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Time-periodic solutions for a driven sixth-order Cahn-Hilliard type equation

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Abstract
We study a driven sixth-order Cahn-Hilliard type equation which arises naturally as a continuum model for the formation of quantum dots and their faceting. Based on the Leray-Schauder fixed point theorem, we prove the existence of time-periodic solutions.
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1 Introduction

In this paper, we are concerned with the following problem for the sixth-order Cahn-Hilliard type equation:

$$\frac{\partial u}{\partial t} - \gamma D^6 u = D^4 \psi(u,t) + \nu u D u + f(x,t), \quad (x,t) \in Q, \tag{1.1}$$

$$u|_{x=0,1} = D^2 u|_{x=0,1} = D^4 u|_{x=0,1} = 0, \quad t \geq 0, \tag{1.2}$$

$$u(x,0) = u(x,T), \quad x \in (0,1), \tag{1.3}$$

where $Q \equiv (0,1) \times (0,+\infty)$, $D = \frac{\partial}{\partial x}$, $\psi(u,t) = -a(t)u^3 + b(t)u$, $a(t)$ and $b(t)$ are Hölder continuous functions defined on $\mathbb{R}^+$ with period T, $f(x,t)$ belongs to the space $C^{\alpha,\frac{\alpha}{4}}(\overline{Q})$ for some $\alpha \in (0,1)$ with $f(x,0) = f(x,T)$. Furthermore, we assume that $\underline{M} \leq a(t) \leq \overline{M}$, $|b(t)| \leq N$, $|a'(t)| \leq L$, $|b'(t)| \leq \Lambda$, where γ, ν, $\underline{M}$, $\overline{M}$, N, L and Λ are positive constants.

Equation (1.1) with $f(x,t) = 0$ arises naturally as a continuum model for the formation of quantum dots and their faceting; see [1]. It can also be used to describe competition and exclusion of biological population [2]. If we consider that the perturbation function $f(x,t)$ (for example, source) has the influence, then we obtain equation (1.1).

Korzec *et al.* [3] studied equation (1.1) with $f(x,t) = 0$. New types of stationary solutions of one-dimensional driven sixth-order Cahn-Hilliard type equation (1.1) are derived by an extension of the method of matched asymptotic expansions that retains exponentially small terms. Liu *et al.* [4] proved that equation (1.1) with $f(x,t) = 0$ possesses a global attractor in the H^k ($k \geq 0$) space, which attracts any bounded subset of $H^k(\Omega)$ in the H^k-norm.

During the past years, many authors have paid much attention to other sixth-order thin film equations such as the existence, uniqueness and regularity of the solutions [5–7].

However, as far as we know, there are few investigations concerned with the time-periodic solutions of equation (1.1), even though there is some literature for population models and Cahn-Hilliard [8, 9]. In fact, it is natural to consider the time-periodic solutions of equation (1.1) when it is used to describe the models of the growth and dispersal in the population which is sensitive to time-periodic factors (for example, seasons). In this paper, we prove the existence of time-periodic solutions of problem (1.1)-(1.3) based on the framework of the Leray-Schauder fixed point theorem which can be found in any standard textbook of PDE (see, for example, [10]). For this purpose, we first introduce an operator $\mathcal{L}$ by considering a linear sixth-order equation with a parameter $\sigma \in [0,1]$. After verifying the compactness of the operator and some necessary *a priori* estimates for the solutions, we then obtain a fixed point of the operator in a suitable functional space with $\sigma = 1$, which is the desired solution of problem (1.1)-(1.3).

The main difficulties for treating problem (1.1)-(1.3) are caused by the nonlinearity of both the fourth-order term and the convective factors. The main method that we use is based on the Schauder-type *a priori* estimates, which here are obtained by means of a modified Campanato space. We note that the Campanato spaces have been widely used for the discussion of partial regularity of solutions of parabolic systems of second order and fourth order. So, in the following section we give a detailed description and the associated properties of such a space, and subsequently, in the next section we prove the existence of classical time-periodic solutions of problem (1.1)-(1.3).

2 Hölder norm estimates

Let $Q_T = (0,1) \times (0,T)$, $y_0 = (x_0, t_0) \in \overline{Q}_T$. For any fixed $R > 0$, we define

$$\begin{aligned}
&B_R = B_R(x_0) = (x_0 - R, x_0 + R), && I_R = I_R(t_0) = \left(t_0 - R^6, t_0 + R^6\right),\\
&Q_R = Q_R(y_0) = I_R(t_0) \times B_R(x_0), && S_R = Q_R \cap Q_T,\\
&E_R = E_R(x_0) = B_R(x_0) \cap (0,1), && J_R = J_R(t_0) = I_R(t_0) \cap (0,T).
\end{aligned}$$

Let u be a function defined on Q_T, and set

$$u_R = u_{y_0,R} = \frac{1}{|S_R|} \iint_{S_R} u \, dx \, dt, \qquad \hat{u}_R = \hat{u}_{y_0,R} = \begin{cases} u_R & \text{if } Q_R \cap \partial_p Q_T = \emptyset, \\ 0 & \text{if } Q_R \cap \partial_p Q_T \neq \emptyset, \end{cases}$$

where $\partial_p Q_T$ denotes the parabolic boundary of Q_T and $|S_R|$ denotes the area of S_R.

For any $u \in C(\overline{Q}_T)$ and $\lambda > 0$, define

$$M^2[u] = \sup_{y_0 \in \overline{Q}_T} \sup_{0<R\leq R_0} \frac{1}{R^\lambda} \iint_{S_R(y_0)} \left|u(x,t) - \hat{u}_{y_0,R}\right|^2 dx \, dt,$$

where $R_0 = \operatorname{diam} Q_T$. By the space $\mathcal{L}^{2,\lambda}(Q_T)$ we mean the subset of $C(\overline{Q}_T)$, each element of which satisfies $M[u] < +\infty$. For $u \in \mathcal{L}_0^{2,\lambda}$, its norm is defined as

$$\|u\|_{\mathcal{L}_0^{2,\lambda}}(Q_T) = \sup_{Q_T}\left|u(x,t)\right| + M[u].$$

Now, we give some useful lemmas.

Lemma 2.1 [11] *Let* $\lambda > 7$,

$$\|u\|_{C^{\alpha,\frac{\alpha}{6}}} \le C(\lambda)\|u\|_{\mathcal{L}_0^{2,\lambda}}(Q_T),$$

where $\alpha = \frac{\lambda-7}{2}$.

Now we consider the following linear periodical problem:

$$\frac{\partial u}{\partial t} - \gamma D^6 u = \Phi(x,t), \quad (x,t) \in Q_T = (0,1) \times (0,T), \tag{2.1}$$

$$u|_{x=0,1} = D^2 u|_{x=0,1} = D^4 u|_{x=0,1} = 0, \quad t \in (0,T), \tag{2.2}$$

$$u(x,0) = u(x,T), \quad x \in (0,1). \tag{2.3}$$

Here we simply assume that $\Phi(x,t)$ is sufficiently smooth. Our main purpose is to find the relation between the Hölder norm of the solution u and $\Phi(x,t)$.

Let $y_0 = (x_0,t_0) \in \overline{Q}_T$ be a fixed point and define

$$\varphi(u,\rho) = \iint_{S_\rho} \left(\left|u(x,t) - \hat{u}_\rho\right|^2 + \rho^6 \left|D^3 u(x,t)\right|^2\right) dx\,dt \quad (\rho > 0).$$

Let u be an arbitrary solution of problem (2.1)-(2.3). We split u on $S_R = S_R(y_0)$ as $u = u_1 + u_2$ so that u_1 solves the problem

$$\frac{\partial u_1}{\partial t} - \gamma D^6 u_1 = 0, \quad (x,t) \in S_R, \tag{2.4}$$

$$\int_{E_R} u_1(x,t)\,dx = \int_{E_R} u(x,t)\,dx, \quad t \in (0,T), \tag{2.5}$$

$$u_1|_{\partial_1 J_R} - u_1|_{\partial_2 J_R} = u|_{\partial_1 J_R} - u|_{\partial_2 J_R}, \qquad P^i(x,D)u_1|_{\partial E_R} = P^i(x,D)u|_{\partial E_R}, \tag{2.6}$$

and u_2 solves the problem

$$\frac{\partial u_2}{\partial t} - \gamma D^6 u_2 = \Phi(x,t), \quad (x,t) \in S_R, \tag{2.7}$$

$$\int_{E_R} u_2(x,t)\,dx = 0, \quad t \in (0,T), \tag{2.8}$$

$$u_2|_{\partial_1 J_R} - u_2|_{\partial_2 J_R} = P^i(x,D)u_2|_{\partial E_R} = 0, \tag{2.9}$$

where

$$P^i(x,D) = \begin{cases} D^i & \text{if } x = 0,1, \\ D^{i+1} & \text{if } x \neq 0,1, \end{cases} \quad i = 0,2,4$$

and $\partial_1 J_R$, $\partial_2 J_R$ are the down-side and up-side points of J_R, and ∂E_R is the boundary of E_R.

Some essential estimates on u_1 and u_2 are based on the following lemmas.

Lemma 2.2 *For the solution* u_2 *of problem* (2.7)-(2.9), *we have*

$$\iint_{S_R} \left(D^i u_2\right)^2 dx\,dt + R^6 \iint_{S_R} \left(D^{i+3} u_2\right)^2 dx\,dt \le CR^{12-2i} \iint_{S_R} \Phi^2\,dx\,dt, \tag{2.10}$$

where C *is a positive constant,* $i = 0,1,2$.

Proof Noticing the condition (2.8) and the boundary value condition (2.9), we use the Poincaré inequality and interpolation method (see Chapter 5 in [12]) and get

$$\begin{aligned}
&\iint_{S_R} u_2^2\,dx\,dt \le CR^2 \iint_{S_R} (Du_2)^2\,dx\,dt, \\
&\iint_{S_R} (Du_2)^2\,dx\,dt \le \frac{\varepsilon}{R^2} \iint_{S_R} u_2^2\,dx\,dt + CR^4 \iint_{S_R} \bigl(D^3u_2\bigr)^2\,dx\,dt,
\end{aligned} \tag{2.11}$$

which implies that

$$\iint_{S_R} u_2^2\,dx\,dt \le CR^6 \iint_{S_R} \bigl(D^3u_2\bigr)^2\,dx\,dt. \tag{2.12}$$

Multiplying equation (2.7) by u_2, integrating the result over S_R and using the boundary value condition (2.9), we have

$$\gamma \iint_{S_R} \bigl(D^3u_2(x,t)\bigr)^2\,dx\,dt = \iint_{S_R} \Phi(x,t)u_2(x,t)\,dx\,dt. \tag{2.13}$$

Using the Young inequality and (2.12), we obtain

$$\begin{aligned}
\left|\iint_{S_R} \Phi(x,t)u_2(x,t)\,dx\,dt\right| &\le \frac{\varepsilon}{R^6} \iint_{S_R} u_2^2\,dx\,dt + CR^6 \iint_{S_R} \Phi^2\,dx\,dt \\
&\le \varepsilon \iint_{S_R} \bigl(D^3u_2\bigr)^2\,dx\,dt + CR^6 \iint_{S_R} \Phi^2\,dx\,dt.
\end{aligned} \tag{2.14}$$

Combining (2.12), (2.13) and (2.14) yields the estimate (2.10) with $i = 0$.

Similarly, multiplying (2.7) by D^2u_2 and D^4u_2, we can obtain the estimates (2.10) with $i = 1$, $i = 2$. □

Lemma 2.3 *For any* $(x_1,t),(x_2,t),(x,t_1),(x,t_2) \in S_\rho$,

$$\bigl|u_1(x_1,t) - u_1(x_2,t)\bigr|^2 \le CM(u_1,\rho)|x_1 - x_2|, \tag{2.15}$$

$$\bigl|u_1(x,t_1) - u_1(x,t_2)\bigr|^2 \le CM(u_1,\rho)|t_1 - t_2|^{1/6}, \tag{2.16}$$

where

$$M(u_1,\rho) = \sup_{t\in J_\rho} \int_{E_\rho} \bigl(Du_1(x,t)\bigr)^2\,dx + \iint_{S_\rho} \bigl(D^4u_1\bigr)^2\,dx\,dt,$$

and C is a constant number. Further, (2.15) *and* (2.16) *still hold if u_1 is replaced by Du_1 or D^2u_1.*

Proof The estimate (2.15) is obvious. In fact, by the Hölder inequality,

$$\begin{aligned}
\bigl|u_1(x_1,t) - u_1(x_2,t)\bigr|^2 &= \left|\int_{x_1}^{x_2} Du_1(x,t)\,dx\right|^2 \\
&\le \left(\int_{x_1}^{x_2} \bigl|Du_1(x,t)\bigr|\,dx\right)^2 \le M(u_1,\rho)|x_1 - x_2|.
\end{aligned}$$

For (2.16), we only consider the case when $\Delta t = t_1 - t_2 > 0$, $x, x + 3(\Delta t)^{1/6} \in E_\rho$. Integrating equation (2.4) over the region $(t_1, t_2) \times (z, z + (\Delta t)^{1/6})$, we have

$$\int_z^{z+(\Delta t)^{1/6}} \bigl[u_1(\xi, t_2) - u_1(\xi, t_1)\bigr]\,d\xi - \gamma \int_{t_1}^{t_2} \bigl[D^5 u_1\bigl(z + (\Delta t)^{1/6}, s\bigr) - D^5 u_1(z, s)\bigr]\,ds = 0.$$

Integrating the above equation with respect to z over $(y, y + (\Delta t)^{1/6})$, and then integrating the result with respect to y over $(x, x + (\Delta t)^{1/6})$, we have

$$\begin{aligned}
&\int_x^{x+(\Delta t)^{1/6}} \int_y^{y+(\Delta t)^{1/6}} \int_z^{z+(\Delta t)^{1/6}} \bigl[u_1(\xi, t_2) - u_1(\xi, t_1)\bigr]\,d\xi\,dz\,dy \\
&\quad = \gamma \int_{t_1}^{t_2} \int_x^{x+(\Delta t)^{1/6}} \bigl[\bigl(D^4 u_1\bigl(y + 2(\Delta t)^{1/6}, s\bigr) - D^4 u_1\bigl(y + (\Delta t)^{1/6}, s\bigr)\bigr) \\
&\qquad - \bigl(D^4 u_1\bigl(y + (\Delta t)^{1/6}, s\bigr) - D^4 u_1(y, s)\bigr)\bigr]\,dy\,ds.
\end{aligned}$$

By virtue of the mean value theorem and the Hölder inequality, we see that there exists $\xi^* \in (x, x + 3(\Delta t)^{1/6})$ such that

$$\begin{aligned}
(\Delta t)^{1/2}\bigl|u_1\bigl(\xi^*, t_1\bigr) - u_1\bigl(\xi^*, t_2\bigr)\bigr| &\le C(\Delta t)^{7/12}\left(\iint_{S_\rho} \bigl(D^4 u_1\bigr)^2\,dx\,dt\right)^{1/2} \\
&\le C(\Delta t)^{7/12}\bigl(M(u_1, \rho)\bigr)^{1/2}.
\end{aligned}$$

Combining the above result with (2.15), it follows that

$$\bigl|u_1(x, t_1) - u_1(x, t_2)\bigr|^2 \le CM(u_1, \rho)|t_1 - t_2|^{1/6}.$$

To prove the results on Du_1 or $D^2 u_1$, we only need to differentiate equation (2.4) once or twice with respect to x. And the next procedures are completely similar to the above argument. □

Lemma 2.4

$$\sup_{t \in J_{\frac{R}{4}}} \int_{E_{\frac{R}{4}}} \bigl(D^i u_1\bigr)^2\,dx + \iint_{S_{\frac{R}{4}}} \bigl(D^{i+3} u_1\bigr)^2\,dx\,dt \le \frac{C}{R^8} \iint_{S_R} \bigl(D^{i-1} u_1 - \lambda\bigr)^2\,dx\,dt, \tag{2.17}$$

where C is a constant and $i = 1, 2, 3$,

$$\lambda = \begin{cases} \textit{arbitrary constant} & \textit{if } Q_R \cap \partial_p Q_T = \emptyset, \\ 0 & \textit{if } Q_R \cap \partial_p Q_T \neq \emptyset. \end{cases}$$

Proof In order to prove (2.17) with $i = 1$, we first prove that

$$\sup_{t \in J_{\frac{R}{4}}} \int_{E_{\frac{R}{4}}} \bigl(u_1(x, t) - \lambda\bigr)^2\,dx + \iint_{S_{\frac{R}{4}}} \bigl(D^3 u_1\bigr)^2\,dx\,dt \le \frac{C}{R^6} \iint_{S_{\frac{R}{2}}} (u_1 - \lambda)^2\,dx\,dt. \tag{2.18}$$

We discuss it in the following two cases.

(I) We first prove (2.18) in the case $0, T \in J_R$. In such a case, $J_R = (0, T)$, $\lambda = 0$. Choose a smooth function $\chi(x)$ satisfying the following requirements.

If $0, 1 \notin E_R$, then $\operatorname{supp}\chi \subset (x_0 - \frac{R}{2}, x_0 + \frac{R}{2})$, $\chi(x) = 1$ when $x \in (x_0 - \frac{R}{4}, x_0 + \frac{R}{4})$, $0 \le \chi(x) \le 1$,

$$\left|\chi'(x)\right| \le \frac{C}{R}, \qquad \left|\chi''(x)\right| \le \frac{C}{R^2}, \qquad \left|\chi'''(x)\right| \le \frac{C}{R^3}, \qquad \left|\chi^{(4)}(x)\right| \le \frac{C}{R^4}.$$

If $0 \in E_R$, then the value of $\chi(x)$ for $x \le x_0$ is changed into 1.

If $1 \in E_R$, then the value of $\chi(x)$ for $x \ge x_0$ is changed into 1.

Multiplying equation (2.4) by $\chi^6 u_1$ and integrating the result over S_R, then using the boundary value condition (2.6), we have

$$\begin{aligned} 0 &= \iint_{S_R} D^3u_1 D^3\left(\chi^6 u_1\right) dx\,dt \\ &= \iint_{S_R} \chi^6\left(D^3u_1\right)^2 dx\,dt + 18\iint_{S_R} \chi^5\chi' D^2u_1 D^3u_1\,dx\,dt + 90\iint_{S_R} \chi^4\chi'^2 Du_1 D^3u_1\,dx\,dt \\ &\quad + 18\iint_{S_R} \chi^5\chi'' Du_1 D^3u_1\,dx\,dt + \iint_{S_R} \left(\chi^6\right)''' u_1 D^3u_1\,dx\,dt. \end{aligned}$$

By the Young inequality and the definition of $\chi(x)$, we have

$$\begin{aligned} &18\iint_{S_R} \chi^5\chi' D^2u_1 D^3u_1\,dx\,dt \\ &\quad \le \varepsilon\iint_{S_R} \chi^6\left(D^3u_1\right)^2 dx\,dt + C\iint_{S_R} \chi^4\chi'^2\left(D^2u_1\right)^2 dx\,dt. \end{aligned}$$

Similarly, we can estimate other three terms. Combining the above expressions yields

$$\begin{aligned} &\iint_{S_R} \chi^6\left(D^3u_1\right)^2 dx\,dt \\ &\quad \le C\biggl(\iint_{S_R} \chi^2\chi'^4(Du_1)^2\,dx\,dt + \iint_{S_R} \chi^4\chi''^2(Du_1)^2\,dx\,dt \\ &\qquad + \iint_{S_R} \chi^4\chi'^2\left(D^2u_1\right)^2 dx\,dt + \frac{1}{R^6}\iint_{S_{\frac{R}{2}}} u_1^2\,dx\,dt\biggr) \\ &\quad \equiv C(I_1 + I_2 + I_3 + I_4). \end{aligned} \tag{2.19}$$

As for I_1, we have

$$\begin{aligned} I_1 &= -\iint_{S_R} u_1 D\left(\chi^2\chi'^4 Du_1\right) dx\,dt \\ &= -\iint_{S_R} \chi^2\chi'^4 u_1 D^2u_1\,dx\,dt - \iint_{S_R} \left(\chi^2\chi'^4\right)' u_1 Du_1\,dx\,dt \\ &\le \varepsilon I_3 + C\iint_{S_R} \chi'^6 u_1^2\,dx\,dt + \frac{1}{2}\iint_{S_R} \left(\chi^2\chi'^4\right)'' u_1^2\,dx\,dt \\ &\le \varepsilon I_3 + CI_4. \end{aligned} \tag{2.20}$$

As for I_2, we have

$$\begin{aligned}
I_2 &= -\iint_{S_R} u_1 D\big(\chi^4 \chi''^2 Du_1\big)\,dx\,dt \\
&= \iint_{S_R} \chi' D\big(\chi^4\chi'' u_1 D^2 u_1\big)\,dx\,dt + \frac{1}{2}\iint_{S_R} \big(\chi^4\chi''^2\big)'' u_1^2\,dx\,dt \\
&\le \iint_{S_R} \chi'\big(\chi^4\chi''\big)' u_1 D^2 u_1\,dx\,dt \\
&\quad + \iint_{S_R} \chi^4\chi'\chi''\big(Du_1D^2u_1 + u_1D^3u_1\big)\,dx\,dt + CI_4 \\
&= \varepsilon I_3 + CI_4 + \varepsilon\iint_{S_R} \chi^6\big(D^3u_1\big)^2\,dx\,dt - \frac{1}{2}\iint_{S_R}\big(\chi^4\chi'\chi''\big)'(Du_1)^2\,dx\,dt \\
&= \varepsilon I_3 + CI_4 + \varepsilon\iint_{S_R} \chi^6\big(D^3u_1\big)^2\,dx\,dt - \frac{1}{2}I_2 \\
&\quad - \frac{1}{2}\iint_{S_R}\big(\chi^4\chi'\chi''' + 4\chi^3\chi'^2\chi''\big)(Du_1)^2\,dx\,dt,
\end{aligned}$$

that is,

$$\begin{aligned}
I_2 &\le \varepsilon I_3 + CI_4 + \varepsilon\iint_{S_R} \chi^6\big(D^3u_1\big)^2\,dx\,dt \\
&\quad - \frac{1}{3}\iint_{S_R}\big(\chi^4\chi'\chi''' + 4\chi^3\chi'^2\chi''\big)(Du_1)^2\,dx\,dt.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&-\iint_{S_R}\big(\chi^4\chi'\chi''' + 4\chi^3\chi'^2\chi''\big)(Du_1)^2\,dx\,dt \\
&\quad = \iint_{S_R}\big(\chi^4\chi'\chi''' + 4\chi^3\chi'^2\chi''\big)u_1D^2u_1\,dx\,dt \\
&\qquad + \iint_{S_R}\big(\big(\chi^4\chi'\chi'''\big)' + 4\big(\chi^3\chi'^2\chi''\big)'\big)u_1Du_1\,dx\,dt \\
&\quad \le \varepsilon I_3 + CI_4.
\end{aligned}$$

Combining the above two yields

$$I_2 \le \varepsilon I_3 + CI_4 + \varepsilon\iint_{S_R} \chi^6\big(D^3u_1\big)^2\,dx\,dt. \tag{2.21}$$

Notice that

$$\begin{aligned}
I_3 &= -\iint_{S_R} \chi^4\chi'^2 Du_1D^3u_1\,dx\,dt \\
&\quad - \iint_{S_R}\big(4\chi^3\chi'^3 + 2\chi^4\chi'\chi''\big)Du_1D^2u_1\,dx\,dt \\
&\le \varepsilon\iint_{S_R} \chi^6\big(D^3u_1\big)^2\,dx\,dt + CI_1 + \varepsilon I_3 + CI_1 + \varepsilon I_3 + CI_2,
\end{aligned}$$

that is,

$$I_3 \leq C(I_1 + I_2) + \varepsilon \iint_{S_R} \chi^6 (D^3 u_1)^2 \, dx \, dt. \tag{2.22}$$

Finally, from (2.20), (2.21) and (2.22), we see that

$$I_i \leq \varepsilon \iint_{S_R} \chi^6 (D^3 u_1)^2 \, dx \, dt + CI_4, \quad i = 1, 2, 3,$$

which combined with (2.19) yields

$$\iint_{S_{\frac{R}{4}}} (D^3 u_1)^2 \, dx \, dt \leq \iint_{S_R} \chi^6 (D^3 u_1)^2 \, dx \, dt \leq CI_4 = \frac{C}{R^6} \iint_{S_{\frac{R}{2}}} u_1^2 \, dx \, dt. \tag{2.23}$$

We can imitate all the above procedures and derive a similar result on E_R, that is,

$$\left| \int_{E_R} D^3 u_1 D^3 (\chi^6 u_1) \, dx \right| \leq \int_{E_R} H(\chi, \chi', \chi'', \chi''', \chi^{(4)}) u_1^2 \, dx, \tag{2.24}$$

where H is a polynomial with respect to χ, χ', χ'', χ''', $\chi^{(4)}$ and satisfies $|H| \leq \frac{C}{R^6}$. Using the Sobolev inequality on J_R, we have

$$\begin{aligned}
\sup_{t \in J_R} \int_{E_R} \chi^6 u_1^2(x,t) \, dx &\leq \frac{C}{R^6} \iint_{S_{\frac{R}{2}}} u_1^2 \, dx \, dt + 2 \int_{J_R} \left| \int_{E_R} \chi^6 u_1 \frac{\partial u_1}{\partial t} \, dx \right| dt \\
&= CI_4 + 2\gamma \int_{J_R} \left| \int_{E_R} \chi^6 u_1 D^6 u_1 \, dx \right| dt \\
&\leq CI_4 + C \int_{J_R} \left| \int_{E_R} D^3 u_1 D^3 (\chi^6 u_1) \, dx \right| dt.
\end{aligned}$$

Combining the above with (2.24) yields

$$\sup_{t \in J_{\frac{R}{4}}} \int_{E_{\frac{R}{4}}} u_1^2(x,t) \, dx \leq \frac{C}{R^6} \iint_{S_{\frac{R}{2}}} u_1^2 \, dx \, dt.$$

Combining the above with (2.23) yields the desired estimate (2.18).

(II) Then we prove (2.18) in the case 0 or $T \notin J_R$. Take the case $0, T \notin J_R$ as an example. Choose another smooth function $\eta(t)$ such that $\eta(t) = 1$ when $x \in (t_0 - (\frac{R}{4})^6, t_0 + (\frac{R}{4})^6)$; $\eta(t) = 0$ when $x \in (0, t_0 - (\frac{R}{2})^6) \cup (t_0 + (\frac{R}{2})^6, T)$; $0 \leq \eta(t) \leq 1$; $|\eta'(t)| \leq \frac{C}{R^6}$ for all $t \in (0, T)$.

With λ stated in the lemma, we multiply (2.4) by $\chi^6 \eta (u_1 - \lambda)$ and integrate the result over S_R. Then we can derive equalities similar to the above argument in which u_1 is replaced by $u_1 - \lambda$ and a term

$$-\frac{1}{2} \iint_{S_R} \chi^6 \eta' (u_1 - \lambda)^2 \, dx \, dt$$

is added. Then following the argument as in Case I, we can complete the proof of (2.18).

Now we multiply (2.4) by $D(\chi^6 Du_1)$ and follow the above argument. Then we derive the same result on Du_1:

$$\sup_{t\in J_{\frac{R}{4}}}\int_{E_{\frac{R}{4}}}\big(Du_1(x,t)\big)^2\,dx+\iint_{S_{\frac{R}{4}}}\big(D^4u_1\big)^2\,dx\,dt\le\frac{C}{R^6}\iint_{S_{\frac{R}{2}}}(Du_1)^2\,dx\,dt. \tag{2.25}$$

Using the interpolation inequality, we have

$$\iint_{S_{\frac{R}{2}}}(Du_1)^2\,dx\,dt\le\frac{C}{R^2}\iint_{S_{\frac{R}{2}}}u_1^2\,dx\,dt+CR^4\iint_{S_{\frac{R}{2}}}\big(D^3u_1\big)^2\,dx\,dt.$$

Replacing R in (2.23) by $2R$, and combining the result with the above inequality, we have

$$\iint_{S_{\frac{R}{2}}}(Du_1)^2\,dx\,dt\le\frac{C}{R^2}\iint_{S_R}u_1^2\,dx\,dt,$$

which together with (2.25) yields (2.17) with $i=1$.

For (2.17) with $i=2$ and $i=3$, we should first multiply (2.4) by $D(\chi^6 Du_1)$ and $D^2(\chi^6 D^2u_1)$ respectively, and the remaining parts are similar and easier. □

Lemma 2.5 *For any* $0<\rho<R$,

$$\varphi(u_1,\rho)\le C\left(\frac{\rho}{R}\right)^8\varphi(u_1,R), \tag{2.26}$$

where C is a constant number. Further, (2.26) *still holds, if* u_1 *is replaced by* Du_1 *or* D^2u_1.

Proof It suffices to show (2.26) for $\rho\le\frac{R}{4}$, otherwise we only need to set $C=4^8$. By Lemma 2.3 and Lemma 2.4, we have

$$\iint_{S_\rho}|u_1-\hat{u}_{1_\rho}|^2\,dx\,dt\le CM\left(u_1,\frac{R}{4}\right)\rho^8\le C\left(\frac{\rho}{R}\right)^8\iint_{S_R}(u_1-\lambda)^2\,dx\,dt.$$

Taking $\lambda=\hat{u}_{1_R}$, we obtain

$$\iint_{S_\rho}|u_1-\hat{u}_{1_\rho}|^2\,dx\,dt\le C\left(\frac{\rho}{R}\right)^8\iint_{S_R}(u_1-\hat{u}_{1_R})^2\,dx\,dt. \tag{2.27}$$

On the other hand, by (2.25),

$$\begin{aligned}
&\iint_{S_\rho}\rho^6\big(D^3u_1(x,t)\big)^2\,dx\,dt\\
&\quad\le C\iint_{S_\rho}\rho^2\big(Du_1(x,t)\big)^2\,dx\,dt+C\iint_{S_\rho}\rho^8\big(D^4u_1(x,t)\big)^2\,dx\,dt\\
&\quad\le C\left(\frac{\rho}{R}\right)^8\left[\iint_{S_R}|u_1-\hat{u}_{1_\rho}|^2\,dx\,dt+\iint_{S_R}R^6\big(D^3u_1\big)^2\,dx\,dt\right]\\
&\quad=C\left(\frac{\rho}{R}\right)^8\varphi(u_1,R),
\end{aligned}$$

which combined with (2.27) implies (2.26). The proofs of the results on Du_1 or D^2u_1 are similar. □

Lemma 2.6 *Let $\varphi(\rho)$ be a nonnegative and nondecreasing function satisfying*

$$\varphi(\rho) \le A\left(\frac{\rho}{R}\right)^{\alpha}\varphi(R) + BR^{\beta}, \quad 0 < \rho \le R \le R_0,$$

where A, B, α, β are positive constants and $\beta < \alpha$. Then there exists a constant C only depending on A, B, α, β such that

$$\varphi(\rho) \le C\left(\frac{\rho}{R}\right)^{\beta}\left[\varphi(R) + BR^{\beta}\right], \quad 0 < \rho \le R \le R_0.$$

The proof of this lemma can be found in [13].

Theorem 2.1 *Let $\Phi(x,t)$ be an appropriately smooth function, and let u be the smooth solution of problem* (2.1)-(2.3). *Then, for any $\alpha \in (0, \frac{1}{2})$, there exists a coefficient K depending only on α, $\iint_{Q_T} u^2\,dx\,dt$, $\iint_{Q_T} (D^3u)^2\,dx\,dt$, $\iint_{Q_T} \Phi^2\,dx\,dt$ such that*

$$\left|u(x_1,t_1) - u(x_2,t_2)\right| \le K\left(|x_1 - x_2|^{\alpha} + |t_1 - t_2|^{\frac{\alpha}{6}}\right). \tag{2.28}$$

Further, (2.28) *still holds if u is replaced by Du or D^2u.*

Proof For any fixed point $(x_0,t_0) \in \overline{Q}_T$, consider the function $\varphi(u,\rho)$, which is clearly nondecreasing with respect to ρ. By Lemma 2.5,

$$\begin{aligned}
\varphi(u,\rho) &\le \varphi(u_1,\rho) + \varphi(u_2,\rho) \\
&\le C\left(\frac{\rho}{R}\right)^{8}\varphi(u_1,R) + \varphi(u_2,R) \\
&\le C\left(\frac{\rho}{R}\right)^{8}\varphi(u,R) + C\varphi(u_2,R)
\end{aligned}$$

holds for any $0 < \rho < R$. By Lemma 2.2,

$$\begin{aligned}
\varphi(u_2,R) &= \iint_{S_R}\left[(u_2 - \hat{u}_{2R})^2 + R^6\left(D^3u_2\right)^2\right]dx\,dt \\
&\le 4\iint_{S_R} u_2^2\,dx\,dt + R^6\iint_{S_R}\left(D^3u_2\right)^2\,dx\,dt \\
&\le CR^{12}\iint_{S_R}\Phi^2\,dx\,dt.
\end{aligned}$$

Thus,

$$\varphi(u,\rho) \le C\left(\frac{\rho}{R}\right)^{8}\varphi(u,R) + CR^{12}\iint_{Q_T}\Phi^2\,dx\,dt.$$

By Lemma 2.6, we have

$$\varphi(u,\rho) \le C\left(\frac{\rho}{R_0}\right)^{\lambda}\left[\varphi(u,R_0) + R_0{}^{\lambda}\iint_{Q_T}\Phi^2\,dx\,dt\right]$$

for some $7 < \lambda < 8$. Hence,

$$M^2[u] \le C\left[\frac{1}{R_0^{\lambda}}\varphi(u,R_0) + \iint_{Q_T}\Phi^2\,dx\,dt\right].$$

Using Lemma 2.1, we immediately obtain (2.28). The proofs of the results on Du_1 or D^2u_1 are similar. □

3 The main result and its proof

In this section, we represent the main result of this paper.

Theorem 3.1 *Problem* (1.1)-(1.3) *admits a time-periodic solution* $u \in C^{6+\alpha,1+\frac{\alpha}{6}}(\overline{Q})$.

To prove the existence of this solution, we employ the Leray-Schauder fixed point theorem which enables us to study the problem by considering the following equation:

$$\frac{\partial u}{\partial t} - \gamma D^6 u = \sigma Dg(x,t) + \sigma f(x,t), \tag{3.1}$$

subject to the conditions (1.2)-(1.3), where σ is a parameter taking value on the interval $[0,1]$, and $g(x,t) \in \mathcal{W}$ is periodic in time t with period T, where $\mathcal{W} \equiv \{w | w \in C^{1+\alpha,\frac{\alpha}{4}}(\overline{Q}_T), w(x,t)$ is periodic in time t with period $T\}$. For any given function $g(x,t) \in \mathcal{W}$, from linear classical theory (see [14]), we see that problems (3.1) and (1.2)-(1.3) admit a unique solution $u \in C^{6+\alpha,1+\frac{\alpha}{6}}(\overline{Q}_T) \subset C^{1+\alpha,\frac{\alpha}{4}}(\overline{Q}_T)$, and hence we can define a mapping $\mathcal{L}$ as follows:

$$\mathcal{L} : \mathcal{W} \times [0,1] \to \mathcal{W}, \qquad (g,\sigma) \mapsto u,$$

together with its composition with $\Psi(v,t) = D^3\psi(v,t) + \nu v^2$, namely

$$\mathcal{L}\big(\Psi(\cdot,\cdot),\cdot\big) : \mathcal{W} \times [0,1] \to \mathcal{W}.$$

Obviously, for any given $v \in \mathcal{W}$, $\mathcal{L}(v,0) = 0$. By virtue of the Leray-Schauder fixed point theorem, to prove the existence of solutions of problem (1.1)-(1.3), we only need to show that the mapping $\mathcal{L}$ is compact and prove that there exists a constant independent of u_σ and σ such that, for any u and σ satisfying $u = \mathcal{L}(\Psi(u),\sigma)$, $\|u_\sigma\|_{C^{4+\alpha,\frac{\alpha}{4}}(\overline{Q}_T)} \le C$. Moreover, it follows from the above arguments that u is a classical solution. Then we consider the problem in $Q_{(T,2T)},\dots,Q_{((n-1)T,nT)},\dots$ in turn. Finally, we know that initial boundary value problem (1.1)-(1.3) admits a classical solution in Q.

Lemma 3.1 *The mapping* $\mathcal{L} : (v,\sigma) \mapsto u$ *is compact.*

This result can be directly obtained by a compact embedding theorem, so we omit the details here.

Lemma 3.2 *Let u_σ be a time-periodic solution of the equation*

$$\frac{\partial u_\sigma}{\partial t} - \gamma D^6 u_\sigma = \sigma D^4 \psi(u_\sigma, t) + \sigma \nu u_\sigma D u_\sigma + \sigma f(x,t), \tag{3.2}$$

subject to the conditions (1.2)-(1.3), *where* $\sigma \in [0,1]$. *Then*

$$\|u_\sigma\|_\infty \le C, \qquad \|Du_\sigma\|_\infty \le C, \qquad \|D^2 u_\sigma\|_\infty \le C, \tag{3.3}$$

where C is a constant independent of the solution u and σ.

Proof First, let $\omega_\sigma(x,t)$ be a time-periodic solution of the problem

$$D^2\omega = u_\sigma, \qquad D\omega|_{x=0,1} = 0, \qquad \int_0^1 \omega\,dx = 0,$$

then from the Poincaré inequality we know that

$$\int_0^1 \omega_\sigma^2\,dx \le \int_0^1 (D\omega_\sigma)^2\,dx \le \int_0^1 \left(D^2\omega_\sigma\right)^2 dx = \int_0^1 u_\sigma^2\,dx. \tag{3.4}$$

Multiplying (3.2) by $\omega_\sigma(x,t)$, integrating the result over Q_T and using the condition (1.2), then using the Young inequality and (3.4), we have

$$\begin{aligned} &\gamma \iint_{Q_T} \left(D^2 u_\sigma\right)^2 dx\,dt + \sigma \iint_{Q_T} 3a(t) u_\sigma^2 (Du_\sigma)^2\,dx\,dt \\ &\quad = \sigma \iint_{Q_T} b(t)(Du_\sigma)^2\,dx\,dt - \sigma \iint_{Q_T} \nu u_\sigma Du_\sigma \omega_\sigma\,dx\,dt - \sigma \iint_{Q_T} f(x,t)\omega_\sigma\,dx\,dt \\ &\quad \le C \iint_{Q_T} (Du_\sigma)^2\,dx\,dt + \frac{\underline{M}\sigma}{2} \iint_{Q_T} u_\sigma^2 (Du_\sigma)^2\,dx\,dt + C \iint_{Q_T} \omega_\sigma^2\,dx\,dt + C, \end{aligned} \tag{3.5}$$

which implies that

$$\iint_{Q_T} \left(D^2 u_\sigma\right)^2 dx\,dt \le C \iint_{Q_T} u_\sigma^2\,dx\,dt + C. \tag{3.6}$$

Moreover,

$$\begin{aligned} \iint_{Q_T} (Du_\sigma)^2\,dx\,dt &= -\iint_{Q_T} u_\sigma D^2 u_\sigma\,dx\,dt \\ &\le C \iint_{Q_T} \left(D^2 u_\sigma\right)^2 dx\,dt + C \iint_{Q_T} u_\sigma^2\,dx\,dt \\ &\le C \iint_{Q_T} u_\sigma^2\,dx\,dt + C. \end{aligned} \tag{3.7}$$

It follows from (3.5) that

$$\begin{aligned} \iint_{Q_T} u_\sigma^2 (Du_\sigma)^2\,dx\,dt &\le \frac{N}{3\underline{M}} \iint_{Q_T} (Du_\sigma)^2\,dx\,dt + \frac{\Lambda}{3\underline{M}} \iint_{Q_T} \omega_\sigma^2\,dx\,dt + C \\ &\le C \iint_{Q_T} u_\sigma^2\,dx\,dt + C. \end{aligned} \tag{3.8}$$

By (1.2), we have

$$u_\sigma^4(x,t) = \int_0^x D\bigl(u_\sigma^4(s,t)\bigr)\,ds \le \int_0^1 \bigl|Du_\sigma^4\bigr|\,dx = 4\int_0^1 \bigl|u_\sigma^3 Du_\sigma\bigr|\,dx.$$

Integrating the above inequality over Q_T and using (3.8) together with the Young inequality, we have

$$\begin{aligned}\iint_{Q_T} u_\sigma^4\,dx\,dt &\le 4\iint_{Q_T} \bigl|u_\sigma^3 Du_\sigma\bigr|\,dx\,dt \\ &\le 4\biggl(\iint_{Q_T} u_\sigma^4\,dx\,dt\biggr)^{1/2}\biggl(\iint_{Q_T} u_\sigma^2(Du_\sigma)^2\,dx\,dt\biggr)^{1/2} \\ &\le 4\biggl(\iint_{Q_T} u_\sigma^4\,dx\,dt\biggr)^{1/2}\biggl(C\iint_{Q_T} u_\sigma^2\,dx\,dt + C\biggr)^{1/2},\end{aligned}$$

that is,

$$\iint_{Q_T} u_\sigma^4\,dx\,dt \le C\iint_{Q_T} u_\sigma^2\,dx\,dt + C.$$

On the other hand, by the Young inequality,

$$\iint_{Q_T} u_\sigma^2\,dx\,dt \le \varepsilon\iint_{Q_T} u_\sigma^4\,dx\,dt + C.$$

Combining the above expressions, we obtain

$$\iint_{Q_T} u_\sigma^4\,dx\,dt \le C, \qquad \iint_{Q_T} u_\sigma^2\,dx\,dt \le C. \tag{3.9}$$

Combining the above with (3.6) and (3.7), we see that

$$\iint_{Q_T} (Du_\sigma)^2\,dx\,dt \le C, \qquad \iint_{Q_T} \bigl(D^2u_\sigma\bigr)^2\,dx\,dt \le C. \tag{3.10}$$

Set

$$F(t) = \int_0^1 \biggl[\frac{\gamma}{2}(Du_\sigma)^2 + \sigma\bigl(H(u_\sigma,t) + \lambda\bigr)\biggr]\,dx,$$

where $H(u,t) = -\int_0^u \psi(s,t)\,ds = \frac{a(t)}{4}u^4 - \frac{b(t)}{2}u^2 \ge -\lambda$, λ is a positive constant depending only on $\underline{M}$ and N. Then $F(t) \ge 0$. Integrating $|F(t)|$ over $(0,T)$, by (3.9) and (3.10), we get

$$\begin{aligned}\int_0^T \bigl|F(t)\bigr|\,dt &\le \iint_{Q_T} \biggl|\frac{\gamma}{2}(Du_\sigma)^2 + \sigma\biggl(\frac{a(t)}{4}u^4 - \frac{b(t)}{2}u^2 + \lambda\biggr)\biggr|\,dx\,dt \\ &\le C\iint_{Q_T} (Du_\sigma)^2\,dx\,dt + C\iint_{Q_T} u_\sigma^4\,dx\,dt + C\iint_{Q_T} u_\sigma^2\,dx\,dt + C \\ &\le C.\end{aligned} \tag{3.11}$$

On the other hand, integrating by parts and using (1.2), we have

$$\begin{aligned}\frac{dF}{dt} &= -\int_0^1 \bigl(\gamma D^4 u_\sigma + \sigma D^2\psi\bigr)^2\,dx - \sigma\int_0^1 \bigl(\gamma D^2 u_\sigma + \sigma\psi\bigr)\nu u_\sigma Du_\sigma\,dx \\ &\quad - \sigma\int_0^1 \bigl(\gamma D^2 u_\sigma + \sigma\psi\bigr)f\,dx + \sigma\int_0^1 \left(\frac{a'(t)}{4}u_\sigma^4 - \frac{b'(t)}{2}u_\sigma^2\right)dx.\end{aligned}$$

Integrating the above equality over $(0,T)$ and noticing the periodicity of F, we have

$$\begin{aligned}&\iint_{Q_T} \bigl(\gamma D^4 u_\sigma + \sigma D^2\psi\bigr)^2\,dx\,dt \\ &\quad = -\sigma\iint_{Q_T} \bigl(\gamma D^2 u_\sigma + \sigma\psi\bigr)\nu u_\sigma Du_\sigma\,dx\,dt - \sigma\iint_{Q_T} \bigl(\gamma D^2 u_\sigma + \sigma\psi\bigr)f\,dx\,dt \\ &\qquad + \sigma\iint_{Q_T} \left(\frac{a'(t)}{4}u_\sigma^4 - \frac{b'(t)}{2}u_\sigma^2\right)dx\,dt.\end{aligned}$$

Integrating $|\frac{dF}{dt}|$ over $(0,T)$, using (3.9) and (3.10), we have

$$\begin{aligned}&\int_0^T \left|\frac{dF}{dt}\right| dt \\ &\quad \le 2\iint_{Q_T} \bigl|\bigl(\gamma D^2 u_\sigma + \sigma\psi\bigr)\nu u_\sigma Du_\sigma\bigr|\,dx\,dt + 2\iint_{Q_T} \bigl|\bigl(\gamma D^2 u_\sigma + \sigma\psi\bigr)f\bigr|\,dx\,dt \\ &\qquad + 2\iint_{Q_T} \left|\frac{a'(t)}{4}u_\sigma^4 - \frac{b'(t)}{2}u_\sigma^2\right| dx\,dt \\ &\quad \le C\iint_{Q_T} \bigl(D^2 u_\sigma\bigr)^2\,dx\,dt + C\iint_{Q_T} u_\sigma^4\,dx\,dt + C\iint_{Q_T} |u_\sigma|^3\,dx\,dt \\ &\qquad + C\iint_{Q_T} u_\sigma^2\,dx\,dt + C \le C. \end{aligned} \tag{3.12}$$

By virtue of (3.11) and (3.12), we have $F(t) \le C$. Noticing the definition of $F(t)$, we get

$$\int_0^1 (Du_\sigma)^2\,dx \le C. \tag{3.13}$$

By (1.2), we know that $\|u_\sigma\|_\infty \le C$.

In order to prove the rest of this lemma, we need to give *a priori* estimate on $D^4 u_\sigma^3$. First, by the Gagliardo-Nirenberg inequality, we can obtain

$$\begin{aligned}&\|Du_\sigma\|_4 \le C\bigl\|D^6 u_\sigma\bigr\|_2^{\frac{1}{20}}\|Du_\sigma\|_2^{\frac{19}{20}}, \\ &\|Du_\sigma\|_8 \le C\bigl\|D^6 u_\sigma\bigr\|_2^{\frac{3}{40}}\|Du_\sigma\|_2^{\frac{37}{40}}, \\ &\bigl\|D^2 u_\sigma\bigr\|_4 \le C\bigl\|D^6 u_\sigma\bigr\|_2^{\frac{1}{4}}\|Du_\sigma\|_2^{\frac{3}{4}}, \\ &\bigl\|D^3 u_\sigma\bigr\|_4 \le C\bigl\|D^6 u_\sigma\bigr\|_2^{\frac{9}{20}}\|Du_\sigma\|_2^{\frac{11}{20}},\end{aligned}$$

where $\|*\|_p$ denotes the L_p norm on $(0,1)$. Regulating the exponents and using the Young inequality for every of the above three expressions, we get

$$\int_0^1 (Du_\sigma)^4\,dx \le \varepsilon \int_0^1 \left(D^6u_\sigma\right)^2 dx + C\left(\int_0^1 (Du_\sigma)^2\,dx\right)^{19/9},$$
$$\int_0^1 (Du_\sigma)^8\,dx \le \varepsilon \int_0^1 \left(D^6u_\sigma\right)^2 dx + C\left(\int_0^1 (Du_\sigma)^2\,dx\right)^{37/7},$$
$$\int_0^1 \left(D^2u_\sigma\right)^4 dx \le \varepsilon \int_0^1 \left(D^6u_\sigma\right)^2 dx + C\left(\int_0^1 (Du_\sigma)^2\,dx\right)^{3},$$
$$\int_0^1 \left(D^3u_\sigma\right)^4 dx \le \varepsilon \int_0^1 \left(D^6u_\sigma\right)^2 dx + C\left(\int_0^1 (Du_\sigma)^2\,dx\right)^{11}.$$

Integrating the above inequalities over $(0,T)$ and noticing (3.13), we see that the terms of left hand side in these inequalities can all be estimated by $\varepsilon\iint_{Q_T}(D^6u_\sigma)^2\,dx\,dt$ and a constant number C. Then by the boundary value condition and (3.10), we have

$$\iint_{Q_T} \left(D^4u_\sigma\right)^2 dx\,dt \le \varepsilon \iint_{Q_T} \left(D^6u_\sigma\right)^2 dx\,dt + C, \tag{3.14}$$

and also, by the above discussion, we have

$$\begin{aligned}\iint_{Q_T} \left(D^4u_\sigma^3\right)^2 dx\,dt &\le C\iint_{Q_T} \left(D^4u_\sigma\right)^2 dx\,dt + C\iint_{Q_T} \left(Du_\sigma D^3u_\sigma\right)^2 dx\,dt \\ &\quad + C\iint_{Q_T} \left(|Du_\sigma|^2 D^2u_\sigma\right)^2 dx\,dt + C\iint_{Q_T} \left(\left|D^2u_\sigma\right|^2\right)^2 dx\,dt \\ &\le \varepsilon \iint_{Q_T} \left(D^6u_\sigma\right)^2 dx\,dt + C.\end{aligned} \tag{3.15}$$

Multiplying (3.2) by D^6u_σ, integrating the result over Q_T, using (3.14), (3.15) and the Young inequality, we get

$$\begin{aligned}\iint_{Q_T} \left(D^6u_\sigma\right)^2 dx\,dt &\le \varepsilon \iint_{Q_T} \left(D^6u_\sigma\right)^2 dx\,dt + C\iint_{Q_T} \left(D^4\psi(u_\sigma,t)\right)^2 dx\,dt + C \\ &\le \varepsilon \iint_{Q_T} \left(D^6u_\sigma\right)^2 dx\,dt + C,\end{aligned} \tag{3.16}$$

that is,

$$\iint_{Q_T} \left(D^6u_\sigma\right)^2 dx\,dt \le C. \tag{3.17}$$

By (3.17) and the approach similar to (3.14), we can derive

$$\begin{aligned}&\iint_{Q_T} \left(D^3u_\sigma\right)^2 dx\,dt \le C, \qquad \iint_{Q_T} \left(D^4u_\sigma\right)^2 dx\,dt \le C, \\ &\iint_{Q_T} \left(D^5u_\sigma\right)^2 dx\,dt \le C.\end{aligned} \tag{3.18}$$

Now we set

$$F_1(t) = \int_0^1 \left(D^2 u_\sigma\right)^2 dx.$$

Obviously,

$$\int_0^T \left|F_1(t)\right| dt \le C. \tag{3.19}$$

On the other hand, by (3.16), (3.17) and (3.18), we have

$$\begin{aligned} \int_0^T \left|\frac{dF_1}{dt}\right| dt &= \int_0^T \left|\int_0^1 D^4 u_\sigma \left(\gamma D^6 u_\sigma + \sigma D^4 \psi + \sigma \nu u_\sigma D u_\sigma + \sigma f\right) dx\right| dt \\ &\le C \iint_{Q_T} \left(D^4 u_\sigma\right)^2 dx\,dt + C \iint_{Q_T} \left(D^6 u_\sigma\right)^2 dx\,dt \\ &\quad + C \iint_{Q_T} \left(D^4 \psi\right)^2 dx\,dt + C \le C. \end{aligned} \tag{3.20}$$

By virtue of (3.19) and (3.20), we have $F_1(t) \le C$. Noticing the definition of $F_1(t)$, we get

$$\int_0^1 \left(D^2 u_\sigma\right)^2 dx \le C.$$

Applying the Poincaré inequality and the Friedrichs inequality [15], we conclude that $\|Du_\sigma\|_\infty \le C$.

Finally, we set

$$F_2(t) = \int_0^1 \left(D^3 u_\sigma\right)^2 dx.$$

By an approach similar to the above argument, we can obtain the last result that $\|D^2 u_\sigma\|_\infty \le C$. The proof of this lemma is complete. □

Proof of Theorem 3.1 Now we apply Theorem 2.1 to complete the proof of Theorem 3.1. For the smooth function $\Phi(x,t)$ in Theorem 2.1, let

$$\Phi(x,t) = \sigma D^4 \psi(u_\sigma, t) + \sigma \nu u_\sigma D u_\sigma + \sigma f.$$

From the proof of Lemma 3.2, we see that $\iint_{Q_T} (D^i u_\sigma)^2\, dx\,dt$ $(i = 0,1,\dots,6)$ and $\iint_{Q_T} \Phi^2\, dx\,dt$ can be all uniformly bounded by a constant number C. Therefore the coefficient K in Theorem 2.1 now only depends on the Hölder exponent α. So, for u_σ, we have

$$\left|D^i u_\sigma(x_1,t_1) - D^i u_\sigma(x_2,t_2)\right| \le K(\alpha)\left(|x_1 - x_2|^\alpha + |t_1 - t_2|^{\frac{\alpha}{6}}\right), \quad i = 0,1,2,$$

which combines with the results of Lemma 3.2. We know that $\|u_\sigma\|_{C^{2+\alpha,\frac{\alpha}{6}}(\overline{Q}_T)} \le C$, where C is independent of u and σ. Then, it follows from the results in [16] that $\|u_\sigma\|_{C^{4+\alpha,\frac{\alpha}{6}}(\overline{Q}_T)} \le C$.

Recalling the discourse in the beginning of this section, we conclude from the Leray-Schauder fixed point theorem that $\mathcal{L}(*, 1)$ admits a fixed point u in the space $C^{6+\alpha,1+\frac{\alpha}{6}}(\overline{Q}_T)$, which is the desired solution of problem (1.1)-(1.3). The proof of Theorem 3.1 is completed. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and read and approved the final manuscript.

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