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Existence of nontrivial solutions for p -Kirchhoff type equations

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Abstract

In this paper, the linking theorem and the mountain pass theorem are used to show the existence of nontrivial solutions for the p -Kirchhoff equations without assuming Ambrosetti-Rabinowitz type growth conditions, nontrivial solutions are obtained.

MSC: 35J60; 35J25

Keywords: linking theorem; mountain pass theorem; nontrivial solutions

1 Introduction

In this paper, we consider the nonlocal elliptic problem of the p -Kirchhoff type given by

$$\begin{cases} [M(\int_{\Omega} |\nabla u|^p dx)]^{p-1}(-\Delta_p u) = f(x, u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (1)$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, and $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is the p -Laplacian with $1 < p < N$.

Recently, the equation

$$\begin{cases} -(a + b \int_{\Omega} |\nabla u|^2 dx) \Delta u = f(x, u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (2)$$

began to attract the attention of several researchers only after Lion [1] had proposed an abstract framework for this problem. Perera and Zhang [2] obtained a nontrivial solution of (2) by using the Yang index and critical group. They revisited (2) via invariant sets of descent flow and obtained the existence of a positive solution, a negative, and a sign-changing solutions in [3].

The study of Kirchhoff-type equations has been extended to the following case involving the p -Laplacian:

$$\begin{cases} -M(\frac{1}{p} \int_{\Omega} |\nabla u|^p dx) \operatorname{div}(|\nabla u|^{p-2} \nabla u) = f(x, u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

for details see [4–6]. One of the authors has done some related work on this field. Liu [7] gave infinite solutions to the following equation via the fountain theorem and the dual fountain theorem:

$$\begin{cases} [M(\int_{\Omega} (|\nabla u|^p + \lambda(x)|u|^p) dx)]^{p-1}(-\Delta_p u + \lambda(x)|u|^{p-2}u) = f(x, u), & \text{in } \Omega, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = \eta |u|^{p-2}, & \text{on } \partial\Omega. \end{cases}$$

However, to the best of our knowledge, there have been few papers dealing with equation (1) using the linking theorem and the mountain pass theorem. This paper will make some contribution to this research field.

It is well known (see [8]) that the eigenvalue problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has the first eigenvalue $\lambda_1 > 0$, which is simple, and has an associated eigenfunction $\phi_1 > 0$. It is also known that λ_1 is an isolated point of $\sigma(-\Delta_p)$, the spectrum of $-\Delta_p$, which contains at least an eigenvalue sequence $\{\lambda_n\}$ and $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_n \rightarrow \infty$.

Let

$$W_0^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) : \int_{\Omega} |\nabla u|^p dx < \infty \text{ and } u|_{\partial\Omega} = 0 \right\}$$

be a Banach space with the norm $\|u\| = \|u\|_p = (\int_{\Omega} |\nabla u|^p dx)^{\frac{1}{p}}$ for $u \in W_0^{1,p}(\Omega)$. $W = \text{span}\{\phi_1\}$ be the one-dimensional eigenspace associated with λ_1 , where $\|\phi_1\| = 1$. Let $W_0^{1,p}(\Omega) = W \oplus V$, where $V = \{u \in W_0^{1,p}(\Omega) : \int_{\Omega} u \phi_1^{p-1} dx = 0\}$, there exists $\bar{\lambda} > \lambda_1$ such that

$$\int_{\Omega} |\nabla u|^p dx \geq \bar{\lambda} \int_{\Omega} |u|^p dx \quad \text{for } u \in V.$$

When $p = 2$, we can take $\bar{\lambda} = \lambda_2$, the second eigenvalue of $-\Delta$ in $H_0^1(\Omega)$.

In this paper, the weak solutions of (1) are the critical points of the energy functional

$$\Phi(u) = \frac{1}{p} \hat{M} \left(\int_{\Omega} |\nabla u|^p dx \right) - \int_{\Omega} F(x, u) dx,$$

where $\hat{M}(u) = \int_0^u [M(s)]^{p-1} ds$, $F(x, u) = \int_0^u f(x, s) ds$. Obviously, $\Phi(u) \in C^1(W_0^{1,p}(\Omega), R)$ and for all $u, v \in W_0^{1,p}(\Omega)$,

$$\langle \Phi'(u), v \rangle = \left[M \left(\int_{\Omega} |\nabla u|^p dx \right) \right]^{p-1} \int_{\Omega} (|\nabla u|^{p-2} \nabla u \nabla v) dx - \int_{\Omega} f(x, u) v dx.$$

In this paper we use the following notation: $L^p(\Omega)$ denotes the Lebesgue space with the norm $|\cdot|_p$; $|\Omega|$ denotes the Lebesgue measure of the set $\Omega \subset R^N$; $\langle \cdot, \cdot \rangle$ is the dual pairing of the space $(W_0^{1,p}(\Omega))^*$ and $W_0^{1,p}(\Omega)$; \rightarrow (resp. \rightharpoonup) denotes strong (resp. weak) convergence. C_0, C_1, C_2, \dots denote positive constants (possibly different).

Definition 1 [9] Let $\Phi \in C^1(X, R)$, we say that Φ satisfies the Cerami condition at the level $c \in R$ if any sequence $\{u_n\} \subset X$, along with

$$\Phi(u_n) \rightarrow c \quad \text{and} \quad (1 + \|u_n\|) \Phi'(u_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

possesses a convergent subsequence; Φ satisfies the (C) condition if Φ satisfies $(C)_c$ for all $c \in R$.

Definition 2 [9] A subset A of E is link (with respect to Φ) to B of E if $A \cap B = \emptyset$, for every $\Gamma \in \Phi$, there is $t \in [0, 1]$ such that $\Gamma(t, A) \cap B \neq \emptyset$.

Theorem 1 [9] (Linking theorem) Let $X = Y \oplus Z$ be a Banach space with $\dim Y < \infty$. Let $\rho > r > 0$, and let $z \in Z$ be such that $\|z\| = r$. Define

$$M = \{u = y + \lambda z : \|u\| \leq \rho, \lambda \geq 0, y \in Y\},$$

$$M_0 = \{u = y + \lambda z : y \in Y, \|u\| = \rho \text{ and } \lambda \geq 0 \text{ or } \|u\| \leq \rho \text{ and } \lambda = 0\},$$

$$N = \{u \in Z : \|u\| = r\}.$$

Let $\Phi \in C^1(X, \mathbb{R})$ be such that

$$b = \inf_N \Phi > a = \max_{M_0} \Phi.$$

If Φ satisfies the (PS) condition with

$$c = \inf_{\gamma \in \Gamma} \max_{u \in M} \Phi(\gamma(u)),$$

$$\Gamma = \{\gamma \in C(M, X) : \gamma|_{M_0} = id\},$$

then c is a critical value of Φ .

Remark 1 If Φ satisfies the (C) condition, then Theorem 1 still holds.

Theorem 2 [10] (Mountain pass theorem) Let X be a real Banach space, and let $\Phi \in C^1(X, \mathbb{R})$ satisfy the (C) condition. Suppose, for some $\alpha < \beta$, $\rho > 0$ and $u \in X$, $\|u\| > \rho$,

$$\max\{\Phi(0), \Phi(u)\} \leq \alpha < \beta \leq \inf_{\|u\|=\rho} \Phi(u).$$

Then Φ has a critical value $c \geq \beta > 0$ characterized by

$$c = \inf_{\gamma \in \Gamma} \max_{\tau \in [0, 1]} \Phi(\gamma(\tau)),$$

where

$$\Gamma = \{\gamma \in C([0, 1], X) : \gamma(0) = 0, \gamma(1) = u\}.$$

2 Main results

In this section, we give our main theorem. Near the origin, we make the following assumptions.

Suppose that $M : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a continuous function satisfying the following conditions:

- (m_0) there exists a constant $m_0 > 0$ such that $M(t) \geq m_0$ for all $t \geq 0$;
- (m_1) there exists a constant $m_1 > 0$ such that $M(t) \leq m_1$ for all $t \geq 0$ and $\hat{M}(t) \geq a[M(t)]^{p-1}t$, $a > 1$.

Caratheodory function f satisfies:

(F₀) For some $p < q < p^* = \frac{Np}{N-p}$, there exists a constant $C > 0$ such that

$$|f(x, t)| \leq C(1 + |t|^{q-1}) \quad \text{for all } x \in \Omega, t \in \mathbb{R}.$$

(F₁) There exist $0 < \alpha < m_0^{p-1} \lambda_1 (a - 1)$ and $\sigma \in [0, p]$ such that

$$\liminf_{|t| \rightarrow \infty} \frac{f(x, t)t - pF(x, t)}{|t|^\sigma} > -\alpha \quad \text{uniformly in } x \in \Omega.$$

(F₂) $\lim_{|t| \rightarrow \infty} \frac{pF(x, t)}{|t|^p} = +\infty$ uniformly in $x \in \Omega$.

(F₂)' $\liminf_{|t| \rightarrow \infty} \frac{pF(x, t)}{|t|^p} > m_1^{p-1} \lambda_1$ uniformly in $x \in \Omega$.

(F₃) $\limsup_{|t| \rightarrow 0} \frac{pF(x, t)}{|t|^p} < m_0^{p-1} \bar{\lambda}$ uniformly in $x \in \Omega$.

(F₃)' $\limsup_{|t| \rightarrow 0} \frac{pF(x, t)}{|t|^p} < m_0^{p-1} \lambda_1$ uniformly in $x \in \Omega$.

(F₄) $F(x, t) \geq \frac{m_1^{p-1} \lambda_1}{p} |t|^p, \forall (x, t) \in \Omega \times \mathbb{R}^1$.

The main results of this paper are the following.

Theorem 3 Assume that $(m_0), (m_1)$ and (F_0) - (F_4) hold, then problem (1) has at least one nontrivial weak solution in $W_0^{1,p}(\Omega)$.

Theorem 4 Assume that $(m_0), (m_1)$ and $(F_0), (F_1), (F_2)', (F_3)'$ hold, then problem (1) has at least one nontrivial weak solution in $W_0^{1,p}(\Omega)$.

3 Proofs of theorems

First, we give several lemmas.

Lemma 1 [7] Under assumptions (m_0) and (F_0) , any bounded sequence $\{u_n\} \subset W_0^{1,p}(\Omega)$ such that $\Phi'(u) \rightarrow 0$ in $(W_0^{1,p}(\Omega))^*$ as $n \rightarrow \infty$ has a convergent subsequence.

Lemma 2 Under assumptions (m_0) and (F_1) , the functional $\Phi(u)$ satisfies the (C) condition.

Proof Let $\{u_n\} \subset W_0^{1,p}(\Omega)$, for every $c > 0$,

$$\Phi(u_n) \rightarrow c \quad \text{and} \quad (1 + \|u_n\|)\Phi'(u_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{3}$$

We claim that $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. For this purpose, we can suppose that $\|u_n\| \rightarrow \infty$. By (F_1) , there exists $r > 1$ such that

$$f(x, t)t - pF(x, t) \geq -\alpha |t|^\sigma, \quad \forall |t| > r. \tag{4}$$

For large n , set $\Omega_n = \{x \in \Omega : |u| > r\}$, (3) and (4) imply that there exists $M_1 > 0$ such that

$$\begin{aligned} p(1 + c) &\geq p\Phi(u_n) - \langle \Phi'(u_n), u_n \rangle \\ &= \hat{M} \left(\int_{\Omega} |\nabla u_n|^p \, dx \right) - \int_{\Omega} pF(x, u_n) \, dx \\ &\quad - \left[\left[M \left(\int_{\Omega} |\nabla u_n|^p \, dx \right) \right]^{p-1} \int_{\Omega} (|\nabla u_n|^p) \, dx - \int_{\Omega} f(x, u_n) u_n \, dx \right] \end{aligned}$$

$$\begin{aligned}
 &\geq (a-1)m_0^{p-1}\|u_n\|^p + \int_{\Omega} [f(x, u_n)u_n - pF(x, u_n)] \, dx \\
 &= (a-1)m_0^{p-1}\|u_n\|^p + \int_{\Omega_n} [f(x, u_n)u_n - pF(x, u_n)] \, dx \\
 &\quad + \int_{\Omega \setminus \Omega_n} [f(x, u_n)u_n - pF(x, u_n)] \, dx \\
 &\geq (a-1)m_0^{p-1}\|u_n\|^p - \int_{\Omega_n} \alpha |u_n|^\sigma \, dx - M_1 \\
 &\geq (a-1)m_0^{p-1}\|u_n\|^p - \int_{\Omega_n} \alpha |u_n|^p \, dx - M_1 \\
 &\geq \left[(a-1)m_0^{p-1} - \frac{\alpha}{\lambda_1} \right] \|u_n\|^p - M_1.
 \end{aligned}$$

This is a contradiction. Then $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. By Lemma 1, we see that $\{u_n\}$ has a convergent subsequence in $W_0^{1,p}(\Omega)$. \square

Proof of Theorem 3 We obtain from assumptions (m_0) , (F_0) and (F_3) that for some $\varepsilon > 0$ small, there exists $C_1 > 0$ such that

$$F(x, u) \leq \frac{m_0^{p-1}(\bar{\lambda} - \varepsilon)}{p} |u|^p + C_1 |u|^q, \quad \forall x \in \Omega.$$

Taking $u \in V$, using the inequality $\int_{\Omega} |\nabla u|^p \, dx \geq \bar{\lambda} \int_{\Omega} |u|^p \, dx$ and the Sobolev inequality $\|u\|_q^q \leq K \|u\|^q$, we have

$$\begin{aligned}
 \Phi(u) &= \frac{1}{p} \hat{M} \left(\int_{\Omega} |\nabla u|^p \, dx \right) - \int_{\Omega} F(x, u) \, dx \\
 &\geq \frac{m_0^{p-1}}{p} \|u\|^p - \frac{m_0^{p-1}(\bar{\lambda} - \varepsilon)}{p} \int_{\Omega} |u|^p \, dx - C_1 \int_{\Omega} |u|^q \, dx \\
 &\geq \frac{m_0^{p-1}}{p} \|u\|^p - \frac{m_0^{p-1}(\bar{\lambda} - \varepsilon)}{p\bar{\lambda}} \|u\|^p - C_1 K \|u\|^q \\
 &= \frac{m_0^{p-1}}{p} \frac{\varepsilon}{\bar{\lambda}} \|u\|^p - C_1 K \|u\|^q.
 \end{aligned}$$

Then there exists $r > 0$ such that $b = \inf_{u \in V, \|u\|=r} \Phi(u) > 0$.

Let $z \in V$, $\|z\| = r$ and $M_0 = \{u = v + \lambda z : v \in W : \|u\| = \rho \text{ and } \lambda \geq 0, \text{ or } \|u\| \leq \rho \text{ and } \lambda = 0\}$, $N = \{u \in V : \|u\| = r\}$.

For every $u \in M_0$, if $u = v + \lambda z$, $\|u\| \leq \rho$ and $\lambda = 0$, then $u = v \in W$. By (F_4) , we know that

$$\begin{aligned}
 \Phi(u) &= \frac{1}{p} \hat{M} \left(\int_{\Omega} |\nabla u|^p \, dx \right) - \int_{\Omega} F(x, u) \, dx \\
 &\leq \frac{m_1^{p-1}}{p} \|u\|^p - \int_{\Omega} F(x, u) \, dx \\
 &\leq \frac{m_1^{p-1}}{p} \lambda_1 \int_{\Omega} |u|^p \, dx - \int_{\Omega} F(x, u) \, dx \\
 &\leq \int_{\Omega} \left[\frac{m_1^{p-1}}{p} \lambda_1 |u|^p \, dx - F(x, u) \right] \, dx \leq 0.
 \end{aligned}$$

Since $\dim W \oplus Rz < \infty$, there exists $C_0 > 1$ such that

$$\|u\|^p \leq C_0 |u|_p^p, \quad \forall u \in W \oplus Rz.$$

By (F_0) and (F_2) , there exists $\rho > 0$ such that $F(x, u) \geq \frac{m_1^{p-1}}{p} C_0^2 |u|^p, \forall |u| > \rho$. Let $M = \max\{0, \inf_{|u| \leq \rho} F(x, u)\}$, then we have

$$F(x, u) \geq \frac{m_1^{p-1}}{p} C_0^2 |u|^p - M.$$

Therefore, $\forall u \in W \oplus Rz$, we have

$$\begin{aligned} \Phi(u) &= \frac{1}{p} \hat{M} \left(\int_{\Omega} |\nabla u|^p \, dx \right) - \int_{\Omega} F(x, u) \, dx \\ &\leq \frac{m_1^{p-1}}{p} \|u\|^p - \frac{m_1^{p-1} C_0^2}{p} \int_{\Omega} |u|^p \, dx + M|\Omega| \\ &\leq \frac{m_1^{p-1}}{p} \|u\|^p - \frac{m_1^{p-1} C_0}{p} \|u\|^p + M|\Omega| \\ &\leq -\frac{(C_0 - 1)m_1^{p-1}}{p} \|u\|^p + M|\Omega|. \end{aligned}$$

Hence, for $\rho = \|u\|$ large enough, we have $\Phi(u) \rightarrow -\infty$.

Then there exists $\rho > r > 0$ such that

$$\inf_N \Phi(u) > 0 = \max_{M_0} \Phi(u).$$

By Lemmas 1 and 2, Φ satisfies the (C) condition. Then the conclusion follows from Theorem 1 and Remark 1. \square

Remark 2 (i) There exists $R > 0$ such that $pF(x, u) \leq uf(x, u), \forall |u| \geq R, x \in \Omega$, which implies $\forall |u| \geq R, x \in \Omega$,

$$f(x, u)u - pF(x, u) \geq 0 > -\alpha |u|^\sigma.$$

Then

$$\liminf_{|t| \rightarrow \infty} \frac{f(x, t)t - pF(x, t)}{|t|^\sigma} > -\alpha \quad \text{uniformly in } x \in \Omega.$$

Hence (F_1) is much weaker than Ambrosetti-Rabinowitz type growth conditions.

(ii) If $f(x, u)u - pF(x, u) \rightarrow +\infty$, as $|u| \rightarrow \infty$ uniformly in $x \in \Omega$, then

$$\liminf_{|t| \rightarrow \infty} \frac{f(x, t)t - pF(x, t)}{|t|^\sigma} \geq 0 > -\alpha \quad \text{uniformly in } x \in \Omega.$$

Example 1 Set

$$f(x, t) = \begin{cases} 0, & t < 0; \\ m_0^{p-1} \tilde{\lambda} t^{p-1}, & 0 \leq t \leq 1; \\ t^{p-1} \ln t + m_0^{p-1} \tilde{\lambda} t^{p-1}, & t > 1, \end{cases}$$

where $\lambda_1 < \tilde{\lambda} < \bar{\lambda}$, $m_1^{p-1}\lambda_1 + 1 < m_0^{p-1}\tilde{\lambda}$. Then it is easy to verify that $f(x, t)$ satisfies (F_0) - (F_4) with $\sigma = p$. When $t < 0$, we can use odd expansion to $f(x, t)$.

Example 2 Set

$$f(x, t) = \begin{cases} 0, & t < 0; \\ m_0^{p-1}\tilde{\lambda}t^{p-1}, & 0 \leq t < 2\pi; \\ \frac{m_1^{p-1}\lambda_1}{p}[q'(x, t)t^p + pq(x, t)t^{p-1}] + m_0^{p-1}\tilde{\lambda}t^{p-1}, & t \geq 2\pi, \end{cases}$$

where

$$q'(x, t) = \begin{cases} t \sin t, & t \in [2k\pi, (2k+1)\pi); \\ \frac{\sin t}{t}, & t \in [(2k+1)\pi, (2k+2)\pi) \end{cases} \quad (k = 1, 2, \dots),$$

$q(x, t) = \int_{2\pi}^t q'(x, s) ds$, $\lambda_1 < \tilde{\lambda} < \bar{\lambda}$, $m_1^{p-1}\lambda_1 < m_0^{p-1}\tilde{\lambda}$. Then it is easy to verify that $f(x, t)$ satisfies (F_0) - (F_4) with $\sigma = p$. When $t < 0$, we can use odd expansion to $f(x, t)$.

Proof of Theorem 4 We obtain from assumptions (m_0) , (F_0) , $(F_3)'$ that for some $\varepsilon > 0$ small, there exists $C_2 > 0$ such that

$$F(x, u) \leq \frac{m_0^{p-1}}{p}(\lambda_1 - \varepsilon)|u|^p + C_2|u|^q \quad \text{for all } x \in \Omega.$$

Taking $u \in W_0^{1,p}(\Omega)$, using the inequality $\int_{\Omega} |\nabla u|^p dx \geq \lambda_1 \int_{\Omega} |u|^p dx$ and the Sobolev inequality $|u|_q^q \leq K\|u\|^q$, we have

$$\begin{aligned} \Phi(u) &= \frac{1}{p} \hat{M} \left(\int_{\Omega} |\nabla u|^p dx \right) - \int_{\Omega} F(x, u) dx \\ &\geq \frac{m_0^{p-1}}{p} \|u\|^p - \frac{m_0^{p-1}(\lambda_1 - \varepsilon)}{p} \int_{\Omega} |u|^p dx - C_2 \int_{\Omega} |u|^q dx \\ &\geq \frac{m_0^{p-1}}{p} \|u\|^p - \frac{m_0^{p-1}(\lambda_1 - \varepsilon)}{p\lambda_1} \|u\|^p - C_2 K \|u\|^q \\ &= \frac{m_0^{p-1}}{p} \frac{\varepsilon}{\lambda_1} \|u\|^p - C_2 K \|u\|^q. \end{aligned}$$

Then there exists $\rho > 0$ such that $\inf_{\|u\|=\rho} \Phi(u) > 0$.

By (F_0) and $(F_2)'$, there exist $\beta > \lambda_1$, $\varepsilon > 0$, $\rho > 0$, and $\beta - \varepsilon > \lambda_1$ such that $|u| > \rho$,

$$F(x, u) \geq \frac{m_1^{p-1}(\beta - \varepsilon)}{p} |u|^p.$$

Let $M_1 = \max\{0, \inf_{\|u\| \leq \rho} F(x, u)\}$, then we have

$$F(x, u) \geq \frac{m_1^{p-1}(\beta - \varepsilon)}{p} |u|^p - M_1.$$

For $\forall u \in W$, we have

$$\begin{aligned} \Phi(u) &= \frac{1}{p} \hat{M} \left(\int_{\Omega} |\nabla u|^p \, dx \right) - \int_{\Omega} F(x, u) \, dx \\ &\leq \frac{m_1^{p-1}}{p} \|u\|^p - \frac{m_1^{p-1}(\beta - \varepsilon)}{p} \int_{\Omega} |u|^p \, dx + M_1 |\Omega|. \end{aligned}$$

Therefore, for $\beta - \varepsilon > \lambda_1$, let $u = t\phi_1$, $t \in R$, we have

$$\begin{aligned} \Phi(t\phi_1) &\leq \frac{m_1^{p-1}}{p} |t|^p - \frac{m_1^{p-1}(\beta - \varepsilon)}{p} \int_{\Omega} |t\phi_1|^p \, dx + M_1 |\Omega| \\ &\leq \frac{m_1^{p-1}}{p} |t|^p - \frac{m_1^{p-1}(\beta - \varepsilon)}{p\lambda_1} |t|^p + M_1 |\Omega| \\ &= \frac{m_1^{p-1}}{p} \left(1 - \frac{\beta - \varepsilon}{\lambda_1} \right) |t|^p + M_1 |\Omega| \rightarrow -\infty \quad \text{as } t \rightarrow \infty. \end{aligned}$$

Hence there exists $u_1 = t_1\phi_1 \in W_0^{1,p}(\Omega)$, $\|u_1\| > \rho$ such that $\Phi(u_1) \leq 0$.

Then

$$\inf_{\|u\|=\rho} \Phi(u) > 0 \geq \max \{ \Phi(0), \Phi(u_1) \}.$$

Summing up Lemma 1 and Lemma 2, $\Phi(u)$ satisfies all the conditions of Theorem 2, then the conclusion follows from Theorem 2. \square

Remark 3 The result of Theorem 1.1 in [11] corresponds to our results for the case $m_0 = m_1 = 1$ and $(F_2^+)_{\mu}$ replaces (F_1) . It is easy to see that (F_1) is much weaker than $(F_2^+)_{\mu}$, hence the results of Theorems 3 and 4 extend the results of [11].

Example 3 Set

$$f(x, t) = \begin{cases} 0, & t < 0; \\ m_0^{p-1} \mu_2 t^{p-1}, & 0 \leq t \leq 1; \\ m_1^{p-1} \mu_1 t^{p-1} - m_1^{p-1} \mu_1 + m_0^{p-1} \mu_2 t, & t > 1, \end{cases}$$

where p is odd, $\mu_1 > \lambda_1 > \mu_2$,

$$f(x, t) = \begin{cases} m_1^{p-1} \mu_1 t^{p-1} + m_1^{p-1} \mu_1 + m_0^{p-1} \mu_2 t, & t < -1; \\ m_0^{p-1} \mu_2 t^{p-1}, & |t| \leq 1; \\ m_1^{p-1} \mu_1 t^{p-1} - m_1^{p-1} \mu_1 + m_0^{p-1} \mu_2 t, & t > 1, \end{cases}$$

where p is even, $\mu_1 > \lambda_1 > \mu_2$.

Then similar to [12], it is easy to verify that $f(x, t)$ satisfies (F_0) , (F_1) , $(F_2)'$, $(F_3)'$ with $\sigma = p$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

CL and JW obtained the results in a joint research. All the authors read and approved the final manuscript.

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