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Blow-up profile for a degenerate parabolic equation with a weighted localized term

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Abstract

In this paper, we investigate the Dirichlet problem for a degenerate parabolic equation $u_t - \Delta u^m = a(x)u^p(0,t) + b(x)u^q(x,t)$. We prove that under certain conditions the solutions have global blow-up, and the rate of blow-up is uniform in all compact subsets of the domain. Moreover, the blow-up profile is precisely determined.

Keywords: degenerate parabolic equation; localized source; uniform blow-up rate

1 Introduction

In this paper, we consider the following parabolic equation with nonlocal and localized reaction:

$$u_t - \triangle u^m = a(x)u^p(0,t) + b(x)u^q(x,t), \quad x \in \Omega, 0 < t < T^*, \tag{1.1}$$

$$u(x,\tau) = 0, \quad x \in \partial\Omega, t > 0, \tag{1.2}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.3}$$

where Ω is an open ball of $\mathbb{R}^N$, $N \geq 2$ with radius R, and $p \geq q > m > 0$.

Many of localized problems arise in applications and have been widely studied. Equations (1.1)-(1.3), as a kind of porous medium equation, can be used to describe some physical phenomena such as chemical reactions due to catalysis and an ignition model for a reaction gas (see [1–3]).

As for our problem (1.1)-(1.3), to our best knowledge, many works have been devoted to the case $m = 1$ (see [4–7]). Let us mention, for instance, when $a(x) = b(x) = 1$, blow-up properties have been investigated by Okada and Fukuda [7]. Moreover, they proved that if $p \geq q > 1$ and $u_0(x)$ is sufficiently large, every radial symmetric solution (maximal solution) has a global blow-up and the solution satisfies

$$C_1\left(T^* - t\right)^{-1/(p-1)} \leq u(x,t) \leq C_2\left(T^* - t\right)^{-1/(p-1)}, \tag{1.4}$$

in all compact subsets of Ω as t is near the blow-up time T^*, where C_1 and C_2 are two positive constants. Souplet [4, 8] investigated that global blow-up solutions have uniform blow-up estimates in all compact subsets of the domain.

The work of this paper is motivated by the localized semi-linear problem

$$u_t - \triangle u^m = \lambda_1 u^p(0,t) + \lambda_2 u^q(x,t), \quad x \in \Omega, 0 < t < T^*, \tag{1.5}$$

with Dirichlet boundary condition (1.2) and initial condition (1.3). In the case of $m = 1$ and $m > 1$, the uniform blow-up profiles were studied in [5, 9] and [10], respectively.

It seems that the result of [5, 9, 10] can be extended to λ_1 and λ_2 are two functions. Motivated by this, in this paper, we extend and improve the results of [5, 9, 10]. Our approach is different from those previously used in blow-up rate studies.

In the following section, we establish the blow-up rate and profile to (1.1)-(1.3).

2 Blow-up rate and profile

Throughout this paper, we assume that the functions $a(x)$, $b(x)$ and $u_0(x)$ satisfy the following two conditions:

(A1) $a(x)$, $b(x)$ and $u_0(x) \in C_0^2(\Omega)$; $a(x)$, $b(x)$ and $u_0(x)$ are positive in Ω.

(A2) $a(x)$, $b(x)$ and $u_0(x)$ are radially symmetric; $a(r)$, $b(r)$ and $u_0(r)$ are non-increasing for $r \in [0, R]$.

Theorem 2.1 *Suppose that $u_0(t)$ satisfies* (A1) *and* (A2). *If* $\max\{p, q\} > m$, *then the solutions of* (1.1)-(1.3) *blow up in finite time for large initial data.*

The proof of this theorem bears much resemblance to the result in [7, 11, 12] and is, therefore, omitted here.

Next we will show that in the situation of localized source dominating ($p > q$), problem (1.1)-(1.3) admits some uniform blow-up profile.

Theorem 2.2 *Assume* (A1) *and* (A2). *Let $u(x,t)$ be the blow-up solution of* (1.1)-(1.3) *and $u(x,t)$ is non-decreasing in time. If $p > \max\{q, m\}$, then we have*

$$\lim_{t\to T^*} u(x,t)\bigl(T^* - t\bigr)^{1/(p-1)} = a(x)\bigl((p-1)a^p(0)\bigr)^{1/(1-p)}, \tag{2.1}$$

uniformly in all compact subsets of Ω.

Throughout this paper, we denote

$$g(t) = u^p(0,t) \quad \text{and} \quad G(t) = \int_0^t g(s)\,\mathrm{d}s.$$

In our consideration, a crucial role is played by the Dirichlet eigenvalue problem

$$\begin{cases} -\triangle\varphi = \lambda\varphi, & \text{in } \Omega, \\ \varphi(x) = 0, & \text{on } \partial\Omega. \end{cases}$$

Denote by λ the first eigenvalue and by φ the corresponding eigenfunction with $\varphi > 0$ in Ω, normalized by $\int_\Omega a(x)\varphi(x)\,\mathrm{d}x = 1$. In the following, C is different from line to line. Also, we will sometimes use the notation $u \sim v$ for $\lim_{t\to T^*} u(t)/v(t) = 1$ with T^* the blow-up time for (1.1)-(1.3).

In order to prove the results of Section 2, first we derive a fact of the following problem:

$$\begin{cases} u_t - \triangle u^m = a(x)g(t), & x \in \Omega, 0 < t < T^*, \\ u(x,t) = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{2.2}$$

Lemma 2.1 *Assume* (A1), (A2) *and* $p > q$. *Let* $u(x,t)$ *be the blow-up solution of* (2.2) *and assume that* $u(x,t)$ *is non-decreasing in time, we then have*

$$\lim_{t\to T^*} \frac{u(x,t)}{G(t)} = a(x) \tag{2.3}$$

uniformly in all compact subsets of Ω.

Proof Assumption (A2) implies $u(0,t) = \max_{x\in\bar{\Omega}} u(x,t)$ and $\triangle u^m(0,t) \le 0$ on $(0,T^*)$. From (2.2), we have

$$u_t(0,t) \le a(0)u^p(0,t) + b(0)u^q(0,t), \quad 0 < t < T^*,$$

which implies

$$\lim_{t\to T^*} \sup \frac{u(0,t)}{G(t)} \le a(0). \tag{2.4}$$

Thus $\lim_{t\to T^*} G(t) = \infty$ and $\lim_{t\to T^*} g(t) = \infty$.

Set $R_1 \in (0,R)$, $\Omega_1 = \{x \in R_N, |x| < R_1\}$ and $b(x) = 1/a(x)$, $x \in \Omega_1$. By $a'(r) \le 0$, we obtain that $b'(r) \ge 0$, for $0 \le r \le R_1$.

Introducing a function

$$w(x,t) = b(x)u(x,t), \quad x \in \Omega_1, 0 < t < T^*.$$

In the following, we only consider $m > 2$. For the case of $0 < m \le 2$, the proof is similar.

A series calculation yields

$$\begin{aligned} b\triangle u^m &= b^{1-m}\triangle(bu)^m + (m-2)bu^{m-2)|\nabla u|^2} - mu^m b^{-1}|\nabla b|^2 \\ &\quad - mu^m\triangle b - 4mu^{m-1}\nabla u\nabla b. \end{aligned} \tag{2.5}$$

In addition, note

$$\nabla u(x,t)\nabla b(x) = u_r(r,t)b'(r) \le 0. \tag{2.6}$$

Now, according to (2.5) and (2.6), it follows that

$$\begin{aligned} w_t &= b(x)u_t \ge b(x)\triangle u^m + g(t) \\ &\ge \big(a(x)\big)^{m-1}\triangle w^m - \big(ma(x)\big(|\nabla b|\big)^2 + m\triangle b\big)u^m + g(t) \end{aligned} \tag{2.7}$$

for $x \in \Omega_1$, $0 < t < T^*$.

Set $m_1 = \min_{x\in\bar{\Omega}_1} |a(x)|^{m-1}$, $m_2 = \max_{x\in\bar{\Omega}_1}\{ma(x)|\nabla b|^2 + m|\triangle b|\}$, $\varepsilon(t) = m_2 u^m(0,t)/g(t)$. Since $p > m$ and note that $g(t) = u^p(0,t)$, then there exists $\tau \in (0,T^*)$ such that $0 < \varepsilon(t) \le 1/2$.

Therefore, in view of (2.7), we observe

$$\begin{aligned} w_t &\ge m_1\triangle w^m + \big(1-\varepsilon(t)\big)g(t) + \varepsilon(t)g(t) - m_2u^m(0,t) \\ &= m_1\triangle w^m + \big(1-\varepsilon(t)\big)g(t), \quad x \in \Omega_1, \tau < t < T^*. \end{aligned}$$

Set $g_1(t) = (1 - \varepsilon(t))g(t)$, $G(t) = \int_\tau^t g_1(s)\,\mathrm{d}s$. We then obtain

$$\lim_{t\to T^*} G_1(t) = \infty \quad \text{and} \quad \lim_{t\to T^*} \frac{G_1(t)}{G(t)} = 1.$$

Clearly, $w(x,t)$ is a sup-solution of the following equation

$$\begin{cases} v_t = m_1 \triangle v^m + g_1(t), & x \in \Omega_1, \tau < t < T^*, \\ v(x,t) = 0, & x \in \partial\Omega_1, t > 0, \\ v(x,0) = v_0(x), & x \in \Omega_1, \end{cases} \tag{2.8}$$

where $0 \le v_0 \le w(x,\tau)$ in Ω_1 and $v_0 \in C^1(\bar{\Omega}_1)$ with $v_0|_{\partial\Omega_1} = 0$. Here we also assume that $v_0(x)$ is a symmetric and non-increasing function of $|x|$ ($r = |x|$).

By the maximum principle, we have $0 \le v(x,t) \le w(x,t)$ and $v_r \le 0$ in Ω_1 for $\tau \le t < T^*$.

Similar to the proof of Theorem 3.1 in [9] that

$$\lim_{t\to T^*} \frac{v(x,t)}{G(t)} = 1$$

uniformly in all compact subsets of Ω.

By the arbitrariness of Ω_1, we obtain that the following limit converges uniformly in all compact subsets of Ω_1

$$\lim_{t\to T^*} \inf \frac{u(x,t)}{G(t)} \ge a(x). \tag{2.9}$$

In particular,

$$\lim_{t\to T^*} \inf \frac{u(0,t)}{G(t)} \ge a(0).$$

This inequality and (2.4) infer that

$$\lim_{t\to T^*} \frac{u(0,t)}{G(t)} = a(0).$$

Multiplying both sides of (2.2) by φ and integrating over $\Omega \times (0,t)$, we have, for $0 < t < T^*$,

$$\int_\Omega u\varphi\,\mathrm{d}x - \int_\Omega u_0\varphi\,\mathrm{d}x = -\lambda \int_0^t \int_\Omega u^m \varphi\,\mathrm{d}x\,\mathrm{d}s + G(t). \tag{2.10}$$

Since $\int_0^t \int_\Omega u^m\varphi\,\mathrm{d}x\,\mathrm{d}s \le \int_\Omega \varphi\,\mathrm{d}x \int_0^t u^m(0,t)\,\mathrm{d}s$ and $\lim_{t\to T^*} u^m(0,t)/g(t) = 0$, it then follows that

$$\lim_{t\to T^*} \frac{\int_\Omega u\varphi\,\mathrm{d}x}{G(t)} = 1. \tag{2.11}$$

Next we prove that

$$\lim_{t\to T^*} \sup \frac{u(x,t)}{G(t)} \le a(x)$$

uniformly in any compact subsets of Ω.

Assume on the contrary that there exists $x_0 \in \Omega$ ($x_0 \neq 0$) such that

$$\lim_{t\to T^*} \sup \frac{u(x_0,t)}{G(t)} = c > a(x_0).$$

Then there exists a sequence $\{t_n\}$, $t_n \to T^*$ such that

$$\lim_{t_n\to T^*} u(x_0,t_n)/G(t_n) = c.$$

Using the continuity of $a(x)$, we see that there exists $x_1 \in \Omega$ ($|x_1| < |x_0|$) such that $c > a(x)$ for $|x_1| \leq |x| \leq |x_0|$. Note that $u_r \leq 0$ and (2.9), we obtain

$$\begin{aligned}
\lim_{t\to T^*} \frac{\int_\Omega u\varphi\,\mathrm{d}x}{G(t)} &= \lim_{t_n\to T^*} \left(\frac{\int_{|x|<|x_1|} u\varphi\,\mathrm{d}x}{G(t_n)} + \frac{\int_{|x_1|<|x|<|x_0|} u\varphi\,\mathrm{d}x}{G(t_n)} + \frac{\int_{|x_0|<|x|<R} u\varphi\,\mathrm{d}x}{G(t_n)} \right) \\
&\geq \int_{|x|<|x_1|} a(x)\varphi(x)\,\mathrm{d}x + c\int_{|x_1|<|x|<|x_0|} \varphi\,\mathrm{d}x + \int_{|x_0|<|x|<R} a(x)\varphi(x)\,\mathrm{d}x \\
&> \int_{|x|<|x_1|} a(x)\varphi(x)\,\mathrm{d}x + \int_{|x_1|<|x|<|x_0|} a(x)\varphi\,\mathrm{d}x + \int_{|x_0|<|x|<R} a(x)\varphi(x)\,\mathrm{d}x = 1.
\end{aligned}$$

This contradicts (2.11) and we then complete the proof of Lemma 2.1. □

Lemma 2.2 *Under the assumption of Theorem* 2.2, *let* $u(x,t)$ *be the blow-up solution of* (1.1)-(1.3), *then it holds that*

$$\lim_{t\to T^*} \frac{u(x,t)}{G(t)} = a(x) \tag{2.12}$$

uniformly in all compact subsets in Ω.

Proof Proceeding as in (2.4), we have

$$\lim_{t\to T^*} \sup \frac{u(0,t)}{G(t)} \leq a(0), \tag{2.13}$$

which implies $\lim_{t\to T^*} G(t) = \infty$ and $\lim_{t\to T^*} g(t) = \infty$.

Now, according to $u_t \geq \triangle u + a(x)g(t)$, it then follows that $u(x,t)$ is a sub-solution of the following equation:

$$\begin{cases} v_t - \triangle v^m = a(x)g(t), & x \in \Omega_1, 0 < t < T^*, \\ v(x,t) = 0, & x \in \partial\Omega_1, t > 0, \\ v(x,0) = u_0(x), & x \in \Omega_1. \end{cases}$$

By the maximum principle, $u(x,t) \geq v(x)$ and $v_r \leq 0$ in $\Omega \times (0,T^*)$. Using Lemma 2.1, it holds that

$$\lim_{t\to T^*} \frac{v(x,t)}{G(t)} = a(x)$$

uniformly in all compact subsets of Ω.

Hence

$$\lim_{t\to T^*} \sup \frac{u(x,t)}{G(t)} \geq a(x) \tag{2.14}$$

uniformly in any compact subsets of Ω, which implies

$$\lim_{t\to T^*} \sup \frac{u(0,t)}{G(t)} = a(0). \tag{2.15}$$

Combining (2.13) with (2.15), we deduce that

$$\lim_{t\to T^*} \sup \frac{u(0,t)}{G(t)} = a(0). \tag{2.16}$$

Multiplying both sides of (1.1) by φ and integrating over $\Omega \times (0,t)$, we find, for $0 < t < T^*$,

$$\int_\Omega u\varphi\,\mathrm{d}x - \int_\Omega u_0\varphi\,\mathrm{d}x = -\lambda\int_0^t\int_\Omega u^m\varphi\,\mathrm{d}x\,\mathrm{d}s + G(t) - \int_0^t\int_\Omega b(x)u^q\varphi\,\mathrm{d}x\,\mathrm{d}s.$$

Since $\int_0^t\int_\Omega u^m\varphi\,\mathrm{d}x\,\mathrm{d}s \leq \int_0^t u(0,s)\,\mathrm{d}s\int_\Omega\varphi\,\mathrm{d}x$ and $p > q$, it then follows that

$$\lim_{t\to T^*}\frac{\int_0^t\int_\Omega v^m\varphi\,\mathrm{d}x\,\mathrm{d}s}{G(t)} = 0 \quad \text{and} \quad \lim_{t\to T^*}\frac{\int_0^t\int_\Omega v^q\varphi\,\mathrm{d}x\,\mathrm{d}s}{G(t)} = 0.$$

Therefore,

$$\lim_{t\to T^*}\frac{\int_\Omega u\varphi\,\mathrm{d}x}{G(t)} = 1.$$

By analogy with the argument taken in Lemma 2.1, we complete the proof of this lemma. □

Proof of Theorem 2.2 By Lemma 2.2, we infer that

$$u(0,t) \sim a(0)G(t) \quad \text{as } t \to T^*,$$

hence

$$G'(t) = g(t) \sim a^p(0)G^p(t) \quad \text{or} \quad \bigl(G^{1-p)}\bigr)' \sim -(p-1)a^p(0).$$

Integrating this equivalence between t and T^*, we obtain

$$G(t) \sim \bigl[(p-1)a^p(0)\bigl(T^* - t\bigr)\bigr]^{\frac{1}{1-p}}. \tag{2.17}$$

The result finally follows by returning (2.17) to (2.12). □

Remark 2.1 It seems that in the case of $p = q > m$, the blow-up rate remains valid in all compact subsets, but we do not know how to treat it. (It is an open problem in this case.)

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All the authors typed, read and approved the final manuscript.

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