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Coupling constant limits of Schrödinger operators with critical potentials

Xiaoyao Jia[1*] and Yan Zhao[2]

*Correspondence:
jiaxiaoyao@gmail.com
[1]Mathematics and Statistics School, Henan University of Science and Technology, No. 263, Luo-Long District, Kai-Yuan Road, Luoyang City, Henan Province 471023, China
Full list of author information is available at the end of the article

Abstract

A family of Schrödinger operators, $P(\lambda) = P_0 + \lambda V$, is studied in this paper. Here $P_0 = -\Delta + f(x)$ with $f(x) \sim \frac{1}{|x|^2}$ when $|x|$ is large enough and $V(x) = O(|x|^{-2-\epsilon})$ for some $\epsilon > 0$. We show that each discrete eigenvalue of $P(\lambda)$ tends to 0 when λ tends to some λ_0. We get asymptotic behavior of the smallest discrete eigenvalue when λ tends to λ_0.

Keywords: Schrödinger operator; critical potential; asymptotic expansion

1 Introduction

In this paper, we consider a family of Schrödinger operators $P(\lambda)$ which are the perturbation of P_0 in the form

$$P(\lambda) = P_0 + \lambda V \quad \text{for } \lambda \geq 0$$

on $L^2(\mathbb{R}^d)$, $d \geq 2$. Here $P_0 = -\Delta + \frac{q(\theta)}{r^2}$. (r, θ) are the polar coordinates on $\mathbb{R}^d$, and $q(\theta)$ is a real continuous function. $V \leq 0$ is a non-zero continuous function satisfying

$$\left|V(x)\right| \leq C\langle x\rangle^{-\rho_0} \quad \text{for some } \rho_0 > 2. \tag{1}$$

Here $\langle x\rangle = (1 + |x|^2)^{1/2}$. Let Δ_s denote the Laplace operator on the sphere $\mathbb{S}^{d-1}$. Assume that

$$-\Delta_s + q(\theta) > -\frac{1}{4}(d-2)^2 \quad \text{on } L^2\bigl(\mathbb{S}^{d-1}\bigr). \tag{2}$$

If (2) holds, then $P_0 \geq 0$ in $L^2(\mathbb{R}^d)$ (see [1]).

Under the assumption on V, we know that $P(\lambda)$ has discrete eigenvalues when λ is large enough, and each discrete eigenvalue tends to zero when λ tends to some λ_0 (see Section 2). We study the asymptotic behaviors of the discrete eigenvalues of $P(\lambda)$ in this paper. The asymptotic behaviors for Schrödinger operators with fast decaying potentials were studied by Klaus and Simon [2]. In [2], they studied the convergence rate of discrete eigenvalues of $H(\lambda) = -\Delta + \lambda V$ when $\lambda \to \lambda_0$. λ_0 is the value at which some discrete eigenvalue $e_i(\lambda)$ tends to zero. The main method they used in their paper is the Birman-Schwinger technique.

In order to use the Birman-Schwinger technique to $P(\lambda)$, we need to get the asymptotic expansion of $(P_0 - \alpha)^{-1}$ for α near zero, $\alpha < 0$, which was studied by Wang [1]. In this paper, we first show that there exists some λ_0 such that when $\lambda > \lambda_0$, $P(\lambda)$ has discrete eigenvalues. Then, we define the Birman-Schwinger kernel $K(\alpha)$ for $P(\lambda)$ and find that there is one-to-one correspondence between the discrete eigenvalues of $K(\alpha)$ and the discrete eigenvalues of $P(\lambda)$. Hence, the asymptotic expansion of the discrete eigenvalue of $P(\lambda)$ can be got through the asymptotic expansion of the discrete eigenvalue of $K(\alpha)$. In our main results, we need to use that $K(\alpha)$ is a bounded operator from L^2 to L^2. To get that, we add a strong condition on V (*i.e.*, $\rho_0 > 6$ in (1)). We show that $K(\alpha)$ is a family of compact operators converging to $K(0)$ and obtain the asymptotic expansions of the discrete eigenvalues of $K(\alpha)$ by functional calculus. After that, the convergence rate of the smallest discrete eigenvalue of $P(\lambda)$ is obtained.

Here is the plan of our work. In Section 2, we recall some results of P_0 and define the Birman-Schwinger kernel $K(\alpha)$ for $P(\lambda)$. The relationship between the eigenvalues of these two kinds of operators is studied. In Section 3, we first study the asymptotic behavior of the discrete eigenvalues of $K(\alpha)$. Then the convergence rate of the smallest discrete eigenvalue of $P(\lambda)$ is obtained. We get the leading term and the estimate of the remainder term of the smallest discrete eigenvalue.

Let us introduce some notations first.

Notation The scalar product on $L^2(\mathbb{R}^+; r^{d-1}\,dr)$ and $L^2(\mathbb{R}^d)$ is denoted by $\langle\cdot,\cdot\rangle$ and that on $L^2(\mathbb{S}^{d-1})$ by $(\cdot,\cdot)$. $H^{r,s}(\mathbb{R}^d)$, $r \in \mathbb{Z}$, $s \in \mathbb{R}$, denotes the weighted Sobolev space of order r with volume element $\langle x\rangle^{2s}\,dx$. The duality between $H^{1,s}$ and $H^{-1,-s}$ is identified with the L^2 product. Denote $H^{0,s} = L^{2,s}$. Notation $\mathcal{L}(H^{r,s}, H^{r',s'})$ stands for the space of continuous linear operators from $H^{r,s}$ to $H^{r',s'}$. The complex plane $\mathbb{C}$ is slit along positive real axis so that $z^\nu = e^{\nu \ln z}$ and $\ln z = \ln|z| + i\arg z$ with $0 < \arg z < 2\pi$ are holomorphic there.

2 Some results for P_0

Assume that (r,θ) are the polar coordinates on $\mathbb{R}^d$. Then the condition

$$-\Delta_s + q(\theta) > -\frac{1}{4}(d-2)^2 \quad \text{on } L^2\left(\mathbb{S}^{d-1}\right)$$

implies

$$-\Delta + \frac{q(\theta)}{r^2} \geq 0 \tag{3}$$

in $L^2(\mathbb{R}^d)$ (see [1]).

Now, we recall some results on the resolvent and the Schrödinger group for the unperturbed operator P_0. Let

$$\sigma_\infty = \left\{\nu; \nu = \sqrt{\lambda + \frac{(d-2)^2}{4}}, \lambda \in \sigma\left(-\Delta_s + q(\theta)\right)\right\}.$$

Denote

$$\sigma_k = \sigma_\infty \cap [0,k], \quad k \in \mathbb{N}.$$

For $\nu \in \sigma_\infty$, let n_ν denote the multiplicity of $\lambda_\nu = \nu^2 - \frac{(d-2)^2}{4}$ as the eigenvalue of $-\Delta_s + q(\theta)$. Let $\varphi_\nu^{(j)}$, $\nu \in \sigma_\infty$, $1 \le j \le n_\nu$ denote an orthogonal basis of $L^2(\mathbb{S}^{d-1})$ consisting of eigenfunctions of $-\Delta_s + q(\theta)$:

$$\bigl(-\Delta_s + q(\theta)\bigr)\varphi_\nu^{(j)} = \lambda_\nu \varphi_\nu^{(j)}, \qquad \bigl(\varphi_\nu^{(i)}, \varphi_\nu^{(j)}\bigr) = \delta_{ij}.$$

Let π_ν denote the orthogonal projection in $L^2(\mathbb{S}^{d-1})$ onto the subspace spanned by the eigenfunctions of $-\Delta_s + q(\theta)$ associated with the eigenvalue λ_ν, and let $\pi_\nu^{(i)}$ denote the orthogonal projection in $L^2(\mathbb{S}^{d-1})$ onto the eigenfunction $\varphi_\nu^{(i)}$:

$$\pi_\nu f = \sum_{j=1}^{n_\nu} \bigl(f, \varphi_\nu^{(j)}\bigr) \otimes \varphi_\nu^{(j)}, \quad f \in L^2\bigl(\mathbb{S}^{n-1}\bigr),$$
$$\pi_\nu^{(i)} f = \bigl(f, \varphi_\nu^{(i)}\bigr) \otimes \varphi_\nu^{(i)}, \quad f \in L^2\bigl(\mathbb{S}^{d-1}\bigr), 1 \le i \le n_\nu.$$

Denote for $\nu \in \sigma_\infty$

$$z_\nu = \begin{cases} z^{\nu'} & \text{if } \nu \notin \mathbb{N}, \\ z \ln z & \text{if } \nu \in \mathbb{N}. \end{cases}$$

Here $\nu' = \nu - [\nu]$, and $[\nu]$ is the largest integer which is not larger than ν. For $\nu > 0$, let $[\nu]_-$ be the largest integer strictly less than ν. When $\nu = 0$, set $[\nu]_- = 0$. Define δ_ν by $\delta_\nu = 1$, if $\nu \in \sigma_\infty \cap \mathbb{N}$, $\delta_\nu = 0$, otherwise. One has $[\nu] = [\nu]_- + \delta_\nu$.

The following is the asymptotic expansion for the resolvent $R_0(z) = (P_0 - z)^{-1}$.

Theorem 2.1 (Theorem 2.2 [1]) *The following asymptotic expansion holds for z near zero with $\Im z > 0$:*

$$R_0(z) = \delta_0 \ln z G_{0,0}\pi_0 + \sum_{j=0}^{N} z^j F_j + \sum_{\nu \in \sigma_N} z_\nu \sum_{j=[\nu]_-}^{N-1} z^j G_{\nu, j+\delta_\nu} \pi_\nu + R_0^{(N)}(z)$$

in $\mathcal{L}(-1, s; 1, -s)$, $s > 2N + 1$. Here

$$G_{\nu,j}(r,\tau) = \begin{cases} b_{\nu',j}(r\tau)^{j+\nu'} f_{j-[\nu]}(r,\tau;\nu'), & \nu \notin \mathbb{N}, \\ -\frac{(ir\tau)^j}{j!} f_{j-[\nu]}(r,\tau;0), & \nu \in \mathbb{N}, \end{cases}$$
$$F_j \in \mathcal{L}(-1, s; 1, -s), \quad s > 2j + 1,$$
$$R_0^{(N)}(z) = O\bigl(|z|^{N+\epsilon}\bigr) \in \mathcal{L}(-1, s; 1, -s), \quad s > 2N + 1, \epsilon > 0.$$

Here

$$b_{\nu',j} = -\frac{i^j e^{-i\nu'\pi/2}\Gamma(1-\nu')}{\nu'(\nu'+1)\cdots(\nu'+j)}$$

for $0 \le \nu' < 1$, and

$$f_j(r,\tau,\nu) = (r\tau)^{-\frac{1}{2}(n-2)} P_{j,\nu}(\rho)$$

with $P_{j,\nu}(\rho)$ a polynomial in ρ of degree j:

$$P_{j,\nu}(\rho) = \frac{i^j a_\nu}{j!} \int_{-1}^{1} \left(\rho + \frac{1}{2}\theta\right)^j \left(1-\theta^2\right)^{\nu-\frac{1}{2}} \, d\theta, \quad a_\nu = -\frac{e^{-i\pi\nu/2}}{2^{2\nu+1}\pi^{1/2}\Gamma(\nu+1/2)}.$$

First, we show that $P(\lambda)$ has discrete eigenvalues when λ is large enough. In fact, we need only to show that there exists a function $\psi \in L^2(\mathbb{R}^d)$ such that $\langle\psi, P(\lambda)\psi\rangle < 0$.

From the assumption on V, we know that there exists a point $x_0 \in \mathbb{R}^d$ such that $V(x_0) = \inf_{x\in\mathbb{R}^d} V(x)$. Choose $\delta > 0$ small enough such that for all $x \in B(x_0, \delta)$, $V(x) < \frac{1}{2}V(x_0)$. For $\psi \in C_0^\infty(\mathbb{R}^d)$, $\|\psi(x)\| = 1$, supp $\psi \subset B(x_0, \delta)$, one has

$$\langle\psi, P(\lambda)\psi\rangle = \langle\psi, P_0\psi\rangle + \lambda\langle\psi, V\psi\rangle < \langle\psi, P_0\psi\rangle + \frac{\lambda}{2}V(x_0),$$

when λ is large enough, one has $\langle\psi, P(\lambda)\psi\rangle < 0$. This means that $P(\lambda)$ has discrete eigenvalues when λ is large enough.

$P(\lambda)$ has a continuous spectrum $[0,\infty)$ for $\lambda \geq 0$ because $\lim_{|x|\to\infty} V(x)$ exists and equals zero (see [3]). We know that $\sigma(P(0)) = \sigma(P_0) = [0,\infty)$. Hence, from the continuity of a discrete spectrum of $P(\lambda)$, we know that there exists some λ_0 such that when $\lambda > \lambda_0$, $P(\lambda)$ has eigenvalues less than zero, and when $\lambda \leq \lambda_0$, $\sigma(P(\lambda)) = [0,\infty)$. So, $P(\lambda)$ has an eigenvalue $e_1(\lambda) < 0$ at the bottom of its spectrum for $\lambda > \lambda_0$. In Section 3 (Proposition 3.1), we prove that $e_1(\lambda)$ is simple and the corresponding eigenfunction can be chosen to be positive everywhere. (There are many results about the simplicity of the smallest eigenvalue of the Schrödinger operator without singularity, but there is no result which can be used directly, because the potential we use in this paper has singularity at zero. Theorem XIII.48 [4] can treat the Schrödinger operator with the potential which has singularity at zero, but the positivity of potential is needed. Hence, we give this result.) From the discussion above and the continuity of a discrete spectrum, one has that $e_1(\lambda)$ tends to zero at some λ. The asymptotic behavior of $e_1(\lambda)$ is studied in this paper.

To study the eigenvalues of $P(\lambda)$, we first define a family of Birman-Schwinger kernel operators. Let

$$K(z) = |V|^{1/2}(P_0 - z)^{-1}|V|^{1/2}$$

for $z \notin \sigma(P_0)$, and

$$K(0) = |V|^{1/2}F_0|V|^{1/2}.$$

Then we have the following result.

Proposition 2.2 *Let $\alpha < 0$. Then*
(a) *Let*

$$A = \{\psi \in L^2(\mathbb{R}^d); (P(\lambda) - \alpha)\psi = 0\},$$
$$B = \{\phi \in L^2(\mathbb{R}^d); K(\alpha)\phi = \lambda^{-1}\phi\}.$$

Then $|V|^{1/2}$ is injective from A to B, and $(P_0 - \alpha)^{-1}|V|^{1/2}$ is injective from B to A.

(b) *The multiplicity of* α *as the eigenvalue of* $P(\lambda)$ *is exactly the multiplicity of* λ^{-1} *as the eigenvalue of* $K(\alpha)$.

Proof

(a) First, we prove that $|V|^{1/2}$ is injective from A to B. Note that if $\psi \in A$, then

$$K(\alpha)\phi = \lambda^{-1}\phi$$

with $\phi = |V|^{1/2}\psi$. And if $\phi = 0$, then

$$\psi = -\lambda(P_0 - \alpha)^{-1}V\psi = \lambda(P_0 - \alpha)^{-1}|V|^{1/2}\phi = 0.$$

It follows that $|V|^{1/2}$ is injective from A to B.

Next, we show that $(P_0 - \alpha)^{-1}|V|^{1/2}$ is injective from B to A. If $\phi \in B$, then

$$\bigl(P(\lambda) - \alpha\bigr)\psi = 0, \quad \text{with } \psi = (P_0 - \alpha)^{-1}|V|^{1/2}\phi.$$

And if $\psi = 0$, then

$$0 = |V|^{1/2}\psi = K(\alpha)\phi = \lambda^{-1}\phi.$$

It follows that $(P_0 - \alpha)^{-1}|V|^{1/2}$ is injective from B to A.

(b) From (a), one has $\dim A = \dim B$. This means that the multiplicity of α as the eigenvalue of $P(\lambda)$ is exactly the multiplicity of λ^{-1} as the eigenvalue of $K(\alpha)$. □

From the last proposition, we know that there exists one-to-one correspondence between the discrete eigenvalues of $P(\lambda)$ and the discrete eigenvalues of $K(\alpha)$. Hence, we can study the eigenvalues of $K(\alpha)$ first.

3 Asymptotic expansion of the eigenvalues

If P_0 and V are defined as above, we show that if $P_0 + V$ has the eigenvalue less than zero, then the smallest eigenvalue of $P_0 + V$ is simple. We use Theorems XIII.44, XIII.45 [4] to prove it.

Proposition 3.1 *Suppose* $P_0 + V$ *has an eigenvalue at the bottom of its spectrum. Then this eigenvalue is simple and the corresponding eigenfunction can be chosen to be a positive function.*

Proof Let $0 \le \chi(t) \le 1$ be a smooth nonincreasing function such that $\chi(t) = 1$ if $|t| < 1$ and $\chi(t) = 0$ if $t > 2$. Let $\chi_n(t) = \chi(t/n)$. Set $V_n = \chi_n(r)\frac{q(\theta)}{r^2} + V - \langle x\rangle$, $H_0 = -\Delta + \langle x\rangle$, $H_n = H_0 + V_n$, $H = P = P_0 + V$. From the proof of Theorem XIII.47 [4], we know that e^{-tH_0} is positivity preserving and $\{e^{-tH_0}\} \cup L^\infty$ acts irreducibly on L^2. Hence, by Theorem XIII.45 [4], if H_n converges to H and $H - V_n$ converges to H_0 in the strong resolvent sense, then e^{-tH} is positivity preserving and $\{e^{-tH}\} \cup L^\infty$ acts irreducibly on L^2. By Theorems XIII.43 and XIII.44 [4], we can get the result. Since $C_0^\infty(\mathbb{R}^d)$ is the core for all P_n and P, and for any $\psi \in C_0^\infty(\mathbb{R}^d)$, $V_n\psi \to (\frac{q(\theta)}{r^2} + V)\psi$ in L^2, then we have the necessary strong resolvent convergence by Theorem VIII.25(a) [5]. This ends the proof. □

Proposition 3.2 *Assume that* $0 \notin \sigma_\infty$. $P_0F_0u = u$ *in* $H^{-1,s}$ *for any* $u \in H^{-1,s}$, $s > 1$.

Proof If $u \in H^{-1,s}$, then $F_0u \in H^{1,-s}$. For any test function $\phi \in C_0^\infty(\mathbb{R}^n)$, we have $\langle P_0F_0u, \phi\rangle = \langle u, F_0P_0\phi\rangle$. If $0 \notin \sigma_\infty$, then we have $\lim_{z\to 0}(P_0 - z)^{-1} = F_0$ in $H^{-1,s}$ for $\Im z > 0$. It follows $\langle u, F_0P_0\phi\rangle = \lim_{z\to 0}\langle u, (P_0 - z)^{-1}P_0\phi\rangle = \lim_{z\to 0}\langle u, \phi - z(P_0 - z)^{-1}\phi\rangle = \langle u, \phi\rangle$ because ϕ and $P_0\phi$ belong to $H^{-1,s}$. Hence, $P_0F_0u = u$ in $H^{-1,s}$. □

Proposition 3.3 *Assume that* $0 \notin \sigma_\infty$. $K(\alpha)$ *is a compact operator for* $\alpha \le 0$. *And* $K(\alpha)$ *converges to* $K(0)$ *in operator norm sense.*

Proof For $\alpha < 0$, $K(\alpha) = |V|^{1/2}(P_0 - \alpha)^{-1}|V|^{1/2}$. Since $(P_0 - \alpha)^{-1}$ is a bounded operator from $L^2(\mathbb{R}^d)$ to $H^1(\mathbb{R}^d)$, and V is a compact operator from $H^1(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$, then $V(P_0 - \alpha)^{-1}$ is a compact operator on $L^2(\mathbb{R}^d)$. Using a similar method to that in Proposition 2.2, we can show that $V(P_0 - \alpha)^{-1}$ and $K(\alpha)$ have the same non-zero eigenvalues, and for the same eigenvalue $e(\alpha)$, the multiplicity of $e(\alpha)$ as the eigenvalue of $V(P_0 - \alpha)^{-1}$ and the multiplicity of $e(\alpha)$ as the eigenvalue of $K(\alpha)$ are the same. Hence, $K(\alpha)$ is a compact operator. Because

$$K(\alpha) - K(0) = |V|^{1/2}\left[(P_0 - \alpha)^{-1} - F_0\right]|V|^{1/2} = |V|^{1/2}R_0^{(0)}|V|^{1/2}$$

and if $\rho_0 > 2$, then $|V|^{1/2}R_0^{(0)}|V|^{1/2} = o(|\alpha|^\epsilon)$ in $L^2(\mathbb{R}^d)$. Hence, $K(\alpha) \to K(0)$ in operator norm sense as $\alpha \to 0$. This means that $K(0)$ is a compact operator. □

Lemma 3.4 *Suppose* A_1, A_2 *are two bounded self-adjoint operators on a Hilbert space* H. *Set*

$$\mu_n(A_i) = \inf_{\phi_1,\dots,\phi_n} \sup_{\|\psi\|=1, \psi\in[\phi_1,\dots,\phi_n]^\perp} (\psi, A_i\psi),$$

then $|\mu_n(A_1) - \mu_n(A_2)| \le \|A_1 - A_2\|$.

Proof By the definition of $\mu_n(A_i)$, one has

$$\begin{aligned}
&\left|\mu_n(A_1) - \mu_n(A_2)\right| \\
&\quad = \left|\inf_{\phi_1,\dots,\phi_n} \sup_{\|\psi\|=1,\psi\in[\phi_1,\dots,\phi_n]^\perp} \langle\psi, A_1\psi\rangle - \inf_{\phi_1,\dots,\phi_n} \sup_{\|\psi\|=1,\psi\in[\phi_1,\dots,\phi_n]^\perp} \langle\psi, A_2\psi\rangle\right| \\
&\quad = \left|-\sup_{\phi_1,\dots,\phi_n}\left(-\sup_{\|\psi\|=1,\psi\in[\phi_1,\dots,\phi_n]^\perp} \langle\psi, A_1\psi\rangle\right) + \sup_{\phi_1,\dots,\phi_n}\left(-\sup_{\|\psi\|=1,\psi\in[\phi_1,\dots,\phi_n]^\perp} \langle\psi, A_2\psi\rangle\right)\right| \\
&\quad \le \left|\sup_{\phi_1,\dots,\phi_n}\left[-\sup_{\|\psi\|=1,\psi\in[\phi_1,\dots,\phi_n]^\perp} \langle\psi, A_1\psi\rangle + \sup_{\|\psi\|=1,\psi\in[\phi_1,\dots,\phi_n]^\perp} \langle\psi, A_2\psi\rangle\right]\right| \\
&\quad = \sup_{\phi_1,\dots,\phi_n}\left|\sup_{\|\psi\|=1,\psi\in[\phi_1,\dots,\phi_n]^\perp} \langle\psi, A_1\psi\rangle - \sup_{\|\psi\|=1,\psi\in[\phi_1,\dots,\phi_n]^\perp} \langle\psi, A_2\psi\rangle\right| \\
&\quad \le \sup_{\phi_1,\dots,\phi_n} \sup_{\|\psi\|=1,[\phi_1,\dots,\phi_n]^\perp} \left|\langle\psi, -A_1\psi\rangle - \langle\psi, -A_2\psi\rangle\right| \\
&\quad \le \|A_1 - A_2\|.
\end{aligned}$$

This ends the proof. □

Lemma 3.5 *Suppose $T(\alpha)$ is a family of compact self-adjoint operators on a separable Hilbert space H, and $T(\alpha) = T_0 + o(|\alpha|^\epsilon)$ for α near zero. Set*

$$\mu_k(\alpha) = \inf_{\phi_1,\dots,\phi_k} \sup_{\|\psi\|=1,\psi\in[\phi_1,\dots,\phi_k]^\perp} \langle \psi, T(\alpha)\psi \rangle.$$

Then:

(a) *$\mu_k(\alpha)$ is an eigenvalue of $T(\alpha)$, and $\mu_k(\alpha)$ converges when $\alpha \to 0$. Moreover, if $\mu_k(\alpha) \to \mu_k$, then μ_k is an eigenvalue of T_0.*

(b) *Suppose that $E_0 \neq 0$ is an eigenvalue of T_0 of the multiplicity of m. Then there are m eigenvalues (counting multiplicity), $E_k(\alpha)$ $(1 \le k \le m)$, of $T(\alpha)$ near E_0. Moreover, we can choose $\{\phi_k(\alpha); 1 \le k \le m\}$ such that $(\phi_k(\alpha), \phi_j(\alpha)) = \delta_{kj}$ $(1 \le k, j \le m)$, $\phi_k(\alpha)$ is the eigenvector of $T(\alpha)$ corresponding to $E_k(\alpha)$ $(E_k(\alpha) \to E_0)$, and $\phi_k(\alpha)$ converges as $\alpha \to 0$. If $\phi_k(\alpha)$ converges to ϕ_k, then ϕ_k is the eigenvector of T_0 corresponding to E_0.*

Proof

(a) By the min-max principle, we know that $\mu_k(\alpha)$ is an eigenvalue of $T(\alpha)$. By Lemma 3.4, one has

$$\left|\mu_k(\alpha) - \mu_k(0)\right| \le \left\|T(\alpha) - T_0\right\| = O\left(|\alpha|^\epsilon\right).$$

It follows that $\mu_k(\alpha)$ converges to the eigenvalue of T_0.

(b) Because T_0 is a compact operator and $E_0 \neq 0$ is an eigenvalue of T_0, then E_0 is a discrete spectrum of T_0. Then there exists a constant $\delta > 0$ small enough such that T_0 has only one eigenvalue E_0 in $B(E_0, \delta)$ $(= \{z \in \mathbb{C}; |z - E_0| < \delta\})$. For α small enough, $T(\alpha)$ has exactly m eigenvalues (counting multiplicity) in $B(E_0, \delta)$ because the eigenvalues of $T(\alpha)$ converge to the eigenvalues of T_0 by part (a) of lemma. Suppose the m eigenvalues, near E_0, of $T(\alpha)$ are $E_1(\alpha), E_2(\alpha), \dots, E_m(\alpha)$, and the corresponding eigenvectors are $\psi_1(\alpha), \psi_2(\alpha), \dots, \psi_m(\alpha)$ such that $\langle \psi_k(\alpha), \psi_j(\alpha)\rangle = \delta_{kj}$. Let

$$P_\alpha = -\frac{1}{2\pi i} \oint_{|E-E_0|=\delta} \left(T(\alpha) - E\right)^{-1} dE.$$

Then $P_\alpha = \sum_{k=1}^m \langle \cdot, \psi_k(\alpha)\rangle \psi_k(\alpha)$. Let $P_\alpha^{(k)} = \langle \cdot, \psi_k(\alpha)\rangle \psi_k(\alpha)$, then $P_\alpha = \sum_{k=1}^m P_\alpha^{(k)}$. For α near zero, one has

$$\begin{aligned}
\left\|P_\alpha - P_0\right\| &= \left\| -\frac{1}{2\pi i} \oint_{|E-E_0|=\delta} \left(T(\alpha) - E\right)^{-1} - (T_0 - E)^{-1}\, dE \right\| \\
&= \left\| -\frac{1}{2\pi i} \oint_{|E-E_0|=\delta} \left(T(\alpha) - E\right)^{-1} (T_0 - T_\alpha)(T_0 - E)^{-1}\, dE \right\| \\
&= O\left(|\alpha|^\epsilon\right).
\end{aligned}$$

Let $A = \{\phi; \|\phi\| = 1, \phi \in \operatorname{Ran} P_0\}$. Let ϕ_k be an element in A such that $\|\phi - \psi_k(\alpha)\|$ acquires the minimum value. Then we have

$$\begin{aligned}
\left\|P_\alpha^{(k)}\phi_k - \phi_k\right\| &\le \left\|P_\alpha^{(k)}\phi_k - \psi_k(\alpha)\right\| + \left\|\psi_k(\alpha) - \phi_k\right\| \\
&= \left\|P_\alpha^{(k)}\phi_k - P_\alpha^{(k)}\psi_k(\alpha)\right\| + \left\|\psi_k(\alpha) - \phi_k\right\| \le 2\left\|\psi_k(\alpha) - \phi_k\right\|,
\end{aligned}$$

and

$$\begin{aligned}\left\|\psi_k(\alpha)-\phi_k\right\| &\le \left\|\frac{P_0\psi_k(\alpha)}{\|P_0\psi_k(\alpha)\|}-\psi_k(\alpha)\right\| \\ &\le \left\|\frac{P_0\psi_k(\alpha)}{\|P_0\psi_k(\alpha)\|}-P_0\psi_k(\alpha)\right\| + \left\|P_0\psi_k(\alpha)-P_\alpha\psi_k(\alpha)\right\| = O\big(|\alpha|^\epsilon\big).\end{aligned}$$

In the last equality, we use the fact

$$\left\|P_0\psi_k(\alpha)\right\| = \left\|\big(P_0\psi_k(\alpha)-\psi_k(\alpha)\big)+\psi_k(\alpha)\right\| = 1+O\big(|\alpha|^\epsilon\big),$$

and

$$\left\|\frac{P_0\psi_k(\alpha)}{\|P_0\psi_k(\alpha)\|}-P_0\psi_k(\alpha)\right\| = \left\|P_0\psi_k(\alpha)\left(\frac{1}{\|P_0\psi_k(\alpha)\|}-1\right)\right\| = O\big(|\alpha|^\epsilon\big).$$

It follows that $\|P_\alpha^{(k)}\phi_k-\phi_k\| = O(|\alpha|^\epsilon)$. Let $\phi_k(\alpha) = \frac{P_\alpha^{(k)}\phi_k}{\|P_\alpha^{(k)}\phi_k\|}$. Then $\langle\phi_k(\alpha),\phi_j(\alpha)\rangle = 0$ for $k \neq j$ because $P_\alpha^{(k)}P_\alpha^{(j)} = 0$ if $k\neq j$, and $\|\phi_k(\alpha)-\phi_k\| \le \|P_\alpha^{(k)}\phi_k-\phi_k\| + \|(1-\frac{1}{\|P_\alpha^{(k)}\phi_k\|})P_\alpha^{(k)}\phi_k\| = O(|\alpha|^\epsilon)$. This ends the proof. □

Let $0<\alpha_1<\alpha_2<\cdots<\alpha_i<\cdots<\alpha_m$ and

$$T(\beta) = T_0+\sum_{i=1}^m \beta^{\alpha_i}(\ln\beta)^{\delta_i}T_i + T_r(\beta).$$

Here, $\delta_i = 0$ or 1, $T_0 \ge 0$, T_i $(1\le i\le m)$, $T_r(\beta)$ are compact operators, and $T_r(\beta) = O(|\beta|^{\alpha_m+\epsilon})$ for β near zero. Set

$$e_s = \inf_{\phi_1,\ldots,\phi_s}\ \sup_{\|\psi\|=1,\psi\in[\phi_1,\ldots,\phi_s]^\perp}\langle\psi,T_0\psi\rangle.$$

Then, by the min-max principle, e_s is an eigenvalue of T_0. Moreover, if $e_s\neq 0$, then e_s is a discrete eigenvalue of T_0 because T_0 is a compact operator. If $e_s\neq 0$ is an eigenvalue of T_0 of multiplicity m, without loss, we can suppose that $e_s = e_{s+1} = \cdots = e_{s+m-1}$. Then there exist exactly m eigenvalues (counting multiplicity), $e_s(\beta), e_{s+1}(\beta), \ldots, e_{s+m-1}(\beta)$, of $T(\beta)$ near e_s. By Lemma 3.5, we know that there exists a family of normalized eigenvectors $\{\phi_j(\beta); j = s, s+1, \ldots, s+m-1\}$ of $T(\beta)$ such that $T(\beta)\phi_j(\beta) = e_j(\beta)\phi_j(\beta)$, $\langle\phi_j(\beta),\phi_k(\beta)\rangle = \delta_{jk}$ $(j,k = s, s+1,\ldots,s+m-1)$, and $\phi_j(\beta)$ $(j = s, s+1, \ldots, s+m-1)$ converge as $\beta\to 0$. Suppose that $\phi_j(\beta)$ converge to ϕ_j for all j such that $e_j\neq 0$. Then $\langle\phi_s,\phi_j\rangle = \delta_{sj}$. $\{\phi_s\}$ can be extended to a standard orthogonal basis. Set

$$T_1(\beta) = \sum_{s=1}^m \beta^{\alpha_s}(\ln\beta)^{\delta_i}T_s + T_r(\beta), \qquad T_{sj}(\beta) = \big\langle\phi_s, T_1(\beta)\phi_j\big\rangle.$$

Then we have the following.

Lemma 3.6 *$T(\beta)$, e_s are given as before. Then the eigenvalue of $T(\beta)$, $e_j(\beta)$ $(j = s, s+1, \ldots, s+m-1)$ has the following form*:

$$e_j(\beta) = e_s + \frac{\sum_{n=0}^{\infty} a_n^{(j)}(\beta)}{\sum_{n=0}^{\infty} b_n^{(j)}(\beta)}.$$

Here

$$\begin{aligned}
a_0^{(j)}(\beta) &= T_{jj}(\beta), \\
a_1^{(j)}(\beta) &= -\sum_{\{k;e_k\neq e_s\}} (e_k - e_s)^{-1} T_{jk}(\beta) T_{kj}(\beta), \\
a_2^{(j)}(\beta) &= \sum_{k\neq j\neq l} (e_k - e_s)^{-1}(e_l - e_s)^{-1} T_{jk}(\beta) T_{kl}(\beta) T_{lj}(\beta) \\
&\quad - 2\sum_{\{k;e_k\neq e_s\}} (e_k - e_s)^{-1} T_{jk}(\beta) T_{kj}(\beta) T_{jj}(\beta), \\
a_n^{(j)}(\beta) &= -\frac{(-1)^n}{2\pi i} \oint_{|E-e_s|=\delta} (e_s - E)^{-1} \sum_{i_1,i_2,\ldots,i_n} (e_{i_1} - E)^{-1} \cdots (e_{i_n} - E)^{-1} \\
&\quad \times T_{ji_1} T_{i_1 i_2} \cdots T_{i_n j}\, dE \quad \textit{for } n > 2, \\
b_0^{(j)}(\beta) &= 1, \\
b_1^{(j)}(\beta) &= 0, \\
b_2^{(j)}(\beta) &= \sum_{\{k;e_k\neq e_s\}} (e_s - e_k)^{-2} T_{jk}(\beta) T_{kj}(\beta), \\
b_n^{(j)}(\beta) &= -\frac{(-1)^n}{2\pi i} \oint_{|E-e_s|=\delta} (e_s - E)^{-2} \sum_{i_1,i_2,\ldots,i_{n-1}} (e_{i_1} - E)^{-1} \cdots (e_{i_{n-1}} - E)^{-1} \\
&\quad \times T_{ji_1} T_{i_1 i_2} \cdots T_{i_{n-1} j}\, dE \quad \textit{for } n > 2.
\end{aligned}$$

Proof If $e_s \neq 0$, then e_s is the discrete eigenvalue of T_0. Suppose that the multiplicity of e_s is m, and suppose that $e_s = e_{s+1} = \cdots = e_{s+m-1}$ as before. Hence, we can choose $\delta > 0$ small enough such that there is only one eigenvalue e_s in $B(e_s, \delta) = \{z \in \mathbb{C}; |z - e_s| < \delta\}$. We know that $e_j(\beta)$ $(j = s, s+1, \ldots, s+m-1)$ converge to e_s. It follows that if δ is small enough, there are exactly m eigenvalues (counting multiplicity) of $T(\beta)$ in $B(e_s, \delta)$ for β small. Set

$$P_\beta \triangleq -\frac{1}{2\pi i} \oint_{|E-e_s|=\delta} \bigl(T(\beta) - E\bigr)^{-1}\, dE.$$

Then

$$\begin{aligned}
e_j(\beta) &= \frac{\langle \phi_j, T(\beta) P_\beta \phi_j\rangle}{\langle \phi_j, P_\beta \phi_j\rangle} \\
&= \frac{\langle \phi_j, T_0 P_\beta \phi_j\rangle}{\langle \phi_j, P_\beta \phi_j\rangle} + \frac{\langle \phi_j, T_1(\beta) P_\beta \phi_j\rangle}{\langle \phi_j, P_\beta \phi_j\rangle} \\
&= e_s + \frac{\langle \phi_j, T_1(\beta) P_\beta \phi_j\rangle}{\langle \phi_j, P_\beta \phi_j\rangle}.
\end{aligned}$$

Since

$$\begin{aligned}\big(T(\beta)-E\big)^{-1} &= (T_0-E)^{-1}\big(I+T_1(\beta)(T_0-E)^{-1}\big)^{-1}\\ &= (T_0-E)^{-1}\sum_{n=0}^{\infty}(-1)^n\big[T_1(\beta)(T_0-E)^{-1}\big]^n,\end{aligned}$$

then

$$\langle\phi_j,P_\beta\phi_j\rangle = -\frac{1}{2\pi i}\oint_{|E-e_s|=\delta}\left\langle\phi_j,(T_0-E)^{-1}\sum_{n=0}^{\infty}(-1)^n\big[\big(T_1(\beta)\big)(T_0-E)^{-1}\big]^n\phi_j\right\rangle dE.$$

Then

$$b_n^{(j)}(\beta) = -\frac{(-1)^n}{2\pi i}\oint_{|E-e_s|=\delta}\big\langle\phi_j,(T_0-E)^{-1}\big[\big(T_1(\beta)\big)(T_0-E)^{-1}\big]^n\phi_j\big\rangle\,dE.$$

In particular,

$$\begin{aligned}
b_0^{(j)}(\beta) &= -\frac{1}{2\pi i}\oint_{|E-e_s|=\delta}\big\langle\phi_j,(T_0-E)^{-1}\phi_j\big\rangle\,dE = 1,\\
b_1^{(j)}(\beta) &= -\frac{-1}{2\pi i}\oint_{|E-e_s|=\delta}\big\langle\phi_j,(T_0-E)^{-1}\big[\big(T_1(\beta)\big)(T_0-E)^{-1}\big]\phi_j\big\rangle\,dE\\
&= -\frac{-1}{2\pi i}\oint_{|E-e_s|=\delta}(e_s-E)^{-2}\big\langle\phi_j,T_1(\beta)\phi_j\big\rangle\,dE = 0,\\
b_2^{(j)}(\beta) &= -\frac{(-1)^2}{2\pi i}\oint_{|E-e_s|=\delta}\big\langle\phi_j,(T_0-E)^{-1}\big[\big(T_1(\beta)\big)(T_0-E)^{-1}\big]^2\phi_j\big\rangle\,dE\\
&= -\frac{1}{2\pi i}\oint_{|E-e_s|=\delta}(e_s-E)^{-2}\big\langle\phi_j,\big[\big(T_1(\beta)\big)(T_0-E)^{-1}\big(T_1(\beta)\big)\big]\phi_j\big\rangle\,dE\\
&= -\frac{1}{2\pi i}\oint_{|E-e_s|=\delta}(e_s-E)^{-2}\sum_k\big\langle\phi_j,\big(T_1(\beta)\big)\phi_k\big\rangle(e_k-E)^{-1}\big\langle\phi_j,\big(T_1(\beta)\big)\phi_k\big\rangle\,dE\\
&= -\frac{1}{2\pi i}\oint_{|E-e_s|=\delta}(e_s-E)^{-2}\sum_{\{k;e_k\neq e_s\}}\big\langle\phi_j,\big(T_1(\beta)\big)\phi_k\big\rangle(e_k-E)^{-1}\big\langle\phi_j,\big(T_1(\beta)\big)\phi_k\big\rangle\,dE\\
&\quad -\frac{1}{2\pi i}\oint_{|E-e_s|=\delta}(e_s-E)^{-2}\sum_{\{k;e_k=e_s\}}\big\langle\phi_j,\big(T_1(\beta)\big)\phi_k\big\rangle(e_k-E)^{-1}\big\langle\phi_j,\big(T_1(\beta)\big)\phi_k\big\rangle\,dE\\
&= \sum_{\{k;e_k\neq e_s\}}(e_s-e_k)^{-2}T_{jk}(\beta)T_{kj}(\beta).
\end{aligned}$$

In the last step, we use that

$$\begin{aligned}\sum_{\{k;e_k\neq e_s\}}\big|(e_k-e_j)^{-2}T_{jk}(\beta)T_{kj}(\beta)\big| &\le C\sum_{\{k;e_k\neq e_s\}}\big(\big|T_{jk}(\beta)\big|^2+\big|T_{kj}(\beta)\big|^2\big)\\ &\le C\big(\big\|T_1^*(\beta)\phi_j\big\|^2+\big\|T_1(\beta)\phi_j\big\|^2\big).\end{aligned}$$

Similarly, we can get

$$\begin{aligned} b_n^{(j)}(\beta) = {} & -\frac{(-1)^n}{2\pi i}\oint_{|E-e_s|=\delta}(e_s-E)^{-2}\sum_{i_1,i_2,\ldots,i_{n-1}}(e_{i_1}-E)^{-1}\cdots(e_{i_{n-1}}-E)^{-1} \\ & \times T_{ji_1}T_{i_1i_2}\cdots T_{i_{n-1}j}\,dE; \end{aligned}$$

and

$$\begin{aligned} a_0^{(j)}(\beta) = {} & T_{jj}(\beta), \\ a_1^{(j)}(\beta) = {} & -\sum_{\{k;e_k\neq e_s\}}(e_k-e_s)^{-1}T_{jk}(\beta)T_{kj}(\beta), \\ a_2^{(j)}(\beta) = {} & \sum_{\{l,k;e_k\neq e_j\neq e_l\}}(e_k-e_s)^{-1}(e_l-e_s)^{-1}T_{jk}(\beta)T_{kl}(\beta)T_{lj}(\beta) \\ & -2\sum_{\{k;e_k\neq e_s\}}(e_k-e_s)^{-1}T_{jk}(\beta)T_{kj}(\beta)T_{jj}(\beta), \\ a_n^{(j)}(\beta) = {} & -\frac{(-1)^n}{2\pi i}\oint_{|E-e_s|=\delta}\bigl\langle\phi_j,\bigl[\bigl(T_1(\beta)\bigr)(T_0-E)^{-1}\bigr]^{n+1}\phi_j\bigr\rangle\,dE \\ = {} & -\frac{(-1)^n}{2\pi i}\oint_{|E-e_s|=\delta}(e_s-E)^{-1}\bigl\langle\phi_j,\bigl[\bigl(T_1(\beta)\bigr)(T_0-E)^{-1}\bigr]^{n}T_1(\beta)\phi_j\bigr\rangle\,dE \\ = {} & -\frac{(-1)^n}{2\pi i}\oint_{|E-e_s|=\delta}(e_s-E)^{-1}\sum_{i_1}\bigl\langle\phi_j,\bigl[\bigl(T_1(\beta)\bigr)(T_0-E)^{-1}\bigr]^{n-1} \\ & \times T_1(\beta)\phi_{i_1}\bigr\rangle(e_s-E)^{-1}T_{i_1j}(\beta)\,dE \\ = {} & \cdots \\ = {} & -\frac{(-1)^n}{2\pi i}\oint_{|E-e_s|=\delta}(e_s-E)^{-1}\sum_{i_1,i_2,\ldots,i_n}(e_{i_1}-E)^{-1}\cdots(e_{i_n}-E)^{-1} \\ & \times T_{ji_1}T_{i_1i_2}\cdots T_{i_nj}\,dE. \end{aligned}$$

□

First, we study the asymptotic expansion of the smallest eigenvalue $e_1(\lambda)$ of $P(\lambda)$. By Proposition 3.1, we know that $e_1(\lambda)$ is a simple eigenvalue of $P(\lambda)$, and the corresponding eigenfunction can be chosen to be positive. We suppose $u(\lambda)$ is a positive eigenfunction corresponding to $e_1(\lambda)$. Then $\tilde{u}(\lambda) = |V|^{1/2}u(\lambda) \in L^2(\mathbb{R}^d)$. Without loss of generality, we can suppose that $\|\tilde{u}(\lambda)\|_{L^2(\mathbb{R}^d)} = 1$. Then we can get the following result.

Lemma 3.7 *Assume that* $0 \notin \sigma_\infty$. *Set* $\nu_0 = \min\{\nu; \nu \in \sigma_\infty\}$. $\tilde{u}$ *is defined as above. Then* $\tilde{u}(\lambda)$ *converges in* $L^2(\mathbb{R}^d)$ *when* $\lambda \to \lambda_0$. *If* $\nu_0 < 1$ *and* $\tilde{u}(\lambda)$ *converges to* ϕ, *then* ϕ *is the eigenfunction of* $K(0)$, *and* $\langle\phi, |V|^{1/2}G_{\nu_0,0}\pi_{\nu_0}|V|^{1/2}\phi\rangle \neq 0$.

Proof By the assumption of $\tilde{u}(\lambda)$, one has $K(e_1(\lambda))\tilde{u}(\lambda) = \lambda^{-1}\tilde{u}(\lambda)$. One can check that $\tilde{u}(\lambda)$ converges in $L^2(\mathbb{R}^d)$ as $\lambda \to \lambda_0$ by Lemma 3.5. And also, by Lemma 3.5, we know that ϕ is the normalized eigenfunction of $K(0)$ corresponding to E_0. ϕ is a positive function since $\tilde{u}(\lambda)$ is a positive function. Let $u = F_0|V|^{1/2}\phi$, then $P(\lambda_0)u = 0$ and u is a positive function because $|V|^{1/2}u = |V|^{1/2}F_0|V|^{1/2}\phi = K(0)\phi = \lambda_0^{-1}\phi$. Then

$$\begin{aligned}
&\langle \phi, |V|^{1/2}G_{\nu_0,0}\pi_{\nu_0}|V|^{1/2}\phi\rangle \\
&\quad = \lambda_0^2\langle |V|^{1/2}u, |V|^{1/2}G_{\nu_0,0}\pi_{\nu_0}|V|^{1/2}|V|^{1/2}u\rangle \\
&\quad = \lambda_0^2\langle Vu, G_{\nu_0,0}\pi_{\nu_0}Vu\rangle \\
&\quad = \lambda_0^2 C_{\nu_0}\left|\langle Vu, |y|^{-\frac{n-2}{2}+\nu_0}\varphi_{\nu_0}\rangle\right|^2 \\
&\quad \neq 0.
\end{aligned}$$

In the last equality, we use the fact that

$$G_{\nu_0,0} = (r\tau)^{\nu_0}b_{\nu_0,0}f_0 = d_{\nu_0}b_{\nu_0,0}(r\tau)^{-\frac{d-2}{2}+\nu_0} = C_{\nu_0}(r\tau)^{-\frac{d-2}{2}+\nu_0}$$

with $d_{\nu_0} = -\frac{e^{-\frac{1}{2}i\pi\nu_0}}{2^{2\nu_0+1}\Gamma(\nu_0+1)}$, and $C_{\nu_0} = d_{\nu_0}b_{\nu_0,0}$. This ends the proof. □

Theorem 3.8 *Assume that* $0 \notin \sigma_\infty$. ϕ *is defined in Lemma* 3.7. *If* $\rho_0 > 6$, *one of three exclusive situations holds*:

(a) *If* $\sigma_1 = \emptyset$, *then*

$$e_1(\lambda) = -c(\lambda - \lambda_0) + o(\lambda - \lambda_0)$$

with $c = (\lambda_0\|F_0|V|^{\frac{1}{2}}\phi\|)^{-2} \neq 0$.

(b) *If* $\nu_0 = 1$, *then*

$$e_1(\lambda) = -c\frac{\lambda - \lambda_0}{\ln(\lambda - \lambda_0)} + o\left(\frac{\lambda - \lambda_0}{\ln(\lambda - \lambda_0)}\right)$$

with $c = \lambda_0^{-2}\langle \phi, |V|^{1/2}G_{1,0}\pi_1|V|^{1/2}\phi\rangle^{-1} \neq 0$.

(c) *If* $\nu_0 < 1$, *then*

$$e_1(\lambda) = c\big((\lambda - \lambda_0)^{\frac{1}{\nu_0}}\big) + o\big((\lambda - \lambda_0)^{\frac{1}{\nu_0}}\big)$$

with $c = \lambda_0^{-2}\langle \phi, |V|^{1/2}G_{\nu_0,0}\pi_{\nu_0}|V|^{1/2}\phi\rangle^{-1} \neq 0$.

Proof

(a) By Theorem 2.1, one has

$$R_0(\alpha) = F_0 + \alpha F_1 + R_0^{(1)}(\alpha), \quad \text{in } \mathcal{L}(-1, s; 1, -s), s > 3.$$

Then if $\rho_0 > 6$, we can get $K(\alpha) = K(0) + |V|^{1/2}(\alpha F_1 + R_0^{(1)}(\alpha))|V|^{1/2}$ in $\mathcal{L}(0,0;0,0)$. Because $e_1(\lambda)$ is the simple eigenvalue of $P(\lambda)$, then λ^{-1} is the simple eigenvalue of $K(e_1(\lambda))$. Since $V \le 0$, one has that $P(\lambda)$ is monotonous with respect to λ and so is the $e_1(\lambda)$. Hence, $K(e_1(\lambda))$ and the eigenvalues of $K(e_1(\lambda))$ are monotonous with respect to λ. Therefore, we have that λ^{-1} is the biggest eigenvalue of $K(e_1(\lambda))$. If not, suppose that $a > \lambda^{-1}$ is an eigenvalue of $K(e_1(\lambda))$, then by the continuity and monotony of the eigenvalue of $K(e_1(\lambda))$ with respect to λ, we know that there exists a constant $\lambda' < \lambda$ such that $\lambda \in \sigma(K(e_1(\lambda')))$.

It follows that $e_1(\lambda') < e_1(\lambda)$ is an eigenvalue of $P_0 + \lambda V$. This is contradictory to that $e_1(\lambda)$ is the smallest eigenvalue. By Lemma 3.7, we know the normalized eigenfunction $\tilde{u}(\lambda)$ of $K(e_1(\lambda))$ converges to ϕ. It follows $\tilde{u}(\lambda) = \frac{P_\lambda \phi}{\|P_\lambda \phi\|}$ with $P_\lambda = -\frac{1}{2\pi i}\oint_{|E-E_0|=\delta}(K(e_1(\lambda)) - E)^{-1}\,dE$. Then

$$\mu\big(e_1(\lambda)\big) = \big\langle \tilde{u}(\lambda), K\big(e_1(\lambda)\big)\tilde{u}(\lambda)\big\rangle = \frac{\langle \phi, K(e_1(\lambda))P_\lambda \phi\rangle}{\langle \phi, P_\lambda \phi\rangle}.$$

Here $\mu(e_1(\lambda))$ is the eigenvalue of $K(e_1(\lambda))$ corresponding to the eigenfunction $\tilde{u}(\lambda)$. By Lemma 3.6, we should compute $\langle \phi, |V|^{1/2}F_1|V|^{1/2}\phi\rangle$. Let $\psi = F_0|V|^{1/2}\phi$. From the definition of ϕ, one has $K(0)\phi = \lambda_0^{-1}\phi$. Hence,

$$(P_0 + \lambda_0 V)\psi = P_0F_0|V|^{1/2}\phi + \lambda_0 VF_0|V|^{1/2}\phi = 0.$$

In the last equality, we use the fact $P_0F_0|V|^{1/2}\phi = |V|^{1/2}\phi$, which can be obtained by Proposition 3.2. Since $\nu_0 > 1$, we have $\psi \in L^2(\mathbb{R}^d)$ by Theorem 3.1 [1]. So, ψ is the ground state of $P(\lambda_0)$. We also have

$$\begin{aligned}
&|V|^{1/2}\psi = K(0)\phi = \lambda_0^{-1}\phi,\\
&(P_0 - \alpha)^{-1}|V|\psi = \lambda_0^{-1}\big(\psi + \alpha R_0(\alpha)\psi\big).
\end{aligned}$$

Hence,

$$\begin{aligned}
|V|^{1/2}F_1|V|^{1/2}\phi &= \lambda_0|V|^{1/2}F_1|V|^{1/2}|V|^{1/2}\psi\\
&= \lambda_0\alpha^{-1}|V|^{1/2}\big(R_0(\alpha) - F_0\big)|V|\psi + O\big(|\alpha|^{\epsilon}\big)\\
&= \lambda_0\alpha^{-1}|V|^{1/2}\lambda_0^{-1}\big(\psi + \alpha R_0(\alpha)\psi - \psi\big) + O\big(|\alpha|^{\epsilon}\big)\\
&= |V|^{1/2}R_0(\alpha)\psi + O\big(|\alpha|^{\epsilon}\big).
\end{aligned}$$

It follows

$$\begin{aligned}
\big\langle \phi, |V|^{1/2}F_1|V|^{1/2}\phi\big\rangle &= \lim_{\alpha\to 0}\big\langle |V|^{1/2}\phi, R_0(\alpha)\psi\big\rangle\\
&= \big\langle F_0|V|^{1/2}\phi, \psi\big\rangle\\
&= \|\psi\|^2 \neq 0.
\end{aligned}$$

So, $\mu(\alpha) = \lambda_0^{-1} + c_1\alpha + o(|\alpha|^{1+\epsilon})$ with $c_1 = \|F_0|V|^{\frac{1}{2}}\phi\|^2$. By the Proposition 2.2, one has $\mu(e_1(\lambda)) = \lambda^{-1}$. It follows

$$\lambda^{-1} = \lambda_0^{-1} + c_1 e_1(\lambda) + O\big(|e_1(\lambda)|^{1+\epsilon}\big).$$

Since $\lambda^{-1} = \lambda_0^{-1} - \lambda_0^{-2}(\lambda - \lambda_0) + O(|\lambda - \lambda_0|^2)$, we can get the leading term of $e_1(\lambda)$ is $-c(\lambda - \lambda_0)$ with $c = (\lambda_0\|F_0|V|^{\frac{1}{2}}\phi\|)^{-2}$.

(b) If $\nu_0 = 1$, then

$$K(\alpha) = K(0) + \alpha\ln\alpha|V|^{1/2}G_{1,0}\pi_1|V|^{1/2} + O(\alpha).$$

By Lemma 3.7, one has

$$\langle\phi, |V|^{1/2} G_{1,0}\pi_1 |V|^{1/2}\phi\rangle \neq 0.$$

Then we have

$$\mu(\alpha) = \lambda_0^{-1} + c_1\alpha \ln\alpha + o(\alpha)$$

with $c_1 = \langle\phi, |V|^{1/2} G_{1,0}\pi_1 |V|^{1/2}\phi\rangle$. As in (a), using $\mu(e_1(\lambda)) = \lambda^{-1}$ and

$$\lambda^{-1} = \lambda_0^{-1} - \lambda_0^{-2}(\lambda - \lambda_0) + O(|\lambda - \lambda_0|^2),$$

one has $-\lambda^{-2}(\lambda - \lambda_0) + O(|\lambda - \lambda_0|^2) = ce_1(\lambda)\ln e_1(\lambda) + o(e_1(\lambda))$. To get the leading term of $e_1(\lambda)$, we can suppose that $e_1(\lambda) = (\lambda - \lambda_0)f(\lambda)$. Then, by comparing the leading term, we can get $f(\lambda) = 1/\ln(\lambda - \lambda_0)$. It follows

$$e_1(\lambda) = -c\frac{\lambda - \lambda_0}{\ln(\lambda - \lambda_0)} + o\left(\frac{\lambda - \lambda_0}{\ln(\lambda - \lambda_0)}\right)$$

with $c = \lambda_0^{-2}\langle\phi, |V|^{1/2} G_{1,0}\pi_1 |V|^{1/2}\phi\rangle^{-1}$.

(c) If $\nu_0 < 1$, one has

$$K(\alpha) = K(0) + \sum_{0<\nu\leq 1} \alpha^\nu |V|^{1/2} G_{\nu,0}\pi_\nu |V|^{1/2} + O(|\alpha|).$$

By Lemma 3.7, we know that $\langle\phi, |V|^{1/2} G_{\nu_0,0}\pi_{\nu_0} |V|^{1/2}\phi\rangle^{-1} \neq 0$. Using the same argument as before, we can conclude

$$\mu(\alpha) = \lambda_0^{-1} + c_1\alpha^{\nu_0} + o(|\alpha|^{\nu_0})$$

with $c_1 = \langle\phi, |V|^{1/2} G_{\nu_0,0}\pi_{\nu_0} |V|^{1/2}\phi\rangle$. As above, we can get that

$$e_1(\lambda) = c(\lambda - \lambda_0)^{\frac{1}{\nu_0}} + o((\lambda - \lambda_0)^{\frac{1}{\nu_0}})$$

with $c = \lambda_0^{-2}\langle\phi, |V|^{1/2} G_{\nu_0,0}\pi_{\nu_0} |V|^{1/2}\phi\rangle^{-1}$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally to the manuscript and read and approved the final manuscript.

Author details
[1] Mathematics and Statistics School, Henan University of Science and Technology, No. 263, Luo-Long District, Kai-Yuan Road, Luoyang City, Henan Province 471023, China. [2] College of Civil Engineering and Architecture, Zhejiang University, B505 Anzhong Building, 866 Yuhangtang Road, Hangzhou, Zhejiang Province 310058, China.

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