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Periodic solution of a quasilinear parabolic equation with nonlocal terms and Neumann boundary conditions

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Abstract
In this article, we study the periodic solution of a quasilinear parabolic equation with nonlocal terms and Neumann boundary conditions. By using the theory of Leray-Schauder degree, we obtain the existence of a nontrivial nonnegative time periodic solution.

1 Introduction

The aim of this work is to consider the following periodic problem for a quasilinear parabolic equation:

$$\frac{\partial u}{\partial t} - D_i\big(a_{ij}(x,t,u)D_j u\big) = \big(m - \Phi[u]\big)u, \quad (x,t) \in Q_T, \tag{1.1}$$

$$\frac{\partial u}{\partial n} = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{1.2}$$

$$u(x,0) = u(x,T), \quad x \in \Omega, \tag{1.3}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $\frac{\partial}{\partial n}$ denotes the outward normal derivative on $\partial\Omega$, $Q_T = \Omega \times (0,T)$, a_{ij} satisfies some suitable smoothness and structure conditions. This model can be used to describe the models for some interesting phenomena in mathematical biology, fisheries and wildlife management. The function $u(x,t)$ gives the number of individuals (per unit area) of the species at position x and time t, where x represents the spatial variable and t represents the time. The term $D_i(a_{ij}(x,t,u)D_j u)$ models a tendency to avoid high density in the habitat, $m - \Phi[u]$ describes the ways in which a given population grows and shrinks over time, as controlled by birth, death, emigration or immigration, and the Neumann boundary condition models the trend of the biology population who survive on the boundary.

In last decades, linear parabolic equations with nonlocal terms have been investigated by numerous researchers [1–4]. A typical model was submitted by Allegretto and Nistri [1] and they proposed the following equation:

$$\frac{\partial u}{\partial t} - \Delta u = f\big(x,t,m,\Phi[u],u\big),$$

with the Dirichlet boundary conditions. Also, according to the actual needs, many authors divert attention to nonlinear diffusion equations with nonlocal terms such as the porous equation [5, 6] with a typical form

$$\frac{\partial u}{\partial t} = \Delta u^m + \big(m - \Phi[u]\big)u, \tag{1.4}$$

and the p-Laplacian equation [7] with a typical form

$$\frac{\partial u}{\partial t} = \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) + \big(m - \Phi[u]\big)u. \tag{1.5}$$

Equation (1.4) is degenerate if $m > 1$ and singular if $0 < m < 1$. Equation (1.5) is degenerate if $p > 2$ and singular if $1 < p < 2$. Only the cases $m > 1$ and $p > 2$ are considered with a few exceptions. All these equations are considered with the Dirichlet boundary condition which describes that the boundary is lethal to the species. Moreover, the methods in these papers are all based on the theory of Leray-Schauder degree. However, results on the quasilinear periodic parabolic equations with nonlocal terms and Neumann boundary conditions are few. In a recent paper [8], Wang and Yin considered the following periodic Neumann boundary value problem:

$$\begin{aligned}
&\frac{\partial u}{\partial t} - \Delta u^m = \big(m - \Phi[u]\big)u, \quad (x,t) \in Q_T,\\
&\frac{\partial u}{\partial n} = 0, \quad (x,t) \in \partial\Omega \times (0,T),\\
&u(x,0) = u(x,T), \quad x \in \Omega,
\end{aligned}$$

where $m > 1$. By the parabolic regularized method and the theory of Leray-Schauder degree, they established the existence of nontrivial nonnegative periodic solutions.

Inspired by the work of [8], we consider the periodic solutions of the Neumann boundary value problem of a quasilinear parabolic equation with nonlocal terms. Compared with the Dirichlet boundary condition, the Neumann boundary condition causes an additional difficulty in establishing some *a priori* estimates. On the other hand, different from the cases of the Dirichlet boundary condition, an auxiliary problem for (1.1)-(1.3) is considered for using the theory of Leray-Schauder degree. We prove that this problem (1.1)-(1.3) admits a nontrivial nonnegative periodic solution, that is, the following theorem.

Theorem 1 *If assumptions* (A1), (A2), (A3) *hold, then problem* (1.1)-(3.3) *admits a nontrivial nonnegative periodic solution* $u \in L^2(0,T;H^1(\Omega)) \cap C_T(\overline{Q_T})$.

The article is organized in the following way. In Section 2, we give some necessary preliminaries including the auxiliary problem. In Section 3, we establish some necessary *a priori* estimations of the solutions of the auxiliary problem. Then we show the proof of the main result of this paper.

2 Preliminaries

In the paper, we assume that

(A1) $a_{ij}(\cdot,\cdot,u) = a_{ji}(\cdot,\cdot,u) \in C_T(\overline{Q}_T)$ and there exist two constants $0 < \lambda \leq \gamma$ such that

$$\lambda|\xi|^2 \leq a_{ij}(x,t,u)\xi_i\xi_j \leq \gamma|\xi|^2 \quad \text{for all } (x,t) \in Q_T, u \in R^+ \text{ and } \xi \in R^n,$$

where $C_T(\overline{Q}_T)$ is the class of functions which are continuous in $\overline{\Omega} \times \mathbb{R}$ and T-periodic with respect to t. Furthermore, $a_{ij}(\cdot,\cdot,u)$ is continuous with respect to u.

(A2) $\Phi[\cdot] : L^2_+(\Omega) \to \mathbb{R}^+$ is a bounded continuous functional satisfying

$$\Phi[u] \le C\|u\|^2_{L^2(\Omega)},$$

where C is a positive constant independent of u, $\mathbb{R}^+ = [0,+\infty)$, $L^2_+(\Omega) = \{u \in L^2(\Omega) | u \ge 0, \text{a.e. in } \Omega\}$.

(A3) $m(x,t) \in C_T(\overline{Q}_T)$ and satisfies that

$$\operatorname*{essinf}_{x\in\Omega} \frac{1}{T}\int_0^T m(x,t)\,dt > \gamma\lambda_1,$$

where λ_1 is the first eigenvalue of the Laplacian equation on ω with zero boundary and $\phi_1(x)$ is the corresponding eigenfunction.

Since the regularity follows from a quite standard approach, we focus on the discussion of weak solutions in the following sense.

Definition 1 A function u is said to be a weak solution of problem (1.1)-(1.3), if $u \in L^2(0,T;H^1(T)) \cap C_T(\overline{Q}_T)$ and satisfies

$$\iint_{Q_T} \left(-u\frac{\partial\varphi}{\partial t} + a_{ij}(x,t,u)D_iuD_j\varphi - \big(m - \Phi[u]\big)u\varphi\right) dx\,dt = 0, \tag{2.1}$$

for any $\varphi \in C^1(\overline{Q}_T)$ with $\varphi(x,0) = \varphi(x,T)$.

In order to use the theory of Leray-Schauder degree, we introduce a map by considering the following auxiliary problem:

$$\frac{\partial u_\varepsilon}{\partial t} - D_i\big((1-\tau)\gamma D_iu_\varepsilon + \tau a_{ij}(x,t,u)D_ju_\varepsilon\big) + \varepsilon u_\varepsilon = f, \quad (x,t) \in Q_T, \tag{2.2}$$

$$\frac{\partial u_\varepsilon}{\partial n} = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{2.3}$$

$$u_\varepsilon(x,0) = u_\varepsilon(x,T), \quad x \in \Omega, \tag{2.4}$$

where ε is a sufficiently small positive constant, $\tau \in [0,1]$ is a parameter and $f \in C_T(\overline{Q_T})$. Then we can define a map $u_\varepsilon = G(\tau,f)$ with $G : [0,1] \times C_T(\overline{Q}_T) \to C_T(\overline{Q}_T)$. Applying classical estimates (see [9]), we can see that $\|u_\varepsilon\|_{L^\infty(Q_T)}$ is bounded by $\|f\|_{L^\infty(Q_T)}$, and u_ε is Hölder continuous in Q_T. Then, by the Arzela-Ascoli theorem, the map G is compact. So, the map G is a compact continuous map. Let $f(u_\varepsilon) = (m - \Phi[u_\varepsilon])u^+_\varepsilon$, where $u^+_\varepsilon = \max\{u_\varepsilon, 0\}$, we can see that the nonnegative solution u_ε of problem (2.2)-(2.4) is also a nonnegative fixed point of the map $u_\varepsilon = G(1,(m - \Phi[u_\varepsilon])u^+_\varepsilon)$. So, we will study the existence of nonnegative fixed points of the map $u = G(1,(m - \Phi[u_\varepsilon])u^+_\varepsilon)$ instead of a nonnegative solution of problem (2.2)-(2.4). And the desired solution u of (1.1)-(1.3) would be obtained as a limit point of u_ε.

3 The proof of the main result

First, by the same method as in [4], we can obtain the nonnegativity of the solutions of problem (2.2)-(2.4).

Lemma 1 *If a nontrivial function* $u_\varepsilon \in C_T(\overline{Q}_T)$ *solves* $u_\varepsilon = G(1, (m - \Phi[u_\varepsilon])u_\varepsilon^+)$, *then*

$$u_\varepsilon(x,t) > 0 \quad \text{for all } (x,t) \in \overline{Q}_T.$$

In the following, we will show some *a priori* estimates for the upper bound of a nonnegative periodic solution of problem (2.2)-(2.4). Here and below, we denote by $\|\cdot\|_p$ $(1 \le p \le \infty)$ the $L^p(\Omega)$ norm.

Lemma 2 *For* $\lambda \in [0,1]$, *let* $u(x,t)$ *be a nonnegative periodic solution which solves* $u_\varepsilon = G(1, \lambda(m - \Phi[u_\varepsilon])u_\varepsilon^+)$, *then there exists a constant* K *independent of* λ, ε *such that*

$$\|u(t)\|_\infty < K, \tag{3.1}$$

where $u(t) = u(\cdot,t)$.

Proof Multiplying (2.2) by u^{m+1} $(m \ge 0)$ and integrating over Ω, we have

$$\frac{1}{m+2}\frac{d}{dt}\|u(t)\|_{m+2}^{m+2} + \frac{4(m+1)}{(m+2)^2}\left\|\nabla\left(|u(t)|^{\frac{m}{2}}u(t)\right)\right\|_2^2 \le \|m(x,t)\|_{L^\infty(\Omega\times(0,T))}\|u(t)\|_{m+2}^{m+2},$$

and hence

$$\frac{d}{dt}\|u(t)\|_{m+2}^{m+2} + C_1\left\|\nabla\left(|u(t)|^{\frac{m}{2}}u(t)\right)\right\|_2^2 \le C_2(m+2)\|u(t)\|_{m+2}^{m+2}, \tag{3.2}$$

where C_i $(i = 1,2)$ are positive constants independent of u and m. Assume that $\|u(t)\|_\infty \ne 0$ and set

$$u_k(t) = |u(t)|^{\frac{m_k}{2}}u(t), \quad m_k = 2^k - 2 \ (k = 1,2,\ldots),$$

then $m_k = 2m_{k-1} + 2$. For convenience, we denote by C a positive constant independent of k and m, which may take different values. From (3.2), we obtain

$$\frac{d}{dt}\|u_k(t)\|_2^2 + C\|\nabla u_k(t)\|_2^2 \le C(m+2)\|u_k(t)\|_2^2. \tag{3.3}$$

By using the Gagliardo-Nirenberg inequality, we have

$$\|u_k(t)\|_2 \le C\|\nabla u_k(t)\|_2^\theta\|u_k(t)\|_1^{1-\theta}, \tag{3.4}$$

with

$$\theta = \frac{N}{N+2} \in (0,1).$$

By inequalities (3.3), (3.4) and the fact that $\|u_k(t)\|_1 = \|u_{k-1}(t)\|_2^2$, we obtain the following differential inequality:

$$\begin{aligned}\frac{d}{dt}\|u_k(t)\|_2^2 &\le -C\|u_k(t)\|_2^{\frac{2}{\theta}}\|u_k(t)\|_1^{\frac{2(\theta-1)}{\theta}} + C(m_k+2)\|u_k(t)\|_2^2\\ &\le -C\|u_k(t)\|_2^{\frac{2}{\theta}}\|u_{k-1}(t)\|_2^{\frac{4(\theta-1)}{\theta}} + C(m_k+2)\|u_k(t)\|_2^2.\end{aligned}$$

Let

$$\lambda_k = \max\Big\{1, \sup_t \|u_k(t)\|_2\Big\},$$

we have

$$\begin{aligned}\frac{d}{dt}\|u_k(t)\|_2^2 \le \|u_k(t)\|_2^{\frac{2(m_k+1)}{m_k+2}} \Big\{ -C\|u_k(t)\|_2^{\frac{2}{\theta}-\frac{2(m_k+1)}{m_k+2}} \lambda_{k-1}^{\frac{4(\theta-1)}{\theta}} \\ + C(m_k+2)\|u_k(t)\|_2^{\frac{2}{m_k+2}}\Big\}.\end{aligned} \tag{3.5}$$

By Young's inequality,

$$ab \le \epsilon a^{p'} + \epsilon^{-\frac{q'}{p'}} b^{q'},$$

where $p' > 1$, $q' > 1$, $a > 0$, $b > 0$, $\epsilon > 0$ and $\frac{1}{p'} + \frac{1}{q'} = 1$. Set

$$a = \|u_k(t)\|_2^{\frac{2}{m_k+2}}, \qquad b = m_k + 2, \qquad \epsilon = \frac{1}{2}\lambda_{k-1}^{\frac{4(\theta-1)}{\theta}},$$
$$p' = l_k = \frac{m_k+2}{\theta} - m_k - 1 = (m_k+2)\frac{N+2}{N} - m_k - 1,$$

then we obtain

$$(m_k+2)\|u_k(t)\|_2^{\frac{2}{m_k+2}} \le \frac{1}{2}\|u_k(t)\|_2^{\frac{2}{\theta}-\frac{2(m_k+1)}{m_k+2}} \lambda_{k-1}^{\frac{4(\theta-1)}{\theta}} + C(m_k+2)^{\frac{l_k}{l_k-1}} \lambda_{k-1}^{4\frac{1-\theta}{\theta}\frac{1}{l_k-1}}. \tag{3.6}$$

Here we have used the fact that $p' = l_k > r > 1$ for some r independent of k. In fact, it is easy to verify that

$$\lim_{k\to\infty} l_k = +\infty.$$

Denoting

$$a_k = \frac{l_k}{l_k-1}, \qquad b_k = \frac{1-\theta}{\theta}\frac{4}{l_k-1},$$

and combining (3.5) with (3.6), we have

$$\frac{d}{dt}\|u_k(t)\|_2^2 \le \|u_k(t)\|_2^{\frac{2(m_k+1)}{m_k+2}} \Big\{ -C\|u_k(t)\|_2^{\frac{2}{\theta}-\frac{2(m_k+1)}{m_k+2}} \lambda_{k-1}^{\frac{4(\theta-1)}{\theta}} + C(m_k+2)^{a_k}\lambda_{k-1}^{b_k}\Big\}. \tag{3.7}$$

Then

$$(m_k+2)\frac{d}{dt}\|u_k(t)\|_2^{\frac{2}{m_k+2}} \le -C\|u_k(t)\|_2^{\frac{2}{\theta}-\frac{2(m_k+1)}{m_k+2}} \lambda_{k-1}^{\frac{4(\theta-1)}{\theta}} + C(m_k+2)^{a_k}\lambda_{k-1}^{b_k}. \tag{3.8}$$

From the periodicity of $u_k(t)$, we know that there exists t_0 at which $\|u_k(t)\|_2$ reaches its maximum and thus the left-hand side of (3.8) vanishes. Then we obtain

$$\|u_k(t)\|_2 \le \Big\{C\Big[(m_k+2)^{a_k}\lambda_{k-1}^{b_k+\frac{4(1-\theta)}{\theta}}\Big]\Big\}^{\frac{1}{\alpha_k}},$$

where

$$\alpha_k = \frac{2}{\theta} - \frac{2(m_k+1)}{m_k+2} = \frac{2l_k}{m_k+2}.$$

Therefore, we conclude that

$$\|u_k(t)\|_2 \le \left\{C(m_k+2)^{a_k}\lambda_{k-1}^{b_k+\frac{4(1-\theta)}{\theta}}\right\}^{\frac{1}{\alpha_k}} = \left\{C(m_k+2)^{a_k}\right\}^{\frac{m_k+2}{2l_k}}\lambda_{k-1}^{\frac{2(1-\theta)(m_k+2)}{(l_k-1)\theta}}.$$

Since $\frac{m_k+2}{(l_k-1)\theta} = \frac{1}{1-\theta}$ and $\frac{m_k+2}{2l_k}$ and α_k are bounded, we get

$$\|u_k(t)\|_2 \le CA^k\lambda_{k-1}^2,$$

where $A > 1$ is a positive constant independent of k. Then we have

$$\ln\|u_k(t)\|_2 \le \ln\lambda_k \le \ln C + k\ln A + 2\ln\lambda_{k-1},$$

thus

$$\begin{aligned}\ln\|u_k(t)\|_2 &\le \ln C\sum_{i=0}^{k-2}2^i + 2^{k-1}\ln\lambda_1 + \ln A\left(\sum_{j=0}^{k-2}(k-j)2^j\right)\\ &\le \left(2^{k-1}-1\right)\ln C + 2^{k-1}\ln\lambda_1 + f(k)\ln A,\end{aligned}$$

or

$$\|u_k(t)\|_{m_k+2} \le \left\{C^{2^{k-1}-1}\lambda_1^{2^{k-1}}Af(k)\right\}^{\frac{2}{m_k+2}},$$

where

$$f(k) = k - 2(k+1) - 2^{k-1} + 2^{k+1}.$$

Letting $k \to \infty$, we obtain

$$\|u(t)\|_\infty \le C\lambda_1 \le C\left(\max\left\{1, \sup_t\|u(t)\|_2\right\}\right). \tag{3.9}$$

Now, we just need to show the estimate of $\|u(t)\|_2$. Multiplying (2.2) by u and integrating by parts over Q_T, by the periodicity of u, we have

$$\iint_{Q_T}\lambda_1|\nabla u|^2 + \varepsilon u^2\,dt\,dx \le \iint_{Q_T}\lambda u^2\left(m - \phi[u]\right)dt\,dx,$$

which implies that

$$\iint_{Q_T}u^2\left(m - \phi[u]\right)dt\,dx \ge 0.$$

Let $M = \max_{(x,t)\in\overline{Q}_T} m(x,t)$, by assumption (A2), we have

$$\begin{aligned}0 &\le \iint_{Q_T} u^2\bigl(m - \phi[u]\bigr)\,dt\,dx \le M\iint_{Q_T} u^2\,dt\,dx - \iint_{Q_T} u^2\phi[u]\,dt\,dx \\ &\le M\iint_{Q_T} u^2\,dt\,dx - C\int_0^T \|u\|_2^2\,dt,\end{aligned}$$

that is,

$$\int_0^T \|u\|_2^4\,dt \le C\int_0^T \|u\|_2^2\,dt,$$

where C is a positive independent of λ. By Young's inequality, we have

$$\int_0^T \|u\|_2^2\,dt \le \int_0^T \frac{1}{4\varepsilon^2} + \varepsilon^2\|u\|_2^4\,dt.$$

Combining with the above inequality, we have

$$\bigl\|u_k(t)\bigr\|_2 \le C,$$

which together with (3.9) implies (3.1), and thus the proof is complete. □

Corollary 1 *There exists a positive constant R independent of ε such that*

$$\deg\bigl(I - G\bigl(1, \bigl(m - \Phi[u_\varepsilon]\bigr)u_\varepsilon^+\bigr), B_R, 0\bigr) = 1,$$

where B_R is a ball centered at the origin with radius R in $L^\infty(Q_T)$.

Proof It follows from Lemma 2 that there exists a positive constant R independent of λ, ε such that

$$u \neq G\bigl(1, \bigl(m - \Phi[u_\varepsilon]\bigr)u_\varepsilon^+\bigr), \quad \forall u \in \partial B_R, \lambda \in [0,1].$$

So, the degree is well defined on B_R. From the homotopy invariance of the Leray-Schauder degree and the existence and uniqueness of the solution of $G(1,0)$, we can see that

$$\begin{aligned}\deg\bigl(1 - G\bigl(1, \bigl(m - \Phi[u_\varepsilon]\bigr)u_\varepsilon^+\bigr), B_R, 0\bigr) &= \deg\bigl(1 - G\bigl(1, \lambda\bigl(m - \Phi[u_\varepsilon]\bigr)u_\varepsilon^+\bigr), B_R, 0\bigr) \\ &= \deg\bigl(1 - G(1,0), B_R, 0\bigr) \\ &= 1.\end{aligned}$$

The proof is completed. □

Lemma 3 *There exist constants $r_0 > 0$ and $\varepsilon > 0$ such that for any $r < r_0$, $\varepsilon < \varepsilon_0$, $u = G(\tau, (m - \Phi[u])u^+ + (1-\tau))$ admits no nontrivial solution u satisfying*

$$0 < \|u\|_{L^\infty(Q_T)} \le r,$$

where r is a positive constant independent of ε.

Proof By contradiction, let u be a nontrivial fixed point of $u = G(\tau, (m - \Phi[u])u^+ + 1 - \tau)$ satisfying $0 < \|u\|_{L^\infty(Q_T)} \leq r$. For any given $\phi(x) \in C_0^\infty(B_\delta(x_0))$, multiplying (2.2) by $\frac{\phi^2}{u}$ and integrating over $Q_T^* = B_\delta(x_0) \times (0, T)$, we have

$$\begin{aligned}
&\iint_{Q_T} \frac{\partial u}{\partial t}\left(\frac{\phi^2}{u}\right) dt\,dx + \iint_{Q_T} (1-\tau)\gamma D_i u D_i\left(\frac{\phi^2}{u}\right) dt\,dx \\
&\quad + \iint_{Q_T} \tau a_{ij}(x,t,u) D_j u D_i\left(\frac{\phi^2}{u}\right) dt\,dx \\
&= \iint_{Q_T} \left(\phi^2\big(m - \varepsilon - \Phi[u]\big) + (1-\tau)\frac{\phi^2}{u}\right) dt\,dx. \qquad (3.10)
\end{aligned}$$

By the periodicity of u, the first term on the left-hand side is zero. The second term on the left-hand side can be rewritten as

$$\begin{aligned}
&\iint_{Q_T} (1-\tau)\gamma D_i u D_i\left(\frac{\phi^2}{u}\right) dt\,dx \\
&= \iint_{Q_T} (1-\tau)\gamma D_i u D_i\left(\phi \cdot \frac{\phi}{u}\right) dt\,dx \\
&= \iint_{Q_T} (1-\tau)\gamma |D\phi|^2\,dt\,dx - \iint_{Q_T} (1-\tau)\gamma u^2 \left|D\left(\frac{\phi}{u}\right)\right|^2 dt\,dx.
\end{aligned}$$

The third term of the left-hand side of equation (3.11) can be rewritten as

$$\begin{aligned}
&\iint_{Q_T} \tau a_{ij}(x,t,u) D_j u D_i\left(\frac{\phi^2}{u}\right) dt\,dx \\
&= \iint_{Q_T} \tau a_{ij}(x,t,u) D_j u D_i\left(\phi \cdot \frac{\phi}{u}\right) dt\,dx \\
&= \iint_{Q_T} \tau a_{ij}(x,t,u) 2 D_j u D_i\left(\frac{\phi}{u}\right) dt\,dx - \iint_{Q_T} \tau a_{ij}(x,t,u)\left(\frac{\phi}{u}\right)^2 D_i(u) D_j(u)\,dt\,dx.
\end{aligned}$$

Then from (3.10), we obtain

$$\begin{aligned}
&\iint_{Q_T} (1-\tau)\gamma |D\phi|^2\,dt\,dx - \iint_{Q_T} (1-\tau)\gamma u^2 \left|D\left(\frac{\phi}{u}\right)\right|^2 dt\,dx \\
&\quad + \iint_{Q_T} \tau a_{ij}(x,t,u) 2 D_j u D_i\left(\frac{\phi}{u}\right) dt\,dx - \iint_{Q_T} \tau a_{ij}(x,t,u)\left(\frac{\phi}{u}\right)^2 D_i(u) D_j(u)\,dt\,dx \\
&= \iint_{Q_T} \phi^2\big(m - \varepsilon - \Phi[u]\big)\,dt\,dx + \iint_{Q_T} (1-\tau)\frac{\phi^2}{u}\,dt\,dx.
\end{aligned}$$

From assumption (A1), we can see that

$$\begin{aligned}
&\iint_{Q_T} (1-\tau)\gamma |D\phi|^2\,dt\,dx + \iint_{Q_T} \tau a_{ij}(x,t,u) 2 D_j u D_i\left(\frac{\phi}{u}\right) dt\,dx \\
&\quad - \iint_{Q_T} \phi^2\big(m - \varepsilon - \Phi[u]\big)\,dt\,dx
\end{aligned}$$

$$\begin{aligned}
&= \iint_{Q_T} (1-\tau)\frac{\phi^2}{u}\,dt\,dx + \iint_{Q_T} (1-\tau)u^2\left|D\left(\frac{\phi}{u}\right)\right|^2 dt\,dx \\
&\quad + \iint_{Q_T} \tau a_{ij}(x,t,u)\left(\frac{\phi}{u}\right)^2 D_i(u)D_j(u)\,dt\,dx \\
&\geq \iint_{Q_T} (1-\tau)\frac{\phi^2}{u}\,dt\,dx + \iint_{Q_T} (1-\tau)\gamma u^2\left|D\left(\frac{\phi}{u}\right)\right|^2 dt\,dx \\
&\quad + \iint_{Q_T} \tau\lambda\left(\frac{\phi}{u}\right)^2 \left|D(u)\right|^2 dt\,dx \\
&\geq 0.
\end{aligned}$$

Since $\tau \in [0,1]$, we have

$$\iint_{Q_T} \gamma|D\phi|^2\,dt\,dx - \iint_{Q_T} \phi^2\bigl(m - \varepsilon - \Phi[u]\bigr)\,dt\,dx \geq 0.$$

By an approaching process, we choose $\phi = \phi_1$, where ϕ_1 is the eigenvector of the first eigenvalue λ_1 in (A3), and then we obtain

$$\begin{aligned}
0 &\leq \iint_{Q_T} \gamma|D\phi_1|^2\,dt\,dx - \iint_{Q_T} \phi_1^2\bigl(m - \varepsilon - \Phi[u]\bigr)\,dt\,dx \\
&= -\iint_{Q_T} \gamma\phi_1\triangle\phi_1\,dt\,dx - \iint_{Q_T} \phi_1^2\bigl(m - \varepsilon - \Phi[u]\bigr)\,dt\,dx \\
&= \iint_{Q_T} \gamma\lambda_1\phi_1^2\,dt\,dx - \iint_{Q_T} \phi_1^2\bigl(m - \varepsilon - \Phi[u]\bigr)\,dt\,dx \\
&= \int_\Omega \phi_1^2 \int_0^T \bigl(\gamma\lambda_1 - m + \varepsilon + \Phi[u]\bigr)\,dt\,dx.
\end{aligned} \tag{3.11}$$

Thus, there exists $x_0 \in \Omega$ such that $\int_0^T (\lambda_1 - m(x_0,t) + \Phi[u(x_0,t)])\,dt \geq 0$, then

$$\frac{1}{T}\int_0^T m(x_0,t)\,dt \leq \gamma\lambda_1 + \varepsilon + \frac{1}{T}\int_0^T \Phi\bigl[u(x_0,t)\bigr]\,dt.$$

From assumption (A2), we can see that

$$\frac{1}{T}\int_0^T m(x_0,t)\,dt \leq \gamma\mu_1 + \varepsilon + Cr^2|\Omega|$$

holds for any sufficiently small r and ε, which is a contradiction to assumption (A3). The proof is complete. □

Corollary 2 *There exists a small positive constant $r < R$ which is independent of ε, τ such that*

$$\deg\bigl(I - G\bigl(1, \bigl(m - \Phi[u_\varepsilon]\bigr)u_\varepsilon^+\bigr), B_r, 0\bigr) = 0,$$

where B_r is a ball centered at the origin with radius r in $L^\infty(Q_T)$.

Proof Similar to Lemma 3, we can see that there exists a positive constant $0 < r < R$ independent of ε such that

$$u_\varepsilon \neq G\big(\tau, \big(m - \Phi[u_\varepsilon]\big)u_\varepsilon^+ + 1 - \tau\big), \quad \forall u \in \partial B_r, \lambda \in [0,1].$$

So, the degree is well defined on B_r. By Lemma 3.3, we can easily see that $u = G(0,(m - \Phi[u])u^+ + 1)$ admits no solution in B_r. Then, by the homotopy invariance of the Leray-Schauder degree, we have

$$\begin{aligned}\deg\big(I - G\big(1, \big(m - \Phi[u_\varepsilon]\big)u_\varepsilon^+\big), B_r, 0\big) &= \deg\big(1 - G\big(0, \big(m - \Phi[u_\varepsilon]\big)u_\varepsilon^+ + 1\big), B_r, 0\big) \\ &= 0.\end{aligned}$$

The proof is completed. □

Now, we show the proof of the main result of this paper.

Proof of Theorem 1 Using Corollaries 1 and 2, we have

$$\deg\big(1 - G\big(f(\cdot)\big), B_R \setminus B_r, 0\big) = 1,$$

where R and r are positive constants and $R > r$. Problem (2.2)-(2.4) admits a nonnegative nontrivial solution u_ε with $r \leq \|u_\varepsilon\|_\infty \leq R$. Combining with the regularity results [9] and a similar argument as in [10], we can prove that the limit function of u_ε is a nonnegative nontrivial periodic solution of problem (1.1)-(1.3). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
RH, JS and BW carried out the proof of the main part of this article, BW corrected the manuscript and participated in its design and coordination. All authors have read and approved the final manuscript.

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