Prasad et al. *Boundary Value Problems* 2013, **2013**:40
http://www.boundaryvalueproblems.com/content/2013/1/40

Boundary Value Problems
a SpringerOpen Journal

RESEARCH **Open Access**

The continuous fractional Bessel wavelet transformation

Akhilesh Prasad[1], Ashutosh Mahato[1], Vishal Kumar Singh[1] and Madan Mohan Dixit[2*]

*Correspondence:
mmdixit1975@yahoo.co.in
[2]Department of Mathematics, NERIST, Nirjuli, India
Full list of author information is available at the end of the article

Abstract
The main objective of this paper is to study the fractional Hankel transformation and the continuous fractional Bessel wavelet transformation and some of their basic properties. Applications of the fractional Hankel transformation (*FrHT*) in solving generalized *n*th order linear nonhomogeneous ordinary differential equations are given. The continuous fractional Bessel wavelet transformation, its inversion formula and Parseval's relation for the continuous fractional Bessel wavelet transformation are also studied.

MSC: 46F12; 26A33

Keywords: Hankel transformation; fractional Hankel transformation; fractional Bessel wavelet transformation; Bessel function

1 Introduction

Pathak and Dixit [1] introduced continuous and discrete Bessel wavelet transformations and studied their properties by exploiting the Hankel convolution of Haimo [2] and Hirschman [3]. Upadhyay *et al.* [4] studied the continuous Bessel wavelet transformation associated with the Hankel-Hausdorff operator.

Let $L^p(\mathbb{R})$ denote the class of measurable functions of ϕ on $\mathbb{R}$ such that the integral $\int_{\mathbb{R}} |\phi(x)|^p\,dt$ is finite. Also, let $L^\infty(\mathbb{R})$ be a collection of almost everywhere bounded functions, hence endowed with the norm

$$\|\phi\|_{L^p} = \begin{cases} (\int_{\mathbb{R}} |\phi(x)|^p\,dx)^{1/p}, & 1 \le p < \infty, \\ \operatorname{ess\,sup}_{x\in\mathbb{R}} |\phi(x)|, & p = \infty. \end{cases}$$

The Hankel transformation h_μ, [5] of a conventional function $\varphi \in L^1(\mathbb{R}_+)$, $\mathbb{R}_+ = (0,\infty)$ is usually defined by

$$\hat{\varphi}(y) = (h_\mu\varphi)(y) = \int_0^\infty (xy)^{\frac{1}{2}} J_\mu(xy)\varphi(x)\,dx, \quad x \in \mathbb{R}_+, \mu \ge -1/2, \tag{1}$$

and its inversion formula is given by

$$\varphi(x) = \big(h_\mu^{-1}\hat{\varphi}(y)\big)(x) = \int_0^\infty (xy)^{\frac{1}{2}} J_\mu(xy)\hat{\varphi}(y)\,dy, \quad y \in \mathbb{R}_+, \tag{2}$$

where J_μ is the Bessel function of the first kind of order μ.

The fractional Hankel transformation is the generalization of the conventional Hankel transformation in the fractional order with parameter θ and is effectively used in the design of lens, analysis of laser cavity study of wave propagation in quadratic refractive index medium when the system is axially symmetric. The earliest work on the fractional Hankel transformation was published by Namias in 1980 [6]. Recently, it has become of importance in various applications in optics [7, 8]. Kerr [9] has developed a theory of fractional power of Hankel transforms in Zemanian spaces. We define a one-dimensional fractional Hankel transformation (*FrHT*) with parameter θ of $\varphi(x)$ for $\mu \geq -1/2$ and $0 < \theta < \pi$ as follows:

$$\hat{\varphi}_\mu^\theta(y) = \left(h_\mu^\theta \varphi\right)(y) = \int_0^\infty K_\mu^\theta(x,y)\varphi(x)\,dx, \tag{3}$$

where the kernel

$$K_\mu^\theta(x,y) = \begin{cases} c_\mu^\theta e^{\frac{i}{2}(x^2+y^2)\cot\theta}(xy\csc\theta)^{\frac{1}{2}} J_\mu(xy\csc\theta), & \theta \neq n\pi, \\ (xy)^{\frac{1}{2}} J_\mu(xy), & \theta = \frac{\pi}{2}, \\ \delta(x-y), & \theta = n\pi, \forall n \in \mathbb{Z}, \end{cases}$$

and

$$c_\mu^\theta = \frac{\exp[i(1+\mu)(\pi/2-\theta)]}{\sin\theta}.$$

The inversion formula of (3) is given by

$$\varphi(x) = \left(\left(h_\mu^\theta\right)^{-1}\hat{\varphi}\right)(x) = \int_0^\infty \overline{K_\mu^\theta(x,y)}\left(h_\mu^\theta\varphi\right)(y)\,dy, \tag{4}$$

where

$$\begin{aligned} \overline{K_\mu^\theta(x,y)} &= \exp\left[-i(1+\mu)(\pi/2-\theta)\right] e^{\frac{-i}{2}(x^2+y^2)\cot\theta}(xy\csc\theta)^{\frac{1}{2}} J_\mu(xy\csc\theta) \\ &= \overline{\left(c_\mu^\theta\right)}\sin\theta\, e^{\frac{-i}{2}(x^2+y^2)\cot\theta}(xy\csc\theta)^{\frac{1}{2}} J_\mu(xy\csc\theta), \end{aligned}$$

and

$$\begin{aligned} &h_\mu^0\varphi = h_\mu^\pi\varphi = \varphi, \\ &h_\mu^{\theta+2\pi}\varphi = h_\mu^\theta\varphi, \quad \theta \in \mathbb{R}. \end{aligned}$$

We assume that throughout this paper $\theta \neq n\pi$, $n \in \mathbb{Z}$.

From [10], wavelets as a family of functions constructed from translation and dilation of a single function ψ are called the mother wavelet defined by

$$\psi_{b,a}(x) = \frac{1}{\sqrt{a}}\psi\left(\frac{x-b}{a}\right), \quad b \in \mathbb{R}, a > 0,$$

where a is called the scaling parameter which measures the degree of compression or scale and b is a translation parameter which determines the time location of the wavelet. Shi *et* *al.* [11] defined the fractional mother wavelet as

$$\psi_{b,a,\theta}(x) = \frac{1}{\sqrt{a}}\psi\left(\frac{x-b}{a}\right)e^{\frac{-i}{2}(x^2-b^2)\cot\theta},$$

for all a, b and θ as above.

As per [2, 12], we defined the fractional Hankel convolution of functions $\varphi, \psi \in L^1(\mathbb{R}_+)$ as follows:

$$\begin{aligned}(\varphi \#_\theta \psi)(x) &= \int_0^\infty \varphi^\theta(x,y)\psi(y)\,dy \\ &= \int_0^\infty \left(\tau_x^\theta \varphi\right)(y)\psi(y)\,dy \\ &= \int_0^\infty \varphi(y)\left(\tau_x^\theta \psi\right)(y)\,dy, \end{aligned} \tag{5}$$

where the fractional Hankel translation of the function $\varphi \in L^1(\mathbb{R}_+)$ is defined by

$$\left(\tau_x^\theta \varphi\right)(y) = \varphi^\theta(x,y) = e^{\frac{-i}{2}(x^2+y^2)\cot\theta}\int_0^\infty \varphi(z)D_\mu^\theta(x,y,z)\,dz, \tag{6}$$

and

$$\begin{aligned} D_\mu^\theta(x,y,z) &= c_\mu^\theta e^{\frac{i}{2}(x^2+y^2+z^2)\cot\theta}\int_0^\infty \xi^{(-\mu-\frac{1}{2})}(x\xi\csc\theta)^{\frac{1}{2}}J_\mu(x\xi\csc\theta) \\ &\quad \times (y\xi\csc\theta)^{\frac{1}{2}}J_\mu(y\xi\csc\theta)(z\xi\csc\theta)^{\frac{1}{2}}J_\mu(z\xi\csc\theta)\,d\xi \\ &= \frac{2^{\mu-1}\,\nabla^{2\mu-1}\,c_\mu^\theta e^{\frac{i}{2}(x^2+y^2+z^2)\cot\theta}}{(xyz)^{\mu-1/2}\sqrt{\pi}\,\Gamma(\mu+1/2)}, \end{aligned} \tag{7}$$

where $\bigtriangledown(x,y,z)$ denotes the area of a triangle with sides x, y, z of such a triangle exists and zero otherwise. Clearly, $|D_\mu^\theta(x,y,z)| \geq 0$ and is symmetric in x, y, z.

Now, setting $\xi = 0$, we have

$$\int_0^\infty \left|D_\mu^\theta(x,y,z)\right|z^{\mu+1/2}\,dz \leq \frac{(xy)^{\mu+1/2}}{2^\mu\Gamma(\mu+1)|(\sin\theta)^{\mu+1/2}|} \tag{8}$$

for $x, y, z \in \mathbb{R}_+$.

Applying the inverse fractional Hankel transformation of $D_\mu^\theta(x,y,z)$, we obtain

$$\begin{aligned}&\overline{\left(c_\mu^\theta\right)}\int_0^\infty e^{\frac{-i}{2}(z^2+\xi^2)\cot\theta}(z\xi\csc\theta)^{\frac{1}{2}}J_\mu(z\xi\csc\theta)D_\mu^\theta(x,y,z)\,dz \\ &\quad = e^{\frac{i}{2}(x^2+y^2-\xi^2)\cot\theta}\xi^{(-\mu-\frac{1}{2})}(x\xi\csc\theta)^{\frac{1}{2}}J_\mu(x\xi\csc\theta)(y\xi\csc\theta)^{\frac{1}{2}}J_\mu(y\xi\csc\theta).\end{aligned}$$

Lemma 1.1 *If* $\varphi \in L^2(\mathbb{R}_+)$, *then*

$$\left\|x^{-\mu-1/2}\left(\tau_x^\theta\varphi\right)(y)\right\|_{L^2} \leq \frac{1}{|(\sin\theta)^{\mu+1/2}|2^\mu\Gamma(\mu+1)}\|\varphi\|_{L^2}.$$

Proof Since

$$\left(\tau_x^\theta \varphi\right)(y) = \varphi^\theta(x,y) = e^{\frac{-i}{2}(x^2+y^2)\cot\theta} \int_0^\infty \varphi(z) D_\mu^\theta(x,y,z)\,dz,$$

using (8), we have

$$\begin{aligned}
\left|\left(\tau_x^\theta \varphi\right)(y)\right| &\le \int_0^\infty \left|\varphi(z) z^{-1/2(\mu+1/2)} \left\{D_\mu^\theta(x,y,z)\right\}^{1/2} z^{1/2(\mu+1/2)} \left\{D_\mu^\theta(x,y,z)\right\}^{1/2}\right|\,dz \\
&\le \left(\int_0^\infty z^{-(\mu+1/2)} \left|\varphi(z)\right|^2 \left|D_\mu^\theta(x,y,z)\right|\,dz\right)^{\frac{1}{2}} \left(\int_0^\infty z^{(\mu+1/2)} \left|D_\mu^\theta(x,y,z)\right|\,dz\right)^{\frac{1}{2}} \\
&\le \left(\frac{(xy)^{\mu+1/2}}{|(\sin\theta)^{\mu+1/2}|2^\mu \Gamma(\mu+1)}\right)^{\frac{1}{2}} \left(\int_0^\infty z^{-(\mu+1/2)} \left|\varphi(z)\right|^2 \left|D_\mu^\theta(x,y,z)\right|\,dz\right)^{\frac{1}{2}}, \qquad (9)
\end{aligned}$$

so that

$$\begin{aligned}
&\int_0^\infty \left|\left(\tau_x^\theta \varphi\right)(y)\right|^2 dy \\
&\quad\le \frac{x^{\mu+1/2}}{|(\sin\theta)^{\mu+1/2}|2^\mu \Gamma(\mu+1)} \int_0^\infty z^{-(\mu+1/2)} \left|\varphi(z)\right|^2 dz \int_0^\infty \left|D_\mu^\theta(x,y,z)\right| y^{\mu+1/2}\,dy \\
&\quad\le \frac{x^{\mu+1/2}}{|(\sin\theta)^{\mu+1/2}|2^\mu \Gamma(\mu+1)} \int_0^\infty z^{-(\mu+1/2)} \left|\varphi(z)\right|^2 \frac{(zx)^{\mu+1/2}}{|(\sin\theta)^{\mu+1/2}|2^\mu \Gamma(\mu+1)}\,dz \\
&\quad= \frac{x^{2(\mu+1/2)}}{(|(\sin\theta)^{\mu+1/2}|2^\mu \Gamma(\mu+1))^2} \int_0^\infty \left|\varphi(z)\right|^2 dz.
\end{aligned}$$

Thus

$$\left\|x^{-\mu-1/2}\left(\tau_x^\theta \varphi\right)(y)\right\|_{L^2} \le \frac{1}{|(\sin\theta)^{\mu+1/2}|2^\mu \Gamma(\mu+1)} \|\varphi\|_{L^2}.$$

□

Remark 1.1 If $\varphi \in L^2(\mathbb{R}_+)$, then

$$\int_0^\infty \left|\left(\tau_y^\theta \varphi\right)(x)\right|^2 dx \le \frac{y^{2(\mu+1/2)}}{(|(\sin\theta)^{\mu+1/2}|2^\mu \Gamma(\mu+1))^2} \int_0^\infty \left|\varphi(z)\right|^2 dz,$$

and

$$\left\|y^{-\mu-1/2}\left(\tau_y^\theta \varphi\right)(x)\right\|_{L^2} \le \frac{1}{|(\sin\theta)^{\mu+1/2}|2^\mu \Gamma(\mu+1)} \|\varphi\|_{L^2}.$$

2 Properties of a fractional Hankel transformation

Zemanian [5, p.129] introduced a function space $H_\mu(\mathbb{R}_+)$ consisting of all complex-valued infinitely differentiable function φ defined on $\mathbb{R}_+ = (0,\infty)$, satisfying

$$\Gamma_{m,k}^\mu(\varphi) = \sup_{x\in\mathbb{R}_+} \left|x^m \left(x^{-1}D\right)^k \left[x^{-\mu-1/2}\varphi(x)\right]\right| < \infty, \quad \forall \mu \in \mathbb{R}, m,k \in \mathbb{N}_0. \qquad (10)$$

Definition 2.1 (Test function space $H_{\mu,\theta}(\mathbb{R}_+)$) The space $H_{\mu,\theta}(\mathbb{R}_+)$ is defined as follows: φ is a member of $H_{\mu,\theta}(\mathbb{R}_+)$ if and only if it is a complex-valued C^∞-function on $\mathbb{R}_+$ and for every choice of m and k of non-negative integers, it satisfies

$$\Upsilon^{\theta}_{m,k}(\varphi) = \sup_{x\in\mathbb{R}_+} \left|x^m \Delta^k_{\mu,x}\varphi(x)\right| < \infty, \quad \forall\theta \neq n\pi, n \in \mathbb{Z}, \tag{11}$$

where

$$\Delta_{\mu,x} = \left[\frac{d^2}{dx^2} + 2ix\cot\theta\frac{d}{dx} + \left(\frac{1-4\mu^2}{4x^2}\right) + i\cot\theta - x^2\cot^2\theta\right], \tag{12}$$

and

$$\Delta^k_{\mu,x} = x^{-2k}\sum_{r=0}^{2k}\left(\sum_{l=0}^{2k} a_l x^{2l}\right)\left(x^{-1}\frac{d}{dx}\right)^r,$$

where the constants a_l depend only on μ and parameter θ. On $H_{\mu,\theta}(\mathbb{R}_+)$, we consider the topology generated by the family $\{\Upsilon^{\theta}_{m,k}\}_{m,k\in\mathbb{N}_0}$ of seminorms.

Proposition 2.1 *Let $K^{\theta}_{\mu}(x,y)$ be the kernel of the fractional Hankel transformation. Then*

(i) $\Delta^r_{\mu,x}K^{\theta}_{\mu}(x,y) = (-y^2\csc^2\theta)^r K^{\theta}_{\mu}(x,y), \quad \forall r \in \mathbb{N}_0,$

(ii) $h^{\theta}_{\mu}((\Delta^*_{\mu,x})^r\varphi(x))(y) = (-y^2\csc^2\theta)^r(h^{\theta}_{\mu}\varphi)(y)$ *and* $\varphi \in H_{\mu}(\mathbb{R}_+)$,

where $\Delta^*_{\mu,x} = [\frac{d^2}{dx^2} - 2ix\cot\theta\frac{d}{dx} + (\frac{1-4\mu^2}{4x^2}) - i\cot\theta - x^2\cot^2\theta]$ *and is known as a fractional Bessel operator with parameter* θ.

Proof See [13]. □

Example 2.1 $h^{\theta}_{\mu}[(\Delta^*_{\mu,x})^r\delta(x-c)](y) = (-y^2\csc^2\theta)^r K^{\theta}_{\mu}(c,y), x, c \in \mathbb{R}_+$.

The result can be easily shown by using Proposition 2.1(ii).

Proposition 2.2 *Let $\varphi \in L^1(\mathbb{R}_+)$. Then $\hat{\varphi}^{\theta}_{\mu}$ satisfies the following*:

(i) $\hat{\varphi}^{\theta}_{\mu} \in L^{\infty}(\mathbb{R}_+)$ *with* $\|\hat{\varphi}^{\theta}_{\mu}\|_{L^{\infty}} \leq A_{\mu,\theta}\|\varphi\|_{L^1}$,

(ii) $\hat{\varphi}^{\theta}_{\mu}(y) \to 0$ *as* $y \to +\infty$ *or* $-\infty$,

(iii) $\hat{\varphi}^{\theta}_{\mu}$ *is continuous on* $\mathbb{R}_+$,

where $A_{\mu,\theta}$ is a positive constant depending on μ and θ.

Proof (i) Clearly, $\hat{\varphi}^{\theta}_{\mu}(y) = \int_0^{\infty} K^{\theta}_{\mu}(x,y)\varphi(x)\,dx$.
So, $\|\hat{\varphi}^{\theta}_{\mu}\|_{L^{\infty}} \leq A_{\mu,\theta}\|\varphi\|_{L^1}$.
(ii) From Proposition 2.1(ii), where $r = 1$, we have

$$\left|\hat{\varphi}^{\theta}_{\mu}(y)\right| = \frac{1}{|(-y^2\csc^2\theta)|}\left|\left(h^{\theta}_{\mu}(\Delta^*_{\mu,x}\varphi(x))\right)(y)\right| \to 0 \quad \text{as } y \to \pm\infty,$$

although $\hat{\varphi}^{\theta}_{\mu}(y) \to 0$ as $y \to \pm\infty$ for every $\varphi \in L^1(\mathbb{R}_+)$.

(iii) Let $h > 0$, consider

$$\begin{aligned}
&\sup_y \left|\hat{\varphi}_\mu^\theta(y+h) - \hat{\varphi}_\mu^\theta(y)\right| \\
&\quad \leq \left|c_\mu^\theta\right| \sup_y \left| \int_0^\infty e^{ih(h\frac{\cot\theta}{2}+y\cot\theta)} \bigl(x(y+h)\csc\theta\bigr)^{1/2} J_\mu\bigl(x(y+h)\csc\theta\bigr) \right. \\
&\qquad \left. - (xy\csc\theta)^{1/2} J_\mu(xy\csc\theta) \right| \left|\varphi(x)\right| dx \\
&\quad \leq A_{\mu,\theta} \int_0^\infty \left|\varphi(x)\right| dx \in L^1(\mathbb{R}_+),
\end{aligned}$$

and

$$\begin{aligned}
&\left|e^{ih(h\frac{\cot\theta}{2}+y\cot\theta)} \bigl(x(y+h)\csc\theta\bigr)^{1/2} J_\mu\bigl(x(y+h)\csc\theta\bigr) - (xy\csc\theta)^{1/2} J_\mu(xy\csc\theta)\right| \\
&\quad \to 0 \quad \text{as } h \to 0.
\end{aligned}$$

So,

$$\sup_y \left|\hat{\varphi}_\mu^\theta(y+h) - \hat{\phi}_\mu^\theta(y)\right| \to 0 \quad \text{as } h \to 0.$$

This proves that $\hat{\varphi}_\mu^\theta(y)$ is continuous in $\mathbb{R}_+$. □

Proposition 2.3 (Parseval's relation) *If $\Phi(y) = (h_\mu^\theta \varphi)(y)$ and $\Psi(y) = (h_\mu^\theta \psi)(y)$ denote the fractional Hankel transformations of $\varphi(x)$ and $\psi(x)$ respectively, then*

$$\begin{aligned}
\int_0^\infty \varphi(x)\overline{\psi(x)}\, dx &= \sin\theta \int_0^\infty \bigl(h_\mu^\theta \varphi\bigr)(y)\overline{\bigl(h_\mu^\theta \psi\bigr)(y)}\, dy \\
&= \sin\theta \int_0^\infty \Phi(y)\overline{\Psi(y)}\, dy, \qquad (13)
\end{aligned}$$

and

$$\int_0^\infty \left|\varphi(x)\right|^2 dx = \sin\theta \int_0^\infty \left|\bigl(h_\mu^\theta \varphi\bigr)(y)\right|^2 dy. \qquad (14)$$

Proof We have

$$\begin{aligned}
\langle \varphi, \psi \rangle &= \int_0^\infty \varphi(x)\overline{\psi(x)}\, dx \\
&= \int_0^\infty \varphi(x) \overline{\left(\bigl(\overline{c_\mu^\theta}\bigr) \sin\theta \int_0^\infty e^{-\frac{i}{2}(x^2+y^2)\cot\theta} (xy\csc\theta)^{\frac{1}{2}} J_\mu(xy\csc\theta) \bigl(h_\mu^\theta \psi\bigr)(y)\, dy \right)}\, dx \\
&= c_\mu^\theta \sin\theta \int_0^\infty \overline{\bigl(h_\mu^\theta \psi\bigr)}(y) \left(\int_0^\infty e^{\frac{i}{2}(x^2+y^2)\cot\theta} (xy\csc\theta)^{\frac{1}{2}} J_\mu(xy\csc\theta)\varphi(x)\, dx \right) dy \\
&= \sin\theta \int_0^\infty \bigl(h_\mu^\theta \varphi\bigr)(y)\overline{\bigl(h_\mu^\theta \psi\bigr)}(y)\, dy \\
&= \sin\theta \int_0^\infty \Phi(y)\overline{\Psi(y)}\, dy.
\end{aligned}$$

If $\varphi = \psi$, then

$$\int_0^\infty |\varphi(x)|^2\,dx = \sin\theta \int_0^\infty \left|\left(h_\mu^\theta \varphi\right)(y)\right|^2\,dy. \qquad \square$$

3 Applications of the fractional Hankel transformation to generalized differential equations

We consider the generalized nth order linear nonhomogeneous ordinary differential equation

$$L\varphi(x) = f(x), \tag{15}$$

where L is the generalized nth order differential operator given by

$$L = a_n\left(\Delta^*_{\mu,x}\right)^n + a_{n-1}\left(\Delta^*_{\mu,x}\right)^{n-1} + \cdots + a_1\left(\Delta^*_{\mu,x}\right) + a_0,$$

where $a_n, a_{n-1}, \ldots, a_0$ are constants and $\Delta^*_{\mu,x}$ is as given in Proposition 2.1.

Applying $FrHT$ to both sides of equation (15), we have

$$\int_0^\infty K_\mu^\theta(x,y)L\varphi(x)\,dx = \int_0^\infty K_\mu^\theta(x,y)f(x)\,dx,$$

$$\left[a_n\left(-y^2\csc^2\theta\right)^n + a_{n-1}\left(-y^2\csc^2\theta\right)^{n-1} + \cdots + a_1\left(-y^2\csc^2\theta\right) + a_0\right]\left(h_\mu^\theta\varphi\right)(y) = \left(h_\mu^\theta f\right)(y),$$

and equivalently,

$$P\left(-y^2\csc^2\theta\right)\left(h_\mu^\theta\varphi\right)(y) = \left(h_\mu^\theta f\right)(y), \quad \text{where } P(z) = \sum_{r=0}^n a_r z^r.$$

Therefore,

$$\left(h_\mu^\theta\varphi\right)(y) = \frac{(h_\mu^\theta f)(y)}{P(-y^2\csc^2\theta)}. \tag{16}$$

Now, an application of the inverse $FrHT$ gives the solution

$$\varphi(x) = \left(h_\mu^\theta\right)^{-1}\left[\frac{(h_\mu^\theta f)(y)}{P(-y^2\csc^2\theta)}\right].$$

Example 3.1 Let us consider $(1 - (\Delta^*_{\mu,x})^2)\varphi(x) = f(x)$. Then we have

$$\varphi(x) = \left(h_\mu^\theta\right)^{-1}\left[\left(1 - y^4\csc^4\theta\right)^{-1}\left(h_\mu^\theta f\right)(y)\right].$$

Example 3.2 Using the $FrHT$, we investigate the solution of the generalized differential equation

$$\frac{\partial^2 u}{\partial x^2} + 2ix\cot\theta\frac{\partial u}{\partial x} + \left(\frac{1}{4x^2} + i\cot\theta - x^2\cot^2\theta\right)u + \frac{\partial^2 u}{\partial z^2} = 0, \quad x \geq 0, z \geq 0, \tag{17}$$

$$\text{with } u(x,z) \to 0 \quad \text{as } z \to \infty \text{ and } x \to \infty, \tag{18}$$

$$\text{and } u(x,0) = f(x), \quad x \geq 0. \tag{19}$$

Let $\hat{u}_0^\theta(y,z)$ be the *FrHT* of order zero of $u(x,z)$ with respect to the variable x. Then, by definition,

$$\hat{u}_0^\theta(y,z) = \int_0^\infty K_0^\theta(x,y)u(x,z)\,dx, \tag{20}$$

where $K_0^\theta(x,y)$ is the kernel of *FrHT* of order zero.

Taking the *FrHT* of order zero of (17), we get

$$\begin{aligned}
&\int_0^\infty \left[\frac{\partial^2 u}{\partial x^2} + 2ix\cot\theta\frac{\partial u}{\partial x} + \left(\frac{1}{4x^2} + i\cot\theta - x^2\cot^2\theta\right)u\right]K_0^\theta(x,y)\,dx \\
&\quad + \int_0^\infty \frac{\partial^2 u}{\partial z^2}K_0^\theta(x,y)\,dx = 0, \\
&-y^2\csc^2\theta\,\hat{u}_0^\theta + \frac{d^2\hat{u}_0^\theta}{dz^2} = 0, \\
&\left(D^2 - y^2\csc^2\theta\right)\hat{u}_0^\theta = 0,
\end{aligned}$$

where $D \equiv \frac{d}{dz}$, whose solution is

$$\hat{u}_0^\theta(y,z) = Ae^{zy\csc\theta} + Be^{-zy\csc\theta}. \tag{21}$$

Taking the *FrHT* of order zero of (18), we have

$$\hat{u}_0^\theta(y,z) \to 0 \quad \text{as } z \to \infty. \tag{22}$$

Condition (22) is satisfied if we have $A = 0$.

Therefore, from (21)

$$\hat{u}_0^\theta(y,z) = Be^{-zy\csc\theta}. \tag{23}$$

Taking the *FrHT* of order zero of (19), we have

$$\begin{aligned}
&\int_0^\infty u(x,0)K_0^\theta(x,y)\,dx = \int_0^\infty f(x)K_0^\theta(x,y)\,dx, \\
&\hat{u}_0^\theta(y,0) = \hat{f}_0^\theta(y),
\end{aligned} \tag{24}$$

where $\hat{f}_0^\theta(y)$ is the *FrHT* of zero order of $f(x)$.

Putting $z = 0$ in (23) and using (24), we get $B = \hat{f}_0^\theta(y)$.

Hence (23) reduces to

$$\hat{u}_0^\theta(y,z) = \hat{f}_0^\theta(y)e^{-zy\csc\theta}.$$

Applying the inversion formula, we have

$$u(x,z) = \int_0^\infty \hat{f}_0^\theta(y)e^{-zy\csc\theta}\overline{K_0^\theta}(x,y)\,dy.$$

4 The continuous fractional Bessel wavelet transformation

The continuous fractional Bessel wavelet transformation (CFrBWT) is a generalization of the ordinary continuous Bessel wavelet transformation (CBWT) with parameter θ, that is, CBWT is a special case of CFrBWT with parameter $\theta = \frac{\pi}{2}$. In this section, we define the continuous fractional Bessel wavelet transformation and study some of its properties using the theory of fractional Hankel convolution (5) corresponding to [10].

A fractional Bessel wavelet is a function $\psi \in L^2(\mathbb{R}_+)$ which satisfies the condition

$$C_{\mu,\psi,\theta} = \int_0^\infty x^{-2\mu-2}\left|(h_\mu^\theta \psi)(x)\right|^2 dx < \infty, \quad \mu \geq -1/2,$$

where $C_{\mu,\psi,\theta}$ is called the *admissibility condition* of the fractional Bessel wavelet and $(h_\mu^\theta \psi)$ is the fractional Hankel transformation of ψ. The fractional Bessel wavelets $\psi_{b,a}^\theta$ are generated from one single function $\psi \in L^2(\mathbb{R}_+)$ by dilation and translation with parameters $a > 0$ and $b \geq 0$ respectively by

$$\begin{aligned}
\psi_{b,a}^\theta(x) &= \frac{1}{\sqrt{a}}\mathcal{D}_a \tau_b^\theta \psi(x) = \frac{1}{\sqrt{a}}\mathcal{D}_a \psi^\theta(b,x) = \frac{1}{\sqrt{a}}\psi^\theta\left(\frac{b}{a},\frac{x}{a}\right) \\
&= \frac{1}{\sqrt{a}} e^{\frac{-i}{2}(\frac{b^2}{a^2}+\frac{x^2}{a^2})\cot\theta} \int_0^\infty \psi(z) D_\mu^\theta\left(\frac{b}{a},\frac{x}{a},z\right) dz.
\end{aligned}$$

Lemma 4.1 *If* $\psi \in L^2(\mathbb{R}_+)$, *then*

$$\|\psi_{b,a}^\theta\|_{L^2} \leq \frac{b^{(\mu+1/2)} a^{-(\mu+1/2)}}{|(\sin\theta)^{\mu+1/2}| 2^\mu \Gamma(\mu+1)} \|\psi\|_{L^2}.$$

Proof We have

$$\psi_{b,a}^\theta(x) = \frac{1}{\sqrt{a}} e^{\frac{-i}{2}(\frac{b^2}{a^2}+\frac{x^2}{a^2})\cot\theta} \int_0^\infty \psi(z) D_\mu^\theta\left(\frac{b}{a},\frac{x}{a},z\right) dz.$$

Now,

$$\begin{aligned}
\left|\psi_{b,a}^\theta(x)\right| &\leq \frac{1}{\sqrt{a}} \int_0^\infty \left|\psi(z)\right| z^{-1/2(\mu+1/2)} \left|\left\{D_\mu^\theta\left(\frac{b}{a},\frac{x}{a},z\right)\right\}^{1/2}\right| z^{1/2(\mu+1/2)} \\
&\quad \times \left|\left\{D_\mu^\theta\left(\frac{b}{a},\frac{x}{a},z\right)\right\}^{1/2}\right| dz \\
&\leq \frac{1}{\sqrt{a}} \left(\int_0^\infty z^{-(\mu+1/2)} \left|\psi(z)\right|^2 \left|D_\mu^\theta\left(\frac{b}{a},\frac{x}{a},z\right)\right| dz\right)^{1/2} \\
&\quad \times \left(\int_0^\infty \left|D_\mu^\theta\left(\frac{b}{a},\frac{x}{a},z\right)\right| z^{(\mu+1/2)} dz\right)^{1/2} \\
&\leq \frac{1}{\sqrt{a}} \left(\frac{(bx)^{\mu+1/2}}{a^{2(\mu+1/2)}|(\sin\theta)^{\mu+1/2}| 2^\mu \Gamma(\mu+1)}\right)^{1/2} \\
&\quad \times \left(\int_0^\infty z^{-(\mu+1/2)} \left|\psi(z)\right|^2 \left|D_\mu^\theta\left(\frac{b}{a},\frac{x}{a},z\right)\right| dz\right)^{1/2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_0^\infty \left|\psi_{b,a}^\theta(x)\right|^2 dx \\
&\quad \le \frac{b^{\mu+1/2}}{a^{2\mu+2}|(\sin\theta)^{\mu+1/2}|2^\mu\Gamma(\mu+1)} \int_0^\infty z^{-(\mu+1/2)}\left|\psi(z)\right|^2 dz \int_0^\infty \left|D_\mu^\theta\left(\frac{b}{a},\frac{x}{a},z\right)\right| x^{\mu+1/2}\,dx \\
&\quad \le \frac{b^{2(\mu+1/2)}}{a^{2(\mu+1/2)}(|(\sin\theta)^{\mu+1/2}|2^\mu\Gamma(\mu+1))^2}\int_0^\infty \left|\psi(z)\right|^2 dz.
\end{aligned}$$

Thus,

$$\left\|\psi_{b,a}^\theta\right\|_{L^2} \le \frac{b^{(\mu+1/2)}a^{-(\mu+1/2)}}{|(\sin\theta)^{\mu+1/2}|2^\mu\Gamma(\mu+1)}\|\psi\|_{L^2}. \qquad \square$$

Theorem 4.1 *Let $f,\psi \in L^2(\mathbb{R}_+)$. Then the continuous fractional Bessel wavelet transformation B_ψ^θ is defined on f by*

$$\begin{aligned}
\left(B_\psi^\theta f\right)(b,a) &= a^{-\mu}\sin\theta\overline{c_\mu^\theta}\int_0^\infty e^{\frac{-i}{2}(\frac{1}{a^2}-1)(ax)^2\cot\theta} x^{-\mu-\frac{1}{2}}(bx\csc\theta)^{\frac{1}{2}} \\
&\quad \times J_\mu(bx\csc\theta)\left(h_\mu^\theta e^{\frac{-i}{2}(\cdot)^2\cot\theta}f\right)(x)\overline{\left(h_\mu^\theta\psi\right)}(ax)\,dx.
\end{aligned}$$

Proof We have

$$\begin{aligned}
&\left(B_\psi^\theta f\right)(b,a) \\
&\quad = \left\langle f,\psi_{b,a}^\theta\right\rangle \\
&\quad = \int_0^\infty f(t)\overline{\psi_{b,a}^\theta(t)}\,dt \\
&\quad = \int_0^\infty f(t)\overline{\left(\frac{1}{\sqrt{a}}e^{\frac{-i}{2}(\frac{b^2}{a^2}+\frac{t^2}{a^2})\cot\theta}\int_0^\infty \psi(z)D_\mu^\theta\left(\frac{b}{a},\frac{t}{a},z\right)dz\right)}\,dt \\
&\quad = \frac{1}{c_\mu^\theta\sqrt{a}}\int_0^\infty\left(c_\mu^\theta\int_0^\infty e^{\frac{i}{2}(t^2+\frac{\xi^2}{a^2})\cot\theta}\left(t\frac{\xi}{a}\csc\theta\right)^{\frac{1}{2}}J_\mu\left(t\frac{\xi}{a}\csc\theta\right)e^{\frac{-i}{2}t^2\cot\theta}f(t)\,dt\right) \\
&\qquad \times e^{\frac{-i}{2}(\frac{1}{a^2}-1)\xi^2\cot\theta}\xi^{(-\mu-1/2)}\left(\frac{b}{a}\xi\csc\theta\right)^{\frac{1}{2}}J_\mu\left(\frac{b}{a}\xi\csc\theta\right)\left(\overline{h_\mu^\theta\psi(z)}\right)(\xi)\,d\xi \\
&\quad = \frac{1}{c_\mu^\theta\sqrt{a}}\int_0^\infty e^{\frac{-i}{2}(\frac{1}{a^2}-1)\xi^2\cot\theta}\xi^{(-\mu-1/2)}\left(\frac{b}{a}\xi\csc\theta\right)^{\frac{1}{2}} \\
&\qquad \times J_\mu\left(\frac{b}{a}\xi\csc\theta\right)\left(h_\mu^\theta e^{\frac{-i}{2}(\cdot)^2\cot\theta}f\right)\left(\frac{\xi}{a}\right)\left(\overline{h_\mu^\theta\psi}\right)(\xi)\,d\xi
\end{aligned}$$

by putting $\frac{\xi}{a}=x$, then the continuous fractional Bessel wavelet transformation can be written as

$$\begin{aligned}
\left(B_\psi^\theta f\right)(b,a) &= a^{-\mu}\sin\theta\overline{c_\mu^\theta}\int_0^\infty e^{\frac{-i}{2}(\frac{1}{a^2}-1)(ax)^2\cot\theta} x^{-\mu-\frac{1}{2}}(bx\csc\theta)^{\frac{1}{2}} \\
&\quad \times J_\mu(bx\csc\theta)\left(h_\mu^\theta e^{\frac{-i}{2}(\cdot)^2\cot\theta}f\right)(x)\overline{\left(h_\mu^\theta\psi\right)}(ax)\,dx.
\end{aligned}$$

This means that

$$h_\mu^\theta\left\{e^{\frac{-i}{2}b^2\cot\theta}\left(B_\psi^\theta f\right)(b,a)\right\} = a^{-\mu}\sin\theta\left(x^{-\mu-\frac{1}{2}}e^{\frac{i}{2}a^2x^2\cot\theta}\left(h_\mu^\theta e^{\frac{-i}{2}(\cdot)^2\cot\theta}f\right)(x)\overline{\left(h_\mu^\theta\psi\right)}(ax)\right).$$

□

Remark 4.1 If $f\in L^2(\mathbb{R}_+)$ is a homogeneous function of degree n, then

$$\left(B_\psi^\theta f\right)(\lambda b,\lambda a) = \lambda^{n+\frac{1}{2}}\left(B_\psi^\theta f\right)(b,a).$$

Theorem 4.2 *If ψ_1 and ψ_2 are two wavelets and $(B_{\psi_1}^\theta f)(b,a)$ and $(B_{\psi_2}^\theta g)(b,a)$ denote the continuous fractional Bessel wavelet transformations of $f,g\in L^2(\mathbb{R}_+)$ respectively, then*

$$\int_0^\infty\int_0^\infty\left(B_{\psi_1}^\theta f\right)(b,a)\overline{\left(B_{\psi_2}^\theta g\right)}(b,a)\frac{db\,da}{a^2} = \sin^2\theta\,C_{\mu,\psi_1,\psi_2,\theta}\langle f,g\rangle,$$

where

$$C_{\mu,\psi_1,\psi_2,\theta} = \int_0^\infty a^{-2\mu-2}\overline{\left(h_\mu^\theta\psi_1\right)}(a)\left(h_\mu^\theta\psi_2\right)(a)\,da<\infty.$$

Proof We have

$$\begin{aligned}\left(B_{\psi_1}^\theta f\right)(b,a) &= a^{-\mu}\sin\theta\,\overline{c_\mu^\theta}\int_0^\infty e^{\frac{-i}{2}(\frac{1}{a^2}-1)(ax)^2\cot\theta}x^{-\mu-\frac{1}{2}}(bx\csc\theta)^{\frac{1}{2}}\\ &\quad\times J_\mu(bx\csc\theta)\left(h_\mu^\theta e^{\frac{-i}{2}(\cdot)^2\cot\theta}f\right)(x)\overline{\left(h_\mu^\theta\psi_1\right)}(ax)\,dx.\end{aligned}$$

Now,

$$\begin{aligned}&\int_0^\infty\int_0^\infty\left(B_{\psi_1}^\theta f\right)(b,a)\overline{\left(B_{\psi_2}^\theta g\right)}(b,a)\frac{db\,da}{a^2}\\ &\quad=\int_0^\infty\int_0^\infty\Bigg[a^{-\mu}\sin\theta\,\overline{c_\mu^\theta}\int_0^\infty e^{\frac{-i}{2}(\frac{1}{a^2}-1)(ax)^2\cot\theta}x^{-\mu-\frac{1}{2}}(bx\csc\theta)^{\frac{1}{2}}\\ &\qquad\times J_\mu(bx\csc\theta)\left(h_\mu^\theta e^{\frac{-i}{2}(\cdot)^2\cot\theta}f\right)(x)\overline{\left(h_\mu^\theta\psi_1\right)}(ax)\,dx\Bigg]\\ &\qquad\times\overline{\Bigg[a^{-\mu}\sin\theta\,\overline{c_\mu^\theta}\int_0^\infty e^{\frac{-i}{2}(\frac{1}{a^2}-1)(ay)^2\cot\theta}y^{-\mu-\frac{1}{2}}(by\csc\theta)^{\frac{1}{2}}}\\ &\qquad\overline{\times J_\mu(by\csc\theta)\left(h_\mu^\theta e^{\frac{-i}{2}(\cdot)^2\cot\theta}g\right)(y)\overline{\left(h_\mu^\theta\psi_2\right)}(ay)\,dy\Bigg]}\,db\frac{da}{a^2}\\ &\quad=\int_0^\infty\int_0^\infty a^{-2\mu-2}\sin^3\theta\,e^{\frac{-i}{2}(2-a^2)x^2\cot\theta}x^{-\mu-\frac{1}{2}}\left(h_\mu^\theta e^{\frac{-i}{2}(\cdot)^2\cot\theta}f\right)(x)\overline{\left(h_\mu^\theta\psi_1\right)}(ax)\\ &\qquad\times\Bigg\{c_\mu^\theta\int_0^\infty e^{\frac{i}{2}(b^2+x^2)\cot\theta}(bx\csc\theta)^{\frac{1}{2}}J_\mu(bx\csc\theta)\\ &\qquad\times\Bigg(\overline{c_\mu^\theta}\int_0^\infty e^{\frac{-i}{2}(b^2+y^2)\cot\theta}(by\csc\theta)^{\frac{1}{2}}J_\mu(by\csc\theta)\\ &\qquad\times e^{\frac{i}{2}(2-a^2)y^2\cot\theta}y^{-\mu-\frac{1}{2}}\overline{\left(h_\mu^\theta e^{\frac{-i}{2}(\cdot)^2\cot\theta}g\right)}(y)\left(h_\mu^\theta\psi_2\right)(ay)\,dy\Bigg)db\Bigg\}\,dx\,da\end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty \int_0^\infty a^{-2\mu-2} \sin^3\theta e^{\frac{-i}{2}(2-a^2)x^2\cot\theta} x^{-\mu-\frac{1}{2}} \bigl(h_\mu^\theta e^{\frac{-i}{2}(\cdot)^2\cot\theta} f\bigr)(x)\overline{\bigl(h_\mu^\theta \psi_1\bigr)}(ax) \\
&\quad \times \biggl\{ c_\mu^\theta \int_0^\infty e^{\frac{i}{2}(b^2+x^2)\cot\theta}(bx\csc\theta)^{\frac{1}{2}} J_\mu(bx\csc\theta)\bigl(h_\mu^\theta\bigr)^{-1}\bigl(e^{\frac{i}{2}(2-a^2)y^2\cot\theta} y^{-\mu-\frac{1}{2}} \\
&\quad \times \overline{\bigl(h_\mu^\theta e^{\frac{-i}{2}(\cdot)^2\cot\theta} g\bigr)}(y)\bigl(h_\mu^\theta\psi_2\bigr)(ay)\bigr)(b)\,db \biggr\} dx\,da \\
&= \sin^3\theta \int_0^\infty \int_0^\infty a^{-2\mu-2} e^{\frac{-i}{2}(2-a^2)x^2\cot\theta} x^{-\mu-\frac{1}{2}} \bigl(h_\mu^\theta e^{\frac{-i}{2}(\cdot)^2\cot\theta} f\bigr)(x)\overline{\bigl(h_\mu^\theta \psi_1\bigr)}(ax) \\
&\quad \times h_\mu^\theta\bigl(h_\mu^\theta\bigr)^{-1}\bigl(e^{\frac{i}{2}(2-a^2)y^2\cot\theta} y^{-\mu-\frac{1}{2}} \overline{\bigl(h_\mu^\theta e^{\frac{-i}{2}(\cdot)^2\cot\theta} g\bigr)}(y)\bigl(h_\mu^\theta\psi_2\bigr)(ay)\bigr)(x)\,dx\,da \\
&= \sin^3\theta \int_0^\infty \int_0^\infty a^{-2\mu-2} x^{-2\mu-1} \bigl(h_\mu^\theta e^{\frac{-i}{2}(\cdot)^2\cot\theta} f\bigr)(x)\overline{\bigl(h_\mu^\theta \psi_1\bigr)}(ax) \\
&\quad \times \overline{\bigl(h_\mu^\theta e^{\frac{-i}{2}(\cdot)^2\cot\theta} g\bigr)}(x)\bigl(h_\mu^\theta\psi_2\bigr)(ax)\,dx\,da \\
&= \sin^3\theta \int_0^\infty \bigl(h_\mu^\theta e^{\frac{-i}{2}(\cdot)^2\cot\theta} f\bigr)(x)\overline{\bigl(h_\mu^\theta e^{\frac{-i}{2}(\cdot)^2\cot\theta} g\bigr)}(x) \\
&\quad \times \biggl(\int_0^\infty (ax)^{-2\mu-2}\overline{\bigl(h_\mu^\theta \psi_1\bigr)}(ax)\bigl(h_\mu^\theta\psi_2\bigr)(ax)x\,da \biggr) dx \\
&= \sin^3\theta C_{\mu,\psi_1,\psi_2,\theta} \int_0^\infty \bigl(h_\mu^\theta f\bigr)(x)\overline{\bigl(h_\mu^\theta g\bigr)}(x)\,dx \\
&= \sin^3\theta C_{\mu,\psi_1,\psi_2,\theta} \bigl\langle h_\mu^\theta f, h_\mu^\theta g\bigr\rangle \\
&= \sin^2\theta C_{\mu,\psi_1,\psi_2,\theta} \langle f, g\rangle. \qquad \square
\end{aligned}
$$

Theorem 4.3 *If ψ is a wavelet and $(B_\psi^\theta f)(b,a)$ and $(B_\psi^\theta g)(b,a)$ are the continuous fractional Bessel wavelet transformations of $f, g \in L^2(\mathbb{R}_+)$ respectively, then*

$$
\int_0^\infty \int_0^\infty \bigl(B_\psi^\theta f\bigr)(b,a)\overline{\bigl(B_\psi^\theta g\bigr)}(b,a)\frac{db\,da}{a^2} = \sin^2\theta C_{\mu,\psi,\theta}\langle f, g\rangle.
$$

Proof The proof of Theorem 4.3 can be easily deduced by setting $\psi_1 = \psi_2 = \psi$ in Theorem 4.2. $\square$

Remark 4.2 If $f = g$ and $\psi_1 = \psi_2 = \psi$, then from Theorem 4.3, we have

$$
\int_0^\infty \int_0^\infty \bigl|\bigl(B_\psi^\theta f\bigr)(b,a)\bigr|^2 \frac{db\,da}{a^2} = \sin^2\theta C_{\mu,\psi,\theta}\|f\|_2^2.
$$

Theorem 4.4 *Let $f \in L^2(\mathbb{R}_+)$. Then f can be reconstructed by the formula*

$$
f(t) = \frac{1}{\sin^2\theta C_{\mu,\psi,\theta}} \int_0^\infty \int_0^\infty \bigl(B_\psi^\theta f\bigr)(b,a)\psi_{b,a}^\theta(t)\frac{db\,da}{a^2}, \quad a > 0.
$$

Proof For any $g \in L^2(\mathbb{R}_+)$, we have

$$
\begin{aligned}
\sin^2\theta C_{\mu,\psi,\theta}\langle f, g\rangle &= \int_0^\infty \int_0^\infty \bigl(B_\psi^\theta f\bigr)(b,a)\overline{\bigl(B_\psi^\theta g\bigr)}(b,a)\frac{db\,da}{a^2} \\
&= \int_0^\infty \int_0^\infty \bigl(B_\psi^\theta f\bigr)(b,a)\biggl(\overline{\int_0^\infty g(t)\overline{\psi_{b,a}^\theta(t)}\,dt}\biggr)\frac{db\,da}{a^2}
\end{aligned}
$$

$$= \int_0^\infty \left[\int_0^\infty \int_0^\infty \left(B_\psi^\theta f\right)(b,a)\psi_{b,a}^\theta(t)\frac{db\,da}{a^2}\right]\overline{g(t)}\,dt$$
$$= \left\langle \int_0^\infty \int_0^\infty \left(B_\psi^\theta f\right)(b,a)\psi_{b,a}^\theta(t)\frac{db\,da}{a^2}, g(t)\right\rangle.$$

Therefore,

$$f(t) = \frac{1}{\sin^2\theta\, C_{\mu,\psi,\theta}}\int_0^\infty \int_0^\infty \left(B_\psi^\theta f\right)(b,a)\psi_{b,a}^\theta(t)\frac{db\,da}{a^2}.$$

□

Theorem 4.5 *If* $\psi \in L^2(\mathbb{R}_+)$, *then*

$$\int_0^\infty \left[\left(B_\psi^\theta f\right)(b,a)\overline{\left(B_\psi^\theta g\right)(b,a)}\right]db = a^{-2\mu}\sin^2\theta\langle F, G\rangle,$$

where

$$\begin{cases} F(x) := e^{\frac{-i}{2}((2-a^2)x^2)\cot\theta}x^{-\mu-\frac{1}{2}}\left(h_\mu^\theta e^{\frac{-i}{2}(\cdot)^2\cot\theta}f\right)(x)\overline{\left(h_\mu^\theta\psi\right)}(ax), \\ G(x) := e^{\frac{-i}{2}((2-a^2)x^2)\cot\theta}x^{-\mu-\frac{1}{2}}\left(h_\mu^\theta e^{\frac{-i}{2}(\cdot)^2\cot\theta}\overline{g}\right)(x)\overline{\left(h_\mu^\theta\psi\right)}(ax). \end{cases}$$

Proof Using Theorem 4.1 and Theorem 4.2, we have

$$\begin{aligned} &\int_0^\infty \left[\left(B_\psi^\theta f\right)(b,a)\overline{\left(B_\psi^\theta g\right)(b,a)}\right]db = \int_0^\infty \langle f, \psi_{b,a}^\theta\rangle\overline{\langle g, \psi_{b,a}^\theta\rangle}\,db \\ &= \int_0^\infty \left(\int_0^\infty f(\omega)\overline{\psi_{b,a}^\theta}(\omega)\,d\omega\right)\left(\int_0^\infty \overline{g}(\sigma)\psi_{b,a}^\theta(\sigma)\,d\sigma\right)db \\ &= \int_0^\infty \left[\int_0^\infty f(\omega)\overline{\left(\frac{1}{\sqrt{a}}e^{\frac{-i}{2}(\frac{b^2}{a^2}+\frac{\omega^2}{a^2})\cot\theta}\int_0^\infty \psi(z)D_\mu^\theta\left(\frac{b}{a},\frac{\omega}{a},z\right)dz\right)}\,d\omega\right] \\ &\quad\times\left[\int_0^\infty \overline{g}(\sigma)\left(\frac{1}{\sqrt{a}}e^{\frac{-i}{2}(\frac{b^2}{a^2}+\frac{\sigma^2}{a^2})\cot\theta}\int_0^\infty \psi(z)D_\mu^\theta\left(\frac{b}{a},\frac{\sigma}{a},z\right)dz\right)d\sigma\right]db \\ &= \sin^3\theta a^{-2\mu}\int_0^\infty h_\mu^\theta\left(e^{\frac{-i}{2}((2-a^2)x^2)\cot\theta}x^{-\mu-\frac{1}{2}}\left(h_\mu^\theta e^{\frac{-i}{2}(\cdot)^2\cot\theta}f\right)(x)\overline{\left(h_\mu^\theta\psi\right)}(ax)\right)(b) \\ &\quad\times\overline{h_\mu^\theta\left(e^{\frac{-i}{2}((2-a^2)y^2)\cot\theta}y^{-\mu-\frac{1}{2}}\left(h_\mu^\theta e^{\frac{-i}{2}(\cdot)^2\cot\theta}\overline{g}\right)(y)\overline{\left(h_\mu^\theta\psi\right)}(ay)\right)}(b)\,db \\ &= \sin^3\theta a^{-2\mu}\int_0^\infty \left(h_\mu^\theta F\right)(b)\overline{\left(h_\mu^\theta G\right)}(b)\,db \\ &= \sin^3\theta a^{-2\mu}\langle h_\mu^\theta F, h_\mu^\theta G\rangle \\ &= \sin^2\theta a^{-2\mu}\langle F, G\rangle. \end{aligned}$$

This completes the proof of the theorem. □

Theorem 4.6 *If* $\psi \in L^2(\mathbb{R}_+)$ *is a Bessel wavelet and* f *is a bounded integrable function, then the convolution* $(\psi \star_\theta f)$ *is a fractional Bessel wavelet, where*

$$(\psi \star_\theta f)(x) = \int_0^\infty \left(\tau_x^\theta\psi\right)(y)y^{-(\mu+1/2)}f(y)\,dy.$$

Proof We have

$$(\psi \star_\theta f)(x) = \int_0^\infty \left(\tau_x^\theta \psi\right)(y) y^{-(\mu+1/2)} f(y)\,dy.$$

Therefore,

$$\begin{aligned}
&\left|(\psi \star_\theta f)(x)\right| \le \int_0^\infty \left|\left(\tau_x^\theta \psi\right)(y) y^{-(\mu+1/2)} f^{1/2}(y)\right|\left|f^{1/2}(y)\right|\,dy,\\
&\int_0^\infty \left|(\psi \star_\theta f)(x)\right|^2 dx\\
&\quad\le \int_0^\infty \left(\int_0^\infty \left|y^{-(\mu+1/2)} f^{1/2}(y)\left(\tau_x^\theta \psi\right)(y)\right|\left|f^{1/2}(y)\right|\,dy\right)^2 dx\\
&\quad\le \int_0^\infty \left[\left(\int_0^\infty \left|f(y)\right|\left|y^{-(\mu+1/2)}\left(\tau_x^\theta \psi\right)(y)\right|^2 dy\right)^{\frac{1}{2}} \left(\int_0^\infty \left|f(y)\right|\,dy\right)^{\frac{1}{2}}\right]^2 dx\\
&\quad= \left(\int_0^\infty \left|f(y)\right|\,dy\right)\left(\int_0^\infty \left(\int_0^\infty \left|y^{-(\mu+1/2)}\left(\tau_x^\theta \psi\right)(y)\right|^2 dx\right)\left|f(y)\right|\,dy\right)\\
&\quad\le \left(\int_0^\infty \left|f(y)\right|\,dy\right)^2 \left(\int_0^\infty \left|y^{-(\mu+1/2)}\left(\tau_y^\theta \psi\right)(x)\right|^2 dx\right).
\end{aligned}$$

This implies that

$$\left\|(\psi \star_\theta f)(x)\right\|_{L^2} \le \frac{1}{|(\sin\theta)^{\mu+1/2}| 2^\mu \Gamma(\mu+1)} \|\psi\|_{L^2} \|f\|_{L^1} < \infty.$$

We have $(\psi \star_\theta f) \in L^2(\mathbb{R}_+)$. Moreover,

$$\begin{aligned}
&\int_0^\infty x^{-2\mu-2}\left|h_\mu^\theta(\psi \star_\theta f)(x)\right|^2 dx\\
&\quad\le |\csc\theta| \int_0^\infty \left|x^{-3\mu-5/2}(h_\mu f)(x\csc\theta)(h_\mu \psi)(x\csc\theta)\right|^2 dx\\
&\quad\le |\csc\theta| \sup_x \left(x^{-\mu-1/2}\left|(h_\mu f)(x\csc\theta)\right|^2\right)\\
&\qquad\times \int_0^\infty x^{-2\mu-2}\left|(h_\mu \psi)(x\csc\theta)\right|^2 dx\\
&\quad= C'_{\mu,\psi,\theta} \sup_x \left(x^{-\mu-1/2}\left|(h_\mu f)(x\csc\theta)\right|^2\right) < \infty.
\end{aligned}$$

Thus, the convolution function $(\psi \star_\theta f)$ is a fractional Bessel wavelet. □

Theorem 4.7 *If $f, \psi \in L^2(\mathbb{R}_+)$ and $(B_\psi^\theta f)(b,a)$ is the continuous fractional Bessel wavelet transformation, then*

(i) $\left(B_\psi^\theta f\right)(b,a)$ *is continuous on* $\mathbb{R}_+ \times \mathbb{R}_+$,

(ii) $\left\|\left(B_\psi^\theta f\right)(b,a)\right\|_{L^\infty} \le \dfrac{b^{(\mu+1/2)} a^{-(\mu+1/2)}}{|(\sin\theta)^{\mu+1/2}| 2^\mu \Gamma(\mu+1)} \|f\|_{L^2} \|\psi\|_{L^2}.$

Proof (i) Let (b_0, a_0) be an arbitrary but fixed point in $\mathbb{R}_+ \times \mathbb{R}_+$. Then, by the Hölder inequality,

$$\begin{aligned}
&\left|\left(B^\theta_\psi f\right)(b,a) - \left(B^\theta_\psi f\right)(b_0,a_0)\right| \\
&\quad\le \frac{1}{\sqrt{a}} \int_0^\infty \int_0^\infty \left| f(x)\psi(z)\left[D^\theta_\mu\left(\frac{b}{a},\frac{x}{a},z\right) - D^\theta_\mu\left(\frac{b_0}{a_0},\frac{x}{a_0},z\right)\right]\right| dx\,dz \\
&\quad= \frac{1}{\sqrt{a}} \left(\int_0^\infty x^{-(\mu+1/2)} \left|f(x)\right|^2 dx \int_0^\infty z^{\mu+1/2}\left[\left|D^\theta_\mu\left(\frac{b}{a},\frac{x}{a},z\right) - D^\theta_\mu\left(\frac{b_0}{a_0},\frac{x}{a_0},z\right)\right|\right] dz\right)^{1/2} \\
&\qquad\times \left(\int_0^\infty z^{-(\mu+1/2)} \left|\psi(z)\right|^2 dz \int_0^\infty x^{\mu+1/2}\left[\left|D^\theta_\mu\left(\frac{b}{a},\frac{x}{a},z\right) - D^\theta_\mu\left(\frac{b_0}{a_0},\frac{x}{a_0},z\right)\right|\right] dx\right)^{1/2}.
\end{aligned}$$

Since

$$\begin{aligned}
&\int_0^\infty z^{\mu+1/2}\left[\left|D^\theta_\mu\left(\frac{b}{a},\frac{x}{a},z\right) - D^\theta_\mu\left(\frac{b_0}{a_0},\frac{x}{a_0},z\right)\right|\right] dz \\
&\quad\le \frac{x^{(\mu+1/2)}}{|(\sin\theta)^{\mu+1/2}|2^\mu\Gamma(\mu+1)}\left[\frac{b^{(\mu+1/2)}}{a^{2(\mu+1/2)}} - \frac{b_0^{(\mu+1/2)}}{a_0^{2(\mu+1/2)}}\right],
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^\infty x^{\mu+1/2}\left[\left|D^\theta_\mu\left(\frac{b}{a},\frac{x}{a},z\right) - D^\theta_\mu\left(\frac{b_0}{a_0},\frac{x}{a_0},z\right)\right|\right] dx \\
&\quad\le \frac{z^{(\mu+1/2)}}{|(\sin\theta)^{\mu+1/2}|2^\mu\Gamma(\mu+1)}\left[ab^{(\mu+1/2)} - a_0 b_0^{(\mu+1/2)}\right],
\end{aligned}$$

by the dominated convergence theorem and the continuity of $D^\theta_\mu(\frac{b}{a},\frac{x}{a},z)$ in the variable b and a, we have

$$\lim_{b\to b_0}\lim_{a\to a_0}\left|\left(B^\theta_\psi f\right)(b,a) - \left(B^\theta_\psi f\right)(b_0,a_0)\right| = 0.$$

This proves that $(B^\theta_\psi f)(b,a)$ is continuous on $\mathbb{R}_+ \times \mathbb{R}_+$.

(ii) We have

$$\left(B^\theta_\psi f\right)(b,a) = \int_0^\infty f(x)\overline{\left(\frac{1}{\sqrt{a}} e^{\frac{-i}{2}(\frac{b^2}{a^2}+\frac{x^2}{a^2})\cot\theta} \int_0^\infty \psi(z) D^\theta_\mu\left(\frac{b}{a},\frac{x}{a},z\right) dz\right)}\, dx.$$

Therefore, by the Hölder inequality, we have

$$\begin{aligned}
\left|\left(B^\theta_\psi f\right)(b,a)\right| &\le \frac{1}{\sqrt{a}}\left(\int_0^\infty\int_0^\infty x^{-(\mu+1/2)}\left|f(x)\right|^2 z^{(\mu+1/2)}\left|D^\theta_\mu\left(\frac{b}{a},\frac{x}{a},z\right)\right| dx\,dz\right)^{1/2} \\
&\quad\times\left(\int_0^\infty\int_0^\infty z^{-(\mu+1/2)}\left|\psi(z)\right|^2 x^{(\mu+1/2)}\left|D^\theta_\mu\left(\frac{b}{a},\frac{x}{a},z\right)\right| dx\,dz\right)^{1/2} \\
&= \frac{1}{\sqrt{a}}\left(\int_0^\infty x^{-(\mu+1/2)}\left|f(x)\right|^2 dx \int_0^\infty \left|D^\theta_\mu\left(\frac{b}{a},\frac{x}{a},z\right)\right| z^{(\mu+1/2)} dz\right)^{1/2} \\
&\quad\times\left(\int_0^\infty z^{-(\mu+1/2)}\left|\psi(z)\right|^2 dz \int_0^\infty \left|D^\theta_\mu\left(\frac{b}{a},\frac{x}{a},z\right)\right| x^{(\mu+1/2)} dx\right)^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\sqrt{a}}\left(\frac{b^{\mu+1/2}}{a^{2(\mu+1/2)}|(\sin\theta)^{\mu+1/2}|2^{\mu}\Gamma(\mu+1)}\right)^{1/2}\left(\int_0^{\infty}\left|f(x)\right|^2 dx\right)^{1/2} \\
&\quad\times\left(\frac{b^{\mu+1/2}}{a^{-1}|(\sin\theta)^{\mu+1/2}|2^{\mu}\Gamma(\mu+1)}\right)^{1/2}\left(\int_0^{\infty}\left|\psi(z)\right|^2 dz\right)^{1/2} \\
&= \frac{b^{(\mu+1/2)}a^{-(\mu+1/2)}}{|(\sin\theta)^{\mu+1/2}|2^{\mu}\Gamma(\mu+1)}\|f\|_{L^2}\|\psi\|_{L^2}. \qquad \square
\end{aligned}
$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

Author details
[1]Department of Applied Mathematics, Indian School of Mines, Dhanbad, 826004, India. [2]Department of Mathematics, NERIST, Nirjuli, India.

Acknowledgements
Dedicated to Professor Hari M Srivastava.

Received: 17 November 2012 Accepted: 25 January 2013 Published: 27 February 2013

References
1. Pathak, RS, Dixit, MM: Continuous and discrete Bessel wavelet transforms. J. Comput. Appl. Math. **160**(1-2), 241-250 (2003)
2. Haimo, DT: Integral equations associated with Hankel convolution. Trans. Am. Math. Soc. **116**, 330-375 (1965)
3. Hirschman, II Jr.: Variation diminishing Hankel transform. J. Anal. Math. **8**, 307-336 (1960-1961)
4. Upadhyay, SK, Yadav, RN, Debnath, L: On continuous Bessel wavelet transformation associated with the Hankel-Hausdorff operator. Integral Transforms Spec. Funct. **23**(5), 315-323 (2012)
5. Zemanian, AH: Generalized Integral Transformations. Interscience, New York (1968)
6. Namias, V: Fractionalization of Hankel transform. J. Inst. Math. Appl. **26**, 187-197 (1980)
7. Sheppard, CJR, Larkin, KG: Similarity theorems for fractional Fourier transforms and fractional Hankel transforms. Opt. Commun. **154**, 173-178 (1998)
8. Zhang, Y, Funaba, T, Tanno, N: Self-fractional Hankel functions and their properties. Opt. Commun. **176**, 71-75 (2000)
9. Kerr, FH: Fractional powers of Hankel transforms in the Zemanian spaces. J. Math. Anal. Appl. **166**, 65-83 (1992)
10. Debnath, L: Wavelet Transforms and Their Applications. Birkhäuser, Boston (2002)
11. Shi, J, Zhang, N, Liu, X: A novel fractional wavelet transform and its applications. Sci. China Inf. Sci. **55**(6), 1270-1279 (2012)
12. Betancor, JJ, Gonzales, BJ: A convolution operation for a distributional Hankel transformation. Stud. Math. **117**(1), 57-72 (1995)
13. Prasad, A, Singh, VK: The fractional Hankel transform of certain tempered distributions and pseudo-differential operators. Ann. Univ. Ferrara (2012). doi:10.1007/s11565-012-0169-1

doi:10.1186/1687-2770-2013-40
Cite this article as: Prasad et al.: **The continuous fractional Bessel wavelet transformation.** *Boundary Value Problems* 2013 **2013**:40.