\Omega, \end{cases}$$

where n is the outward unit normal to Ω_1^ε. The set Ω_2^ε is a disconnected union of ε-periodic open sets. We suppose that the elasticity tensor, which defines the strains σ, is of order 1 in Ω_1^ε and of order ε^2 in the inclusions Ω_2^ε.

The jump conditions on the surface model in fact a layer of a soft material that surrounds the particles; the layer is here modeled as a surface, thus not only the tangential, but also the normal component of the displacement can have a jump.

Using the periodic unfolding method, we prove some convergence results and describe the homogenized problems.

More precisely, first, in Theorem 11 we describe the homogenized problem in the variables x and y, as is usually done. Then, in Theorem 12 we identify the homogenized prob-

lem in Ω, and we show that

$$\tilde{u}^{1\varepsilon} \rightharpoonup |Y_1| \cdot u^1 \quad \text{weakly in } L^2(\Omega)^N,$$

$$\tilde{u}^{2\varepsilon} \rightharpoonup |Y_2| \cdot u^1 + g_l \cdot q^l \quad \text{weakly in } L^2(\Omega)^N,$$

where, for $\alpha \in \{1, 2\}$, $\tilde{u}^{\alpha\varepsilon}$ is the extension by zero of $u^{\alpha\varepsilon}$ to the whole Ω and u^1 is the unique solution of the problem

$$\begin{cases} -\frac{\partial}{\partial x_j}(a^*_{ijkh} \frac{\partial u^1_k}{\partial x_h}) = g_i & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The homogenized tensor A^* is the same as that obtained for the usual problem stated in the perforated domain Ω^ε_1 with a homogeneous Neumann boundary condition.

The presence of the tensor of order ε^2 in the second component does not contribute to the homogenized tensor A^*. However, it gives rise to the additional term $g_l \cdot q^l$ in the limit of $\tilde{u}^{2\varepsilon}$, where $q^l_i = \int_{Y_2} w^l_{2i} \, dy$ for $i, l = 1, \ldots, N$ and $w^l_2 \in H^1(Y_2)^N$ is the unique solution of the cell problem

$$\begin{cases} -\frac{\partial}{\partial y_j}(a_{ijkh} \frac{\partial w^l_{2k}}{\partial y_h}) = \delta_{il} & \text{in } Y_2, \\ a_{ijkh} \frac{\partial w^l_{2k}}{\partial y_h} n_j = h w^l_{2i} & \text{on } \Gamma. \end{cases}$$

The periodic unfolding method was introduced by Cioranescu, Damlamian and Griso in [1] (see [2] for a general presentation) and later it was extended to periodically perforated domains by Cioranescu, Damlamian, Donato, Griso and Zaki in [3] and [4]. In [5] Donato *et al.* use the periodic unfolding method for a two-component domain similar to the one considered in this paper. Hence, they introduce two unfolding operators. The first one is denoted by $\mathcal{T}^\varepsilon_1$ and deals with functions defined on Ω^ε_1 which is the same operator introduced in [1]. The second one concerns the functions defined on Ω^ε_2 and is denoted by $\mathcal{T}^\varepsilon_2$. Similar properties for $\mathcal{T}^\varepsilon_2$ and also the relations between the two operators and, in particular, the properties of their traces on the common boundary are investigated.

Here, we adapt the ideas therein to our problem and, because the elasticity tensor in the inclusion is of order ε^2, we define a suitable functional space endowed with an adequate norm (see Remark 2). Also, when we apply the unfolding method, we need to construct proper test functions, which is a key step in obtaining the homogenized problem.

The pioneering paper for the heat diffusion in a two-component domain with similar jump conditions on the interface is due to Auriault and Ene [6] (see also [7], where the results were proved using asymptotic expansions). Later, in [8], Ene and Poliševski gave a rigorous proof of the convergence for one of the cases studied in [6] using the two-scale convergence method introduced in [9] and developed in [10]. Several cases of the same problem were treated by Monsurrò in [11] and Donato and Monsurrò in [12]. Successively, Donato *et al.* in [5] treated the problem by the periodic unfolding method. In [13] Poliševski added ε^2 to the diffusion matrix in the inclusions.

In the present paper, we extend the results of [13] to the case of the linearized elasticity. The main difficulty when applying the periodic unfolding method consists in finding suitable test functions adapted to our elasticity tensor and to the interface term appearing

in the variational formulation. Let us mention that a similar model is treated via asymptotic expansions by Smyshlyaev in [14], where, in the inclusions, the tensor is of the form $\varepsilon^2 A(x/\varepsilon) + B(x/\varepsilon)$ with B being only non-negative. Other related elasticity problems were studied by Damlamian, Cioranescu and Orlik in [15] and [16].

For classical results about the homogenization of the linearized elasticity system in composites media, see, for instance, [17–20] and references therein. For the case of periodically perforated domains, we refer to Léné [21] (see also [22]).

2 The domain

Let Ω be an open bounded subset of $\mathbb{R}^N$ ($N \geq 2$) with a Lipschitz continuous boundary $\partial\Omega$ and $Y = (0,1)^N$ the unit cube in $\mathbb{R}^N$. We suppose that Y_2 is a subset of Y such that $\bar{Y}_2 \subset Y$ and its boundary Γ is also Lipschitz continuous. Moreover, we define $Y_1 = Y \setminus \bar{Y}_2$. One can see that repeating Y by periodicity, the union of all $\bar{Y}_1$ is a connected domain in $\mathbb{R}^N$ which will be denoted by $\mathbb{R}_1^N$. Furthermore, $\mathbb{R}_2^N = \mathbb{R}^N \setminus \mathbb{R}_1^N$.

In the following, the parameter $\varepsilon \in (0,1)$ takes its values in a sequence of real numbers which, in the homogenizing process, will tend to zero. For each $k \in \mathbb{Z}^N$, we define $Y^k = k + Y$ and $Y_\alpha^k = k + Y_\alpha$, where $\alpha \in \{1,2\}$. We also define, for each ε,

$$\mathbb{Z}_\varepsilon = \left\{ k \in \mathbb{Z}^N : \varepsilon \bar{Y}_2^k \subset \Omega \right\}, \tag{2.1}$$

and we set (see Figure 1)

$$\Omega_2^\varepsilon = \bigcup_{k \in \mathbb{Z}_\varepsilon} \left(\varepsilon Y_2^k \right) \quad \text{and} \quad \Omega_1^\varepsilon = \Omega \setminus \bar{\Omega}_2^\varepsilon. \tag{2.2}$$
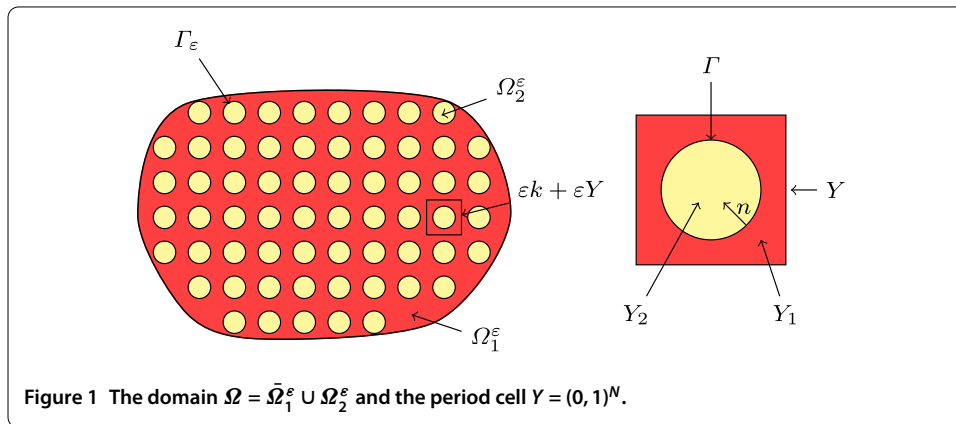
The boundary of Ω_2^ε will be denoted by Γ_ε and n will be the normal on Γ_ε exterior to Ω_1^ε.

We introduce now, for each ε, the jump factor $h_\varepsilon(x) = h(x/\varepsilon)$ and the fourth-order elasticity tensor A^ε defined by

$$A^\varepsilon(x) = \begin{cases} A(x/\varepsilon) & \text{in } Y_1, \\ \varepsilon^2 A(x/\varepsilon) & \text{in } Y_2, \end{cases} \tag{2.3}$$

where $h \in L^\infty(\Gamma)$ and the components $a_{ijkh} \in L^\infty(Y)$ are smooth, real, Y-periodic functions, with the property that there exists $\lambda > 0$ such that

$$h(y) \geq \lambda \quad \text{and} \quad a_{ijkh}(y)\xi_{ij}\xi_{kh} \geq \lambda \xi_{ij}\xi_{ij} \tag{2.4}$$

Figure 1 The domain $\Omega = \bar{\Omega}_1^\varepsilon \cup \Omega_2^\varepsilon$ and the period cell $Y = (0,1)^N$.

for any $y \in Y$ and any symmetric tensor ξ_{ij}. We also assume the symmetry condition for the elasticity tensor, namely

$$a_{ijkh} = a_{jikh} = a_{ijhk} = a_{khij}.$$

Finally, for $\alpha \in \{1, 2\}$ and a function $u^{\alpha\varepsilon}$ defined on $\Omega_\alpha^\varepsilon$, we denote the stress tensors by $\sigma_{ij}^{\alpha\varepsilon} = a_{ijkh}^\varepsilon e_{kh}(u^{\alpha\varepsilon})$, where $e_{kh}(u^{\alpha\varepsilon})$ represent the components of the deformation tensor defined for any function v by

$$e_{kh}(v) = \frac{1}{2}\left(\frac{\partial v_k}{\partial x_h} + \frac{\partial v_h}{\partial x_k}\right). \tag{2.5}$$

One can see that due to the symmetry of the deformation tensor, we have $\sigma_{ij}^{\alpha\varepsilon} = a_{ijkh}^\varepsilon \frac{\partial u_k^{\alpha\varepsilon}}{\partial x_h}$.

For a general exposition of the elasticity theory, we refer, for instance, to Ciarlet [23], Duvaut and Lions [24] (see also [25]).

3 The problem

Our goal is to describe the asymptotic behavior, as $\varepsilon \to 0$, of the problem

$$-\frac{\partial \sigma_{ij}^{\alpha\varepsilon}}{\partial x_j} = g_i \quad \text{in } \Omega_\alpha^\varepsilon, \alpha \in \{1, 2\}, \tag{3.1}$$

where g_i are the components of a vector field $g \in L^2(\Omega)^N$ which represent the given body forces together with the boundary conditions

$$\sigma_{ij}^{1\varepsilon} n_j = \sigma_{ij}^{2\varepsilon} n_j \quad \text{on } \Gamma_\varepsilon, \tag{3.2}$$

$$\sigma_{ij}^{1\varepsilon} n_j = \varepsilon h_\varepsilon \left(u_i^{2\varepsilon} - u_i^{1\varepsilon}\right) \quad \text{on } \Gamma_\varepsilon, \tag{3.3}$$

$$u^{1\varepsilon} = 0 \quad \text{on } \partial\Omega. \tag{3.4}$$

Remark 1 Everywhere in this paper we use the Einstein summation convention, except for the cases where we mention that some product $u_i v_i$ does not follow this convention. Also, if there is no mention of the contrary, the writing $\|v_i\|^2$ will follow the same convention since $\|v_i\|^2 = \|v_i\| \times \|v_i\|$.

We introduce the space $V_\varepsilon = \{v \in H^1(\Omega_1^\varepsilon), v = 0 \text{ on } \partial\Omega\}$ equipped with the L^2-norm of the gradient and the Hilbert space

$$W^\varepsilon = V_\varepsilon^N \times H^1\left(\Omega_2^\varepsilon\right)^N \tag{3.5}$$

endowed with the scalar product

$$(u, v)_{W^\varepsilon} = \int_{\Omega_1^\varepsilon} \nabla u_i^1 \nabla v_i^1 \, dx + \varepsilon^2 \int_{\Omega_2^\varepsilon} \nabla u_i^2 \nabla v_i^2 \, dx + \varepsilon \int_{\Gamma_\varepsilon} \left(u_i^2 - u_i^1\right)\left(v_i^2 - v_i^1\right) d\sigma_x, \tag{3.6}$$

where the elements of any $u \in W^\varepsilon$ are denoted by $u = (u^1, u^2)$.

The norm generated by the scalar product (3.6) is given by

$$\|v\|^2_{W^\varepsilon} = \left\|\nabla v^1_i\right\|^2_{L^2(\Omega^\varepsilon_2)} + \varepsilon^2 \left\|\nabla v^2_i\right\|^2_{L^2(\Omega^\varepsilon_2)} + \varepsilon \left\|v^2_i - v^1_i\right\|^2_{L^2(\Gamma_\varepsilon)}. \tag{3.7}$$

Remark 2 Due to the fact that the elasticity tensor is of order ε^2 in the inclusion Ω^ε_2, the functional space W^ε and the norm defined by (3.7) are different from those used in [5]. More precisely, in [5] the space used was $H^\varepsilon_\gamma = V_\varepsilon \times H^1(\Omega^\varepsilon_2)$ and it was equipped with the norm

$$\|v\|^2_{H^\varepsilon_\gamma} = \left\|\nabla v^1\right\|^2_{L^2(\Omega^\varepsilon_2)} + \left\|\nabla v^2\right\|^2_{L^2(\Omega^\varepsilon_2)} + \varepsilon^\gamma \left\|v^2 - v^1\right\|^2_{L^2(\Gamma_\varepsilon)}, \tag{3.8}$$

where γ was the order of ε in the boundary condition (3.3), the case studied in the present paper being $\gamma = 1$.

For any $u, v \in W^\varepsilon$, we also use the notation

$$a(u, v) = \sum_{\alpha = 1,2} \int_{\Omega^\varepsilon_\alpha} a^\varepsilon_{ijkh} \frac{\partial u^\alpha_k}{\partial x_h} \frac{\partial v^\alpha_i}{\partial x_j} \, dx + \varepsilon \int_{\Gamma_\varepsilon} h_\varepsilon \left(u^2_i - u^1_i\right)\left(v^2_i - v^1_i\right) d\sigma_x, \tag{3.9}$$

and we introduce the following variational formulation of problem (3.1):

Find $u^\varepsilon \in W^\varepsilon$ such that

$$a(u_\varepsilon, v) = \int_{\Omega^\varepsilon_1} g_i v^1_i \, dx + \int_{\Omega^\varepsilon_2} g_i v^2_i \, dx, \quad \forall v \in W^\varepsilon. \tag{3.10}$$

Theorem 3 *For any $\varepsilon \in (0, 1)$, problem (3.10) has a unique solution $u^\varepsilon \in W^\varepsilon$. Moreover, there exists a constant $C > 0$ independent of ε such that, for $\alpha \in \{1, 2\}$ and each $i = 1, \ldots, N$, we have*

$$\left\|u^{\varepsilon\alpha}_i\right\|_{L^2(\Omega^\varepsilon_\alpha)} \leq C, \qquad \left\|\nabla u^{1\varepsilon}_i\right\|_{L^2(\Omega^\varepsilon_1)} \leq C \quad \text{and} \quad \varepsilon \left\|\nabla u^{2\varepsilon}_i\right\|_{L^2(\Omega^\varepsilon_2)} \leq C, \tag{3.11}$$

$$\left\|u^{2\varepsilon}_i - u^{1\varepsilon}_i\right\|_{L^2(\Gamma_\varepsilon)} \leq C\varepsilon^{-1/2}. \tag{3.12}$$

Proof The result is proved by applying the Lax-Milgram theorem. The coerciveness of the form $a(\cdot, \cdot)$ can be easily shown using definition (3.9) and the properties (2.4) which give

$$a(v, v) \geq \lambda \left(\int_{\Omega^\varepsilon_1} \nabla v^1_i \nabla v^1_i \, dx + \varepsilon^2 \int_{\Omega^\varepsilon_2} \nabla v^2_i \nabla v^2_i \, dx + \varepsilon \int_{\Gamma_\varepsilon} \left(v^2_i - v^1_i\right)^2 d\sigma_x \right)$$

$$= \lambda \|v\|^2_{W^\varepsilon}.$$

Let us prove now the continuity of the right-hand side of (3.10). Applying the Cauchy-Schwarz inequality, we obtain

$$\int_{\Omega^\varepsilon_1} g_i v^1_i \, dx + \int_{\Omega^\varepsilon_2} g_i v^2_i \, dx \leq \|g_i\|_{L^2(\Omega^\varepsilon_1)} \left\|v^1_i\right\|_{L^2(\Omega^\varepsilon_1)} + \|g_i\|_{L^2(\Omega^\varepsilon_2)} \left\|v^2_i\right\|_{L^2(\Omega^\varepsilon_2)}$$

$$\leq C\left(\left\|v^1_i\right\|_{L^2(\Omega^\varepsilon_1)} + \left\|v^2_i\right\|_{L^2(\Omega^\varepsilon_2)}\right).$$

It remains to show that $\|v_i^1\|_{L^2(\Omega_1^\varepsilon)} + \|v_i^2\|_{L^2(\Omega_2^\varepsilon)} \leq C\|v\|_{W^\varepsilon}$. It is known (see [8, 11]) that there exists some constant $C > 0$ independent of ε such that for each $i = 1, \ldots, N$,

$$\|v_i^2\|_{L^2(\Omega_2^\varepsilon)} \leq C\big(\varepsilon\|\nabla v_i^2\|_{L^2(\Omega_2^\varepsilon)} + \varepsilon^{1/2}\|v_i^2\|_{L^2(\Gamma_\varepsilon)}\big), \tag{3.13}$$

$$\varepsilon^{1/2}\|v_i^1\|_{L^2(\Gamma_\varepsilon)} \leq C\big(\varepsilon\|\nabla v_i^1\|_{L^2(\Omega_1^\varepsilon)} + \|v_i^1\|_{L^2(\Omega_1^\varepsilon)}\big), \tag{3.14}$$

$$\|v_i^1\|_{L^2(\Omega_1^\varepsilon)} \leq C\|\nabla v_i^1\|_{L^2(\Omega_1^\varepsilon)}. \tag{3.15}$$

From (3.13) we have

$$\|v_i^2\|_{L^2(\Omega_2^\varepsilon)} \leq C\big(\varepsilon\|\nabla v_i^2\|_{L^2(\Omega_2^\varepsilon)} + \varepsilon^{1/2}\|v_i^2 - v_i^1\|_{L^2(\Gamma_\varepsilon)} + \varepsilon^{1/2}\|v_i^1\|_{L^2(\Gamma_\varepsilon)}\big). \tag{3.16}$$

Using now (3.14) and (3.15), we obtain

$$\|v_i^1\|_{L^2(\Omega_1^\varepsilon)} + \|v_i^2\|_{L^2(\Omega_2^\varepsilon)} \leq C\|v\|_{W^\varepsilon}. \tag{3.17}$$

In order to give now the *a priori* estimates for the solution of (3.10), we choose $v = u^\varepsilon$ in (3.10). Using the coerciveness of $a(\cdot, \cdot)$, the Cauchy-Schwarz inequality and (3.17), we find $C > 0$ independent of ε such that

$$\lambda\|u^\varepsilon\|_{W^\varepsilon}^2 \leq a\big(u^\varepsilon, u^\varepsilon\big) = \int_{\Omega_1^\varepsilon} g_i v_i^1 \, dx + \int_{\Omega_2^\varepsilon} g_i v_i^2 \, dx$$

$$\leq \|g_i\|_{L^2(\Omega)}\big(\|v_i^1\|_{L^2(\Omega_1^\varepsilon)} + \|v_i^2\|_{L^2(\Omega_2^\varepsilon)}\big) \leq C\|u^\varepsilon\|_{W^\varepsilon}.$$

Therefore

$$\big\{u^\varepsilon\big\}_\varepsilon \text{ is bounded in } W^\varepsilon. \tag{3.18}$$

Moreover, the definition of the norm in W^ε implies that for $\alpha \in \{1, 2\}$, estimates (3.11) and (3.12) hold. □

4 Periodic unfolding operators in two-component domain

In this section we present the definitions of the unfolding operators for a two-component domain introduced by Donato *et al.* in [5] and their main properties. We will prove only the results that bring some other properties than those already proved in the paper quoted above. The important characteristic of these operators is that they map functions defined on the oscillating domains Ω_1^ε and Ω_2^ε into functions defined on the fixed domains $\Omega \times Y_1$ and $\Omega \times Y_2$ respectively.

In the following, for $x \in \mathbb{R}^N$, we denote by $[x]_Y$ its integer part $k \in \mathbb{Z}^N$ such that $x - [x]_Y \in Y$ and set

$$\{x\}_Y = x - [x]_Y \quad \text{for a.e. } x \in \mathbb{R}^N.$$

Then, for almost every $x \in \mathbb{R}^N$,

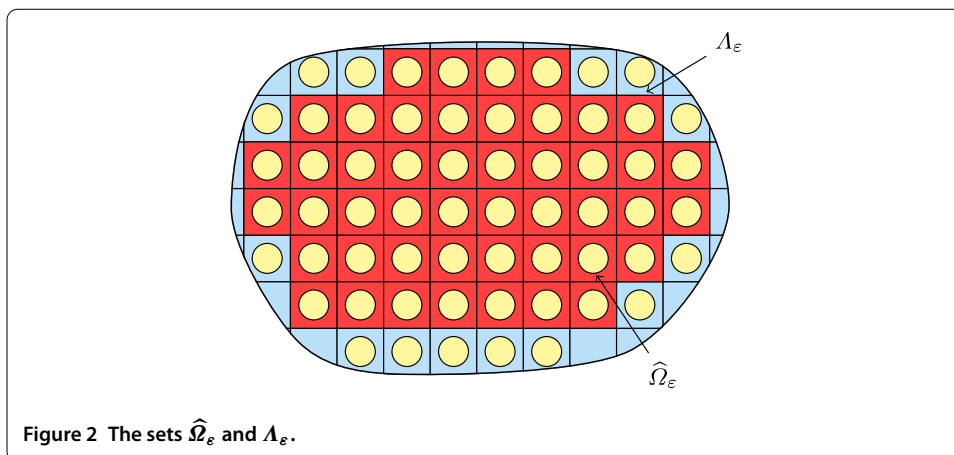$$x = \varepsilon\left(\left[\frac{x}{\varepsilon}\right]_Y + \left\{\frac{x}{\varepsilon}\right\}_Y\right).$$

Figure 2 The sets $\widehat{\Omega}_\varepsilon$ and Λ_ε.

We introduce now the sets (see Figure 2)

$$\widehat{\mathbb{Z}}_\varepsilon = \big\{ k \in \mathbb{Z}^N : \varepsilon Y^k \subset \Omega \big\}, \qquad \widehat{\Omega}_\varepsilon = \operatorname{int} \bigcup_{k \in \widehat{\mathbb{Z}}_\varepsilon} \big(\varepsilon \bar{Y}^k \big), \qquad \Lambda_\varepsilon = \Omega \setminus \widehat{\Omega}_\varepsilon,$$

$$\widehat{\Omega}_\alpha^\varepsilon = \bigcup_{k \in \widehat{\mathbb{Z}}_\varepsilon} \big(\varepsilon Y_\alpha^k \big), \qquad \Lambda_\alpha^\varepsilon = \Omega_\alpha^\varepsilon \setminus \widehat{\Omega}_\alpha^\varepsilon, \qquad \widehat{\Gamma}_\varepsilon = \partial \widehat{\Omega}_2^\varepsilon.$$

For a given set $D \subset \mathbb{R}^N$ and $v \in L^1(D)$, we also denote $\langle v \rangle_D = \frac{1}{|D|} \int_D v(y) \, \mathrm{d}y$. If there is no place for confusion, we will simply use $\langle v \rangle$ instead of $\langle v \rangle_Y$. If v is a function defined on $\Omega_\alpha^\varepsilon$, $\alpha \in \{1, 2\}$, then $\tilde{v}$ will denote the extension by zero to the whole Ω. Furthermore, χ_α will be the characteristic function of Ω_α for each $\alpha \in \{1, 2\}$, and we introduce the spaces

$$H_{\mathrm{per}}^1(Y_\alpha) = \big\{ v \in H_{\mathrm{loc}}^1\big(\mathbb{R}_\alpha^N\big) : v \text{ is } Y\text{-periodic} \big\},$$

$$\tilde{H}_{\mathrm{per}}^1(Y_\alpha) = \big\{ v \in H_{\mathrm{per}}^1(Y_\alpha) : \langle v \rangle_Y = 0 \big\}.$$

Definition 4 For any Lebesgue measurable function φ on $\Omega_\alpha^\varepsilon$, $\alpha \in \{1, 2\}$, we define the periodic unfolding operator by the formula

$$\mathcal{T}_\alpha^\varepsilon(\varphi)(x, y) = \begin{cases} \varphi\big(\varepsilon\big[\frac{x}{\varepsilon}\big]_Y + \varepsilon y\big) & \text{for a.e. } (x, y) \in \widehat{\Omega}_\varepsilon \times Y_\alpha, \\ 0 & \text{for a.e. } (x, y) \in \Lambda_\varepsilon \times Y_\alpha. \end{cases}$$

Remark 5 If φ is a function defined in Ω, then, for the sake of simplicity, we write $\mathcal{T}_\alpha^\varepsilon(\varphi)$ instead of $\mathcal{T}_\alpha^\varepsilon(\varphi|_{\Omega_\alpha^\varepsilon})$.

Proposition 6 *For $p \in [1, \infty)$ and $\alpha \in \{1, 2\}$, the operators $\mathcal{T}_\alpha^\varepsilon$ are continuous from $L^p(\Omega_\alpha^\varepsilon)$ to $L^p(\Omega \times Y_\alpha)$. Moreover,*

(i) *if φ and ψ are two Lebesgue measurable functions on $\Omega_\alpha^\varepsilon$, one has*
$$\mathcal{T}_\alpha^\varepsilon(\varphi\psi) = \mathcal{T}_\alpha^\varepsilon(\varphi) \cdot \mathcal{T}_\alpha^\varepsilon(\psi),$$

(ii) *for every $\varphi \in L^1(\Omega_\alpha^\varepsilon)$,*

$$\int_{\Omega \times Y_\alpha} \mathcal{T}_\alpha^\varepsilon(\varphi)(x, y) \, \mathrm{d}x \, \mathrm{d}y = \int_{\widehat{\Omega}_\alpha^\varepsilon} \varphi(x) \, \mathrm{d}x = \int_{\Omega_\alpha^\varepsilon} \varphi(x) \, \mathrm{d}x - \int_{\Lambda_\varepsilon} \varphi(x) \, \mathrm{d}x,$$

(iii) $\|\mathcal{T}_\alpha^\varepsilon(\varphi)\|_{L^p(\Omega \times Y_\alpha)} \le \|\varphi\|_{L^p(\Omega_\alpha^\varepsilon)}$ *for every* $\varphi \in L^p(\Omega_\alpha^\varepsilon)$,

(iv) $\mathcal{T}_\alpha^\varepsilon(\varphi) \longrightarrow \varphi$ *strongly in* $L^p(\Omega \times Y_\alpha)$ *for* $\varphi \in L^p(\Omega)$,

(v) *if* $\{\varphi_\varepsilon\}_\varepsilon \subset L^p(\Omega)$ *is a sequence such that* $\varphi_\varepsilon \longrightarrow \varphi$ *strongly in* $L^p(\Omega)$, *then*
$\mathcal{T}_\alpha^\varepsilon(\varphi_\varepsilon) \longrightarrow \varphi$ *strongly in* $L^p(\Omega \times Y_\alpha)$,

(vi) *if* $\varphi \in L^p(Y_\alpha)$ *is* Y-*periodic and* $\varphi_\varepsilon(x) = \varphi(x/\varepsilon)$, *then* $\mathcal{T}_\alpha^\varepsilon(\varphi_\varepsilon) \longrightarrow \varphi$ *strongly in*
$L^p(\Omega \times Y_\alpha)$,

(vii) *if* $\varphi_\varepsilon \in L^p(\Omega_\alpha^\varepsilon)$ *satisfies* $\|\varphi_\varepsilon\|_{L^p(\Omega_\alpha^\varepsilon)} \le C$ *and* $\mathcal{T}_\alpha^\varepsilon(\varphi_\varepsilon) \rightharpoonup \hat{\varphi}$ *weakly in* $L^p(\Omega \times Y_\alpha)$, *then*
$\tilde{\varphi}_\varepsilon \rightharpoonup |Y_\alpha| \langle \hat{\varphi} \rangle_\alpha$ *weakly in* $L^p(\Omega)$,

(viii) *if* $\varphi \in W^{1,p}(\Omega_\alpha^\varepsilon)$, *then* $\nabla_y(\mathcal{T}_\alpha^\varepsilon(\varphi)) = \varepsilon \mathcal{T}_\alpha^\varepsilon(\nabla \varphi)$ *and* $\mathcal{T}_\alpha^\varepsilon(\varphi)$ *belongs to* $L^2(\Omega; W^{1,p}(Y_\alpha))$.

Lemma 7 *If* $u^\varepsilon = (u^{1\varepsilon}, u^{2\varepsilon})$ *is a sequence in* W^ε, *then, for each* $i = 1, \dots, N$,

$$\frac{1}{\varepsilon} \int_{\Omega \times \Gamma} \left| \mathcal{T}_2^\varepsilon(u_i^{2\varepsilon}) - \mathcal{T}_1^\varepsilon(u_i^{1\varepsilon}) \right|^2 dx \, d\sigma_y \le \int_{\Gamma_\varepsilon} \left| u_i^{2\varepsilon} - u_i^{1\varepsilon} \right|^2 d\sigma_x. \tag{4.1}$$

Remark 8 From (3.11), (3.12) and Proposition 6(iii), it is easy to see that for each $i = 1, \dots, N$,

$$\begin{aligned}
& \left\| \mathcal{T}_1^\varepsilon(\nabla u_i^{1\varepsilon}) \right\|_{L^2(\Omega \times Y_1)} \le C, \\
& \varepsilon \left\| \mathcal{T}_2^\varepsilon(\nabla u_i^{2\varepsilon}) \right\|_{L^2(\Omega \times Y_2)} \le C, \\
& \left\| \mathcal{T}_2^\varepsilon(u_i^{2\varepsilon}) - \mathcal{T}_1^\varepsilon(u_i^{1\varepsilon}) \right\|_{L^2(\Omega \times \Gamma)} \le C.
\end{aligned} \tag{4.2}$$

Lemma 9 *If* $\varphi \in \mathcal{D}(\Omega)^N$ *and* $u^\varepsilon = (u^{1\varepsilon}, u^{2\varepsilon}) \in W^\varepsilon$, *then, for* ε *small enough, for each* $i = 1, \dots, N$, *we have*

$$\varepsilon \int_{\Gamma_\varepsilon} h_\varepsilon(u_i^{2\varepsilon} - u_i^{1\varepsilon}) \varphi_i \, d\sigma_x = \int_{\Omega \times \Gamma} h(y) \left(\mathcal{T}_2^\varepsilon(u_i^{2\varepsilon}) - \mathcal{T}_1^\varepsilon(u_i^{1\varepsilon}) \right) \mathcal{T}_1^\varepsilon(\varphi_i) \, dx \, d\sigma_y.$$

Theorem 10 *Let* $u^\varepsilon = (u^{1\varepsilon}, u^{2\varepsilon})$ *be a bounded sequence in* W^ε. *Then there exist a subsequence (still denoted by* ε), $u^1 \in H_0^1(\Omega)^N$, $\hat{u}^1 \in L^2(\Omega; H_{per}^1(Y_1))^N$ *and* $\hat{u}^2 \in L^2(\Omega; H^1(Y_1))^N$ *such that for each* $i = 1, \dots, N$,

$$\begin{aligned}
& \mathcal{T}_1^\varepsilon(u_i^{1\varepsilon}) \longrightarrow u_i^1 \quad \textit{strongly in } L^2(\Omega; H^1(Y_1)), \\
& \mathcal{T}_1^\varepsilon(\nabla u_i^{1\varepsilon}) \rightharpoonup \nabla u_i^1 + \nabla_y \hat{u}_i^1 \quad \textit{weakly in } L^2(\Omega \times Y_1), \\
& \mathcal{T}_2^\varepsilon(u_i^{2\varepsilon}) \rightharpoonup \hat{u}_i^2 \quad \textit{weakly in } L^2(\Omega; H^1(Y_2)), \\
& \varepsilon \mathcal{T}_2^\varepsilon(\nabla u_i^{2\varepsilon}) \rightharpoonup \nabla_y \hat{u}_i^2 \quad \textit{weakly in } L^2(\Omega \times Y_2),
\end{aligned} \tag{4.3}$$

where $\langle \hat{u}_i^1 \rangle_\Gamma = 0$ *for almost every* $x \in \Omega$. *Furthermore,*

$$Z_1^\varepsilon = \frac{1}{\varepsilon} \left[\mathcal{T}_1^\varepsilon(u_i^{1\varepsilon}) - \langle \mathcal{T}_1^\varepsilon(u_i^{1\varepsilon}) \rangle_\Gamma \right] \rightharpoonup y_\Gamma \nabla u_i^1 + \hat{u}_i^1, \tag{4.4}$$

weakly in $L^2(\Omega; H^1(Y_1))$, *where* $y_\Gamma = y - \langle y \rangle_\Gamma$.

Proof Convergences (4.3)$_{1,2}$ and (4.4) have already been proved in [4] and [5] respectively. From (3.17) we get that $\|u_i^{2\varepsilon}\|_{L^2(\Omega_2^\varepsilon)} \le C$ and again Proposition 6(iii) implies that

$$\left\| \mathcal{T}_2^\varepsilon(u_i^{2\varepsilon}) \right\|_{L^2(\Omega \times Y_2)} \le C.$$

From Proposition 6(viii) and $(4.2)_2$,

$$\left\| \nabla_y \big(\mathcal{T}_2^\varepsilon \big(u_i^{2\varepsilon} \big) \big) \right\|_{L^2(\Omega \times Y_2)} = \left\| \varepsilon \mathcal{T}_2^\varepsilon \big(\nabla u_i^{2\varepsilon} \big) \right\|_{L^2(\Omega \times Y_2)} \leq C.$$

Hence $\{\mathcal{T}_2^\varepsilon(u_i^{2\varepsilon})\}$ is bounded in $L^2(\Omega; H^1(Y_2))$, *i.e.*,

$$\left\| \mathcal{T}_2^\varepsilon \big(u_i^{2\varepsilon} \big) \right\|_{L^2(\Omega; H^1(Y_2))} \leq C,$$

which means that there exists $\hat{u}^2 \in L^2(\Omega; H^1(Y_2))^N$ such that $(4.3)_3$ holds. Furthermore, it is easy to check that

$$\nabla_y \big(\mathcal{T}_2^\varepsilon \big(u_i^{2\varepsilon} \big) \big) \rightharpoonup \nabla_y \hat{u}_i^2 \quad \text{weakly in } L^2\big(\Omega; L^2(Y_2) \big),$$

and the fact that $\nabla_y(\mathcal{T}_2^\varepsilon(u_i^{2\varepsilon})) = \varepsilon \mathcal{T}_2^\varepsilon(\nabla u_i^{2\varepsilon})$ ends the proof of $(4.3)_4$. $\qquad\square$

5 Homogenization results

In the following we will use the notation

$$V = H_0^1(\Omega)^N \times L^2\big(\Omega; H_{\mathrm{per}}^1(Y_1) \big)^N \times L^2\big(\Omega; H^1(Y_2) \big)^N.$$

Theorem 11 *If $u^\varepsilon = (u^{1\varepsilon}, u^{2\varepsilon})$ is the solution of problem* (3.1), *then*

$$
\begin{aligned}
&\tilde{u}^{1\varepsilon} \rightharpoonup |Y_1| \cdot u^1 \quad \text{weakly in } L^2(\Omega)^N, \\
&\tilde{u}^{2\varepsilon} \rightharpoonup |Y_2| \cdot \big\langle \hat{u}^2 \big\rangle_{Y_2} \quad \text{weakly in } L^2(\Omega)^N, \\
&\mathcal{T}_1^\varepsilon \big(u^{1\varepsilon} \big) \longrightarrow u^1 \quad \text{strongly in } L^2\big(\Omega; H^1(Y_1) \big)^N, \\
&\mathcal{T}_2^\varepsilon \big(u^{2\varepsilon} \big) \rightharpoonup \hat{u}^2 \quad \text{weakly in } L^2\big(\Omega; H^1(Y_2) \big)^N, \\
&\mathcal{T}_1^\varepsilon \big(e_{kh}\big(u^{1\varepsilon} \big) \big) \rightharpoonup e_{kh}\big(u^1 \big) + e_{kh}^y\big(\hat{u}^1 \big) \quad \text{weakly in } L^2(\Omega \times Y_1), \\
&\varepsilon \mathcal{T}_2^\varepsilon \big(e_{kh}\big(u^{2\varepsilon} \big) \big) \rightharpoonup e_{kh}^y\big(\hat{u}^2 \big) \quad \text{weakly in } L^2(\Omega \times Y_2),
\end{aligned}
\tag{5.1}
$$

where the triplet $(u^1, \hat{u}^1, \hat{u}^2) \in V$ with $\langle \hat{u}_i^1 \rangle_\Gamma = 0$ for a.e. $x \in \Omega$ is the unique solution of the problem

$$
\begin{aligned}
\int_{\Omega \times Y_1} & a_{ijkh} \left(\frac{\partial u_k^1}{\partial x_h} + \frac{\partial \hat{u}_k^1}{\partial y_h} \right) \left(\frac{\partial \varphi_i}{\partial x_j} + \frac{\partial \Phi_i^1}{\partial y_j} \right) \mathrm{d}x\, \mathrm{d}y \\
& + \int_{\Omega \times Y_2} a_{ijkh} \frac{\partial \hat{u}_k^2}{\partial y_h} \frac{\partial \Phi_i^2}{\partial y_j} \mathrm{d}x\, \mathrm{d}y + \int_{\Omega \times \Gamma} h\big(\hat{u}_i^2 - u_i^1 \big)\big(\Phi_i^2 - \varphi_i \big)\, \mathrm{d}x\, \mathrm{d}\sigma_y \\
& = \int_{\Omega \times Y_1} g_i \varphi_i\, \mathrm{d}x\, \mathrm{d}y + \int_{\Omega \times Y_2} g_i \Phi_i^2\, \mathrm{d}x\, \mathrm{d}y, \quad \forall(\varphi, \Phi^1, \Phi^2) \in V.
\end{aligned}
\tag{5.2}
$$

Proof Convergences $(5.1)_{3-6}$ are a consequence of Theorem 10. From $(5.1)_{3,4}$ and Proposition 6(vii), we derive that

$$
\begin{aligned}
&\tilde{u}^{1\varepsilon} \rightharpoonup |Y_1| \cdot \big\langle u^1 \big\rangle_{Y_1} \quad \text{weakly in } L^2(\Omega)^N, \\
&\tilde{u}^{2\varepsilon} \rightharpoonup |Y_2| \cdot \big\langle \hat{u}^2 \big\rangle_{Y_2} \quad \text{weakly in } L^2(\Omega)^N.
\end{aligned}
$$

Since u^1 is constant with respect to y, we deduce that

$$\tilde{u}^{1\varepsilon} \rightharpoonup |Y_1| \cdot u^1 \quad \text{weakly in } L^2(\Omega)^N. \tag{5.3}$$

To obtain the limit problem, we take as test functions in (3.10)

$$v_i^1(x) = \varphi_i(x) + \varepsilon \omega_i^1(x) \psi_i^{1\varepsilon}(x), \tag{5.4}$$

$$v_i^2(x) = \omega_i^2(x) \psi_i^{2\varepsilon}(x), \tag{5.5}$$

with no summation of repeated index where $\varphi, \omega^\alpha \in \mathcal{D}(\Omega)^N$, $\psi^\alpha \in H^1_{\text{per}}(Y_\alpha)^N$ and $\psi^{\alpha\varepsilon}(x) = \psi^\alpha(x/\varepsilon)$ ($\alpha \in \{1, 2\}$). Everywhere in the sequel, the product $\omega_i^\alpha \psi_i^{\alpha\varepsilon}$ will not follow the Einstein summation convention and will represent the ith component of a vector field.

We obtain

$$\int_{\Omega_1^\varepsilon} a_{ijkh}^\varepsilon \frac{\partial u_k^{1\varepsilon}}{\partial x_h} \left(\frac{\partial \varphi_i}{\partial x_j} + \frac{\partial(\varepsilon \omega_i^1 \psi_i^{1\varepsilon})}{\partial x_j} \right) dx + \varepsilon \int_{\Omega_2^\varepsilon} a_{ijkh}^\varepsilon \frac{\partial u_k^{2\varepsilon}}{\partial x_h} \frac{\partial(\varepsilon \omega_i^2 \psi_i^{2\varepsilon})}{\partial x_j} dx$$

$$+ \varepsilon \int_{\Gamma_\varepsilon} h_\varepsilon (u_i^{2\varepsilon} - u_i^{1\varepsilon})(\omega_i^2 \psi_i^{2\varepsilon} - \varphi_i) \, d\sigma_x - \varepsilon^2 \int_{\Gamma_\varepsilon} h_\varepsilon (u_i^{2\varepsilon} - u_i^{1\varepsilon}) \omega_i^1 \psi_i^{1\varepsilon} \, d\sigma_x$$

$$= \int_{\Omega_1^\varepsilon} g_i \varphi_i \, dx + \varepsilon \int_{\Omega_1^\varepsilon} g_i \omega_i^1 \psi_i^{1\varepsilon} \, dx + \int_{\Omega_2^\varepsilon} g_i \omega_i^2 \psi_i^{2\varepsilon} \, dx.$$

Using suggestive notations, this can be written as

$$I_1^\varepsilon + I_2^\varepsilon + I_3^\varepsilon - I_4^\varepsilon = I_5^\varepsilon + I_6^\varepsilon + I_7^\varepsilon. \tag{5.6}$$

We will now apply the appropriate unfolding operator to each term of (5.6) and pass to the limit as $\varepsilon \to 0$ in order to obtain the limit problem (5.2).

First, observe that $\varepsilon \omega^1 \psi^{1\varepsilon}$ strongly converges to zero in $L^2(\Omega)^N$. Hence, using Proposition 6(v), we obtain

$$\mathcal{T}_1^\varepsilon (\varepsilon \omega^1 \psi^{1\varepsilon}) \longrightarrow 0 \quad \text{strongly in } L^2(\Omega \times Y_1)^N. \tag{5.7}$$

Moreover, since $e_{ij}(\varepsilon \omega^\alpha \psi^{\alpha\varepsilon})(x) = \varepsilon \psi_i^\alpha(x/\varepsilon) e_{ij}(\omega^\alpha)(x) + \omega_i^\alpha(x) e_{ij}^y(\psi^\alpha)(x/\varepsilon)$, for $\alpha \in \{1, 2\}$, it is easy to see that

$$\mathcal{T}_\alpha^\varepsilon \big(e_{ij}(\varepsilon \omega^\alpha \psi^{\alpha\varepsilon}) \big) = \varepsilon \psi_i^\alpha \mathcal{T}_\alpha^\varepsilon \big(e_{ij}(\omega^\alpha) \big) + e_{ij}^y(\psi^\alpha) \mathcal{T}_\alpha^\varepsilon (\omega_i^\alpha) \longrightarrow e_{ij}^y(\Phi^\alpha) \tag{5.8}$$

strongly in $L^2(\Omega \times Y_\alpha)$, where $\Phi_i^\alpha(x, y) = \omega_i^\alpha(x) \psi_i^\alpha(y)$ (no summation).

Applying the unfolding operator to I_1^ε, we obtain that

$$I_1^\varepsilon = \int_{\Omega \times Y_1} \mathcal{T}_1^\varepsilon (a_{ijkh}^\varepsilon) \mathcal{T}_1^\varepsilon (e_{kh}(u^{1\varepsilon})) \big(\mathcal{T}_1^\varepsilon (e_{ij}(\varphi)) + \mathcal{T}_2^\varepsilon (e_{ij}(\varepsilon \omega^1 \psi^{1\varepsilon})) \big) \, dx \, dy,$$

thus, using (5.1)$_5$ and (5.8), we get

$$I_1^\varepsilon \longrightarrow \int_{\Omega \times Y_1} a_{ijkh}(y) \big(e_{kh}(u^1) + e_{kh}^y(\hat{u}^1) \big) \big(e_{ij}(\varphi) + e_{ij}^y(\Phi^1) \big) \, dx \, dy. \tag{5.9}$$

Similarly, by unfolding, we have

$$I_2^\varepsilon = \int_{\Omega \times Y_2} \mathcal{T}_2^\varepsilon(a_{ijkh}^\varepsilon) \varepsilon \mathcal{T}_2^\varepsilon(e_{kh}(u^{2\varepsilon})) \mathcal{T}_2^\varepsilon(e_{ij}(\varepsilon\omega^2\psi^{2\varepsilon}))\, dx\, dy,$$

and using $(5.1)_6$ and (5.8) yields

$$I_2^\varepsilon \longrightarrow \int_{\Omega \times Y_2} a_{ijkh}(y) e_{kh}^y(\hat{u}^2) e_{ij}^y(\Phi^2)\, dx\, dy. \tag{5.10}$$

Also, from Lemma 9, we get

$$I_3^\varepsilon = \int_{\Omega \times \Gamma} h(y)\big(\mathcal{T}_2^\varepsilon(u_i^{2\varepsilon}) - \mathcal{T}_1^\varepsilon(u_i^{1\varepsilon})\big)\big(\psi_i^2 \mathcal{T}_2^\varepsilon(\omega_i^2) - \mathcal{T}_1^\varepsilon(\varphi_i)\big)\, dx\, dy,$$

so that from $(5.1)_{2,3}$

$$I_3^\varepsilon \longrightarrow \int_{\Omega \times \Gamma} h(y)\big(\hat{u}_i^2 - u_i^1\big)\big(\psi_i^2\omega_i^2 - \varphi_i\big)\, dx\, dy. \tag{5.11}$$

On the other hand, again Lemma 9 yields

$$I_4^\varepsilon = \varepsilon \int_{\Omega \times \Gamma} h(y)\big(\mathcal{T}_2^\varepsilon(u_i^{2\varepsilon}) - \mathcal{T}_1^\varepsilon(u_i^{1\varepsilon})\big) \mathcal{T}_1^\varepsilon(\omega_i^1)\psi_i^1\, dx\, dy$$

$$\leq \varepsilon C \cdot \big\| \mathcal{T}_2^\varepsilon(u_i^{2\varepsilon}) - \mathcal{T}_1^\varepsilon(u_i^{1\varepsilon})\big\|_{L^2(\Omega \times \Gamma)} \cdot \big\| \mathcal{T}_1^\varepsilon(\omega_i^1)\psi_i^1\big\|_{L^2(\Omega \times \Gamma)},$$

and from $(4.2)_3$ we have $I_4^\varepsilon \leq \varepsilon C$. Consequently,

$$I_4^\varepsilon \longrightarrow 0. \tag{5.12}$$

Finally, we apply the appropriate unfolding operator to $I_5^\varepsilon, I_6^\varepsilon, I_7^\varepsilon$ and for I_6^ε we make use of (5.7). Thus,

$$I_5^\varepsilon \longrightarrow \int_{\Omega \times Y_1} g_i \varphi_i\, dx\, dy, \qquad I_6^\varepsilon \longrightarrow 0 \quad \text{and} \quad I_7^\varepsilon \longrightarrow \int_{\Omega \times Y_2} g_i \Phi_i^2\, dx\, dy. \tag{5.13}$$

The last thing to do in order to obtain the limit problem is to sum up (5.10)-(5.13) and use the density of $\mathcal{D}(\Omega)$ in $H_0^1(\Omega)$, that of $\mathcal{D}(\Omega; H_{\text{per}}^1(Y_1))$ in $L^2(\Omega; H_{\text{per}}^1(Y_1))$ and the density of $\mathcal{D}(\Omega; H^1(Y_2))$ in $L^2(\Omega; H^1(Y_2))$.

To conclude the proof, observe that endowing the Hilbert space V with the norm

$$\big\|(\varphi, \Phi^1, \Phi^2)\big\|_V^2 = \|\nabla\varphi_i\|_{L^2(\Omega)}^2 + \big\|\nabla_y\Phi_i^1\big\|_{L^2(\Omega \times Y_1)}^2$$

$$+ \big\|\nabla_y\Phi_i^2\big\|_{L^2(\Omega \times Y_2)}^2 + \big\|\Phi_i^2 - \varphi_i\big\|_{L^2(\Omega \times \Gamma)}^2, \tag{5.14}$$

the uniqueness of the solution of problem (5.2) can be proved applying the Lax-Milgram theorem with the norm (5.14) and the bilinear form defined by the left-hand side of (5.2).

□

We introduce now, for $l, m = 1, \ldots, N$, the unique solutions $w_1^{lm} \in \tilde{H}_{\text{per}}^1(Y_1)^N$ of the problems

$$
\begin{cases}
-\frac{\partial}{\partial y_j}\left(a_{ijlm} + a_{ijkh}\frac{\partial w_{1k}^{lm}}{\partial y_h}\right) = 0 & \text{in } Y_1, \\
\left(a_{ijlm} + a_{ijkh}\frac{\partial w_{1k}^{lm}}{\partial y_h}\right)n_j = 0 & \text{on } \Gamma,
\end{cases}
\tag{5.15}
$$

and for $l = 1, \ldots, N$, the unique solutions $w_2^l \in H^1(Y_2)^N$ of the problems

$$
\begin{cases}
-\frac{\partial}{\partial y_j}\left(a_{ijkh}\frac{\partial w_{2k}^l}{\partial y_h}\right) = \delta_{il} & \text{in } Y_2, \\
a_{ijkh}\frac{\partial w_{2k}^l}{\partial y_h}n_j = hw_{2i}^l & \text{on } \Gamma,
\end{cases}
\tag{5.16}
$$

and finally, we define the homogenized coefficients,

$$
a_{ijlm}^* = \int_{Y_1}\left(a_{ijlm} + a_{ijkh}\frac{\partial w_{1k}^{lm}}{\partial y_h}\right)dy.
\tag{5.17}
$$

One can verify that if $(u^1, \hat{u}^1, \hat{u}^2) \in V$ is the unique solution of (5.2), then

$$
\hat{u}_k^1(x, y) = w_{1k}^{lm}(y) \cdot \frac{\partial u_l^1}{\partial x_m}(x) \quad \text{in } \Omega \times Y_1,
\tag{5.18}
$$

$$
\hat{u}_k^2(x, y) = u_k^1(x) + g_l(x)w_{2k}^l(y) \quad \text{in } \Omega \times Y_2.
\tag{5.19}
$$

Theorem 12 *If $u^\varepsilon \in W_\varepsilon$ is the solution of (3.10), then*

$$
\tilde{u}^{1\varepsilon} \rightharpoonup |Y_1| \cdot u^1 \quad \text{weakly in } L^2(\Omega)^N,
\tag{5.20}
$$

$$
\tilde{u}^{2\varepsilon} \rightharpoonup |Y_2| \cdot u^1 + g_l \cdot q^l \quad \text{weakly in } L^2(\Omega)^N,
\tag{5.21}
$$

where u^1 is the unique solution of the problem

$$
\begin{cases}
-\frac{\partial}{\partial x_j}\left(a_{ijkh}^*\frac{\partial u_k^1}{\partial x_h}\right) = g_i & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega,
\end{cases}
\tag{5.22}
$$

and the components of q^l are given by $q_i^l = \int_{Y_2} w_{2i}^l \, dy$.

Proof Using suggestive notations, the homogenized problem (5.2) can be written

$$
J_1 + J_2 + J_3 = J_4 + J_5.
\tag{5.23}
$$

We will study now each term of (5.23).

From (5.18) we have

$$
\begin{aligned}
J_1 = {}& \int_\Omega \frac{\partial u_l^1}{\partial x_m}\frac{\partial \varphi_i}{\partial x_j}\int_{Y_1}\left(a_{ijlm} + a_{ijkh}\frac{\partial w_{1k}^{lm}}{\partial y_h}\right)dx\,dy \\
& + \int_\Omega \frac{\partial u_l^1}{\partial x_m}\int_{Y_1}\left(a_{ijlm} + a_{ijkh}\frac{\partial w_{1k}^{lm}}{\partial y_h}\right)\frac{\partial \Phi_i^1}{\partial y_j}dx\,dy
\end{aligned}
$$

and taking into account (5.17) formula and the variational formulation of (5.15), we get

$$J_1 = \int_\Omega a^*_{ijkh} \frac{\partial u^1_k}{\partial x_h} \frac{\partial \varphi_i}{\partial x_j} \, dx.$$

For J_2 we use (5.19), thus

$$J_2 = \int_\Omega g_l \int_{Y_2} a_{ijkh} \frac{\partial w^l_{2k}}{\partial y_h} \frac{\partial \Phi^2_i}{\partial y_j} \, dx \, dy$$

and an integration by parts yields

$$J_2 = \int_\Omega g_l \int_{Y_2} -\frac{\partial}{\partial y_j} \left(a_{ijkh} \frac{\partial w^l_{2k}}{\partial y_h} \right) \Phi^2_i \, dx \, dy + \int_\Omega g_l \int_\Gamma a_{ijkh} \frac{\partial w^l_{2k}}{\partial y_h} \Phi^2_i n_{2j} \, dx \, d\sigma_y,$$

where n_2 is the normal on Γ exterior to Y_2. Using now (5.16) and taking into account that $n_2 = -n$, J_2 becomes

$$J_2 = \int_{\Omega \times Y_2} g_i \Phi^2_i \, dx \, dy - \int_{\Omega \times \Gamma} h g_l w^l_{2i} \Phi^2_i \, dx \, d\sigma_y.$$

Obviously, using (5.19) and the boundary condition from (5.16), we get

$$J_3 = \int_{\Omega \times \Gamma} h g_l w^l_{2i} \Phi^2_i \, dx \, d\sigma_y - \int_\Omega g_l \varphi_i \int_\Gamma a_{ijkh} \frac{\partial w^l_{2k}}{\partial y_h} n_j \, dx \, d\sigma_y,$$

and, due to Gauss-Ostrogradsky formula and the fact that $n_2 = -n$,

$$J_3 = \int_{\Omega \times \Gamma} h g_l w^l_{2i} \Phi^2_i \, dx \, d\sigma_y + \int_\Omega g_l \varphi_i \int_{Y_2} \frac{\partial}{\partial y_j} \left(a_{ijkh} \frac{\partial w^l_{2k}}{\partial y_h} \right) dx \, dy$$

$$= \int_{\Omega \times \Gamma} h g_l w^l_{2i} \Phi^2_i \, dx \, d\sigma_y - |Y_2| \int_\Omega g_i \varphi_i \, dx.$$

Hence (5.23) becomes

$$\int_\Omega a^*_{ijkh} \frac{\partial u^1_k}{\partial x_h} \frac{\partial \varphi_i}{\partial x_j} \, dx - |Y_2| \int_\Omega g_i \varphi_i \, dx = |Y_1| \int_\Omega g_i \varphi_i \, dx,$$

which means that

$$\int_\Omega a^*_{ijkh} \frac{\partial u^1_k}{\partial x_h} \frac{\partial \varphi_i}{\partial x_j} \, dx = \int_\Omega g_i \varphi_i \, dx,$$

and finally (5.22) is proved. The convergences (5.20) and (5.21) are given directly by $(5.1)_{1,2}$. □

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