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Least energy solutions for a quasilinear Schrödinger equation with potential well

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Abstract

In this paper, we consider the existence of least energy solutions for the following quasilinear Schrödinger equation:

$$-\Delta u + (\lambda a(x) + 1)u - \frac{1}{2}(\Delta |u|^2)u = f(u), \quad x \in \mathbb{R}^N, \tag{E_λ}$$

with $a(x) \geq 0$ having a potential well, where $N \geq 3$ and $\lambda > 0$ is a parameter. Under suitable hypotheses, we obtain the existence of a least energy solution u_λ of (E_λ) which localizes near the potential well $\operatorname{int} a^{-1}(0)$ for λ large enough by using the variational method and the concentration compactness method in an Orlicz space.

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Keywords: quasilinear Schrödinger equation; least energy solution; Orlicz space; concentration compactness method; variational method

1 Introduction

Let us consider the following quasilinear Schrödinger equation:

$$-\Delta u + \big(\lambda a(x) + 1\big)u - \frac{1}{2}\big(\Delta |u|^2\big)u = f(u), \quad x \in \mathbb{R}^N, \tag{E_λ}$$

for sufficiently large λ, where $N \geq 3$.

Our assumptions on $a(x)$ are as follows:

(a_1) $a(x) \in C(\mathbb{R}^N, [0, +\infty))$, the potential well $\Omega := \operatorname{int} a^{-1}(0)$ is a non-empty set and $\overline{\Omega} = a^{-1}(0)$;

(a_2) There exists a constant $M_0 > 0$ such that $\mu(\{x \in \mathbb{R}^N | a(x) \leq M_0\}) < \infty$, where μ denotes the Lebesgue measure on $\mathbb{R}^N$.

Condition (a_2) is very weak in dealing with the operator $-\Delta + (\lambda a(x) + 1)I$ on $\mathbb{R}^N$, which was firstly used by Bartsch and Wang [1] in dealing with the semilinear Schrödinger equation.

Remark 1.1 $\Omega := \operatorname{int} a^{-1}(0)$ can be unbounded.

For $f(u)$, we assume that f is continuous and satisfies the following conditions:

(f_1) $\lim_{u \to 0^+} \frac{f(u)}{u} = 0$;

(f_2) $0 \le f(u) \le C(1+u^p)$ for $u \ge 0$, where $C > 0$ is a constant and $4 < p+1 < 2\cdot 2^*$, where $2^* = \frac{2N}{N-2}$;

(f_3) There is a number $4 < \theta \le p+1$ such that for all $u > 0$, we have $f(u)\cdot u \ge \theta F(u)$, where $F(u) = \int_0^u f(t)\,dt$.

Hypotheses (a_1), (a_2) and (f_1), (f_2), (f_3) will be maintained throughout this paper.

Solutions of (E_λ) are related to the existence of the standing wave solutions of the following quasilinear Schrödinger equation:

$$i\partial_t z = -\triangle z + V(x)z - f\big(|z|^2\big)z - k\triangle h\big(|z|^2\big)h'\big(|z|^2\big)z, \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $V(x)$ is a given potential, k is a real constant and f, h are real functions. We would like to mention that (1.1) appears more naturally in mathematical physics and has been derived as models of several physical phenomena corresponding to various types of h. For instance, the case $h(s) = s$ was used for the superfluid film equation in plasma physics by Kurihara [2] (see also [3]); in the case of $h(s) = (1+s)^{\frac{1}{2}}$, (1.1) was used as a model of the self-changing of a high-power ultrashort laser in matter (see [4–7] and references therein).

In recent years, much attention has been devoted to the quasilinear Schrödinger equation of the following form:

$$-\triangle u + \lambda V(x)u - k\triangle\big(u^2\big)u = |u|^{p-2}u, \quad x \in \mathbb{R}^N. \tag{1.2}$$

For example, by using a constrained minimization argument, the existence of positive ground state solution was proved by Poppenberg, Schmitt and Wang [8]. Using a change of variables, Liu, Wang and Wang [9] used an Orlicz space to prove the existence of soliton solution of (1.2) via the mountain pass theorem. Colin and Jeanjean [10] also made use of a change of variables but worked in the Sobolev space $H^1(\mathbb{R}^N)$, they proved the existence of a positive solution for (1.2) from the classical results given by Berestycki and Lions [11]. By using the Nehari manifold method and the concentration compactness principle (see [12]) in the Orlicz space, Guo and Tang [13] considered the following equation:

$$-\triangle u + \big(\lambda a(x) + 1\big)u - \frac{1}{2}\big(\triangle|u|^2\big)u = u^p, \quad u > 0, x \in \mathbb{R}^N, \tag{1.3}$$

with $a(x) \ge 0$ having a potential well and $4 < p+1 < 2\cdot 2^*$, where $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent, and they proved the existence of a ground state solution of (1.3) which localizes near the potential well $\operatorname{int} a^{-1}(0)$ for λ large enough. In [14], Guo and Tang also considered ground state solutions of the corresponding quasilinear Schrödinger systems for (1.3) by the same methods and obtained similar results. For the stability and instability results for the special case of (1.2), one can also see the paper by Colin, Jeanjean and Squassina [15].

It is worth pointing out that the existence of one-bump or multi-bump bound state solutions for the related semilinear Schrödinger equation (1.3) for $k = 0$ has been extensively studied. One can see Bartsch and Wang [1], Ambrosetti, Badiale and Cingolani [16], Ambrosetti, Malchiodi and Secchi [17], Byeon and Wang [18], Cingolani and Lazzo [19], Cingolani and Nolasco [20], Del Pino and Felmer [21, 22], Floer and Weinstein [23], Oh [24, 25] and the references therein.

In this paper, based on the idea from Liu, Wang and Wang [9], we consider the more general equation (E_λ), the existence of least energy solutions for equation (E_λ) with a potential well $\operatorname{int} a^{-1}(0)$ for λ large is proved under the conditions (a_1), (a_2) and (f_1), (f_2), (f_3).

The paper is organized as follows. In Section 2, we describe our main result (Theorem 2.1). In Section 3, we give some preliminaries that will be used for the proof of the main result. Finally, Theorem 2.1 will be proved in Section 4.

Throughout this paper, we use the same C to denote different universal constants.

2 Main result

Let $V_\lambda(x) = \lambda a(x) + 1$. Formally, we define the following functional:

$$J_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^N} \big(1+u^2\big)|\nabla u|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^N} V_\lambda(x)u^2\,dx - \int_{\mathbb{R}^N} F(u)\,dx \tag{2.1}$$

for $u \in X := \{u \in H^1(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} V_\lambda(x)u^2 < \infty\}$. Note that under our assumptions, the functional J_λ is not well defined on X.

We follow the idea of [9] and make the following change of variable.

Let $v = h(u) = \frac{1}{2}u\sqrt{1+u^2} + \frac{1}{2}\ln(u+\sqrt{1+u^2})$, then $dv = \sqrt{1+u^2}\,du$. Moreover, $h(u)$ satisfies

$$h(u) \sim \begin{cases} u, & |u| \ll 1, \\ \frac{1}{2}u|u|, & |u| \gg 1. \end{cases}$$

Since $h'(u) = \sqrt{1+u^2} > 0$, $h(u)$ is strictly monotone and hence has an inverse function denoted by $u = g(v)$. Obviously,

$$g(v) \sim \begin{cases} v, & |v| \ll 1, \\ \sqrt{\frac{2}{|v|}}\,v, & |v| \gg 1, \end{cases} \qquad g'(v) = \frac{1}{\sqrt{1+g^2(v)}}.$$

Let $G(v) = g^2(v)$. Then it holds that

$$G(v) = g^2(v) \sim \begin{cases} v^2, & |v| \ll 1, \\ 2|v|, & |v| \gg 1 \end{cases}$$

and $G(v)$ is convex. Moreover, there exists $C_0 > 0$ such that $G(2v) \le C_0 G(v)$,

$$G'(v) = \frac{2g(v)}{\sqrt{1+g^2(v)}}, \qquad G''(v) = \frac{2}{(1+g^2(v))^2} > 0.$$

Now we introduce the Orlicz space (see [26])

$$E_G^\lambda = \left\{ v \,\Big|\, \int_{\mathbb{R}^N} V_\lambda G(v)\,dx < +\infty \right\}$$

equipped with the norm

$$|v|_G^\lambda := \inf_{\xi > 0} \xi \left(1 + \int_{\mathbb{R}^N} V_\lambda G\big(\xi^{-1}v\big)\,dx \right).$$

Then E_G^λ is a Banach space.

Let

$$H_G^\lambda := \left\{ v \in E_G^\lambda \,\middle|\, \int_{\mathbb{R}^N} |\nabla v|^2 \, dx < +\infty \right\}$$

equipped with the norm

$$\|v\|_\lambda = \|\nabla v\|_{L^2} + |v|_G^\lambda.$$

Using the change of variable, we define the functional Φ_λ on H_G^λ by

$$\Phi_\lambda(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 \, dx + \frac{1}{2}\int_{\mathbb{R}^N} V_\lambda g^2(v) \, dx - \int_{\mathbb{R}^N} F\big(g(v_+)\big) \, dx, \tag{2.2}$$

where $v_+ = \max\{v, 0\}$ is the positive part of v.

Let

$$N_\lambda := \left\{ v \in H_G^\lambda \backslash \{0\} | \langle \Phi'_\lambda(v), v \rangle = 0 \right\}$$

be the Nehari manifold and let

$$c_\lambda := \inf_{v \in N_\lambda} \Phi_\lambda(v)$$

be the infimum of Φ_λ on the Nehari manifold N_λ, where $\Phi'_\lambda(v)$ is the Gateaux derivative (see Proposition 3.3).

We say that $u_\lambda = g(v_\lambda)$ is a least energy solution of (E_λ) if $v_\lambda \in N_\lambda$ such that c_λ is achieved.

Note that under our assumptions, for λ large enough, the following Dirichlet problem is a kind of a 'limit' problem:

$$\begin{cases} -\triangle u + u - \frac{1}{2}(\triangle |u|^2)u = f(u), & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{D}$$

where $\Omega := \operatorname{int} a^{-1}(0)$.

Similar to the definition of the least energy solution of (E_λ), we can define the least energy solution of (D) which will be given in Section 4.

Our main result is as follows.

Theorem 2.1 *Assume that* (a_1), (a_2) *and* (f_1), (f_2), (f_3) *are satisfied. Then for* λ *large,* c_λ *is achieved by a critical point* v_λ *of* Φ_λ *such that* $u_\lambda = g(v_\lambda)$ *is a least energy solution of* (E_λ). *Furthermore, for any sequence* $\lambda_n \to \infty$, $\{v_{\lambda_n}\}$ *has a subsequence converging to* v *such that* $u = g(v)$ *is a least energy solution of* (D).

3 Preliminaries

In order to obtain the compactness of the functional Φ_λ, we recall the following Lemmas 3.1 and 3.2 which can be found in [13].

Lemma 3.1 *There exist two constants* $C_1 > 0$, $C_2 > 0$ *such that*

$$C_1 \min\{\|v\|_\lambda, \|v\|_\lambda^2\} \le \int_{\mathbb{R}^N} |\nabla v|^2\,dx + \int_{\mathbb{R}^N} V_\lambda(x) g^2(v)\,dx \le C_2 \max\{\|v\|_\lambda, \|v\|_\lambda^2\} \tag{3.1}$$

for any $v \in H_G^\lambda$.

Lemma 3.2 *The map*: $v \to g(v)$ *from* H_G^λ *into* $L^q(\mathbb{R}^N)$ *is continuous for* $2 \le q \le 2 \cdot 2^*$.

Now we consider the functional Φ_λ defined on H_G^λ by (2.2), the following Proposition 3.3 is due to [9].

Proposition 3.3

(i) Φ_λ *is well defined on* H_G^λ;

(ii) Φ_λ *is continuous in* H_G^λ;

(iii) Φ_λ *is Gateaux differentiable, the Gateaux derivative* $\Phi'_\lambda(v)$ *for* $v \in H_G^\lambda$ *is a linear functional and* $\Phi'_\lambda(v)$ *is continuous in* v *in the strong-weak topology, that is, if* $v_n \to v$ *strongly in* H_G^λ, *then* $\Phi'_\lambda(v_n) \rightharpoonup \Phi'_\lambda(v)$ *weakly. Moreover, the Gateaux derivative* $\Phi'_\lambda(v)$ *has the form*

$$\begin{aligned}\langle \Phi'_\lambda(v), w\rangle &= \int_{\mathbb{R}^N} \nabla v \nabla w\,dx + \int_{\mathbb{R}^N} V_\lambda(x) g(v) g'(v) w\,dx \\ &\quad - \int_{\mathbb{R}^N} f(g(v_+)) g'(v_+) w\,dx.\end{aligned} \tag{3.2}$$

Recall that $\{v_n\} \subset H_G^\lambda$ is called a Palais-Smale sequence ($(\mathrm{PS})_c$ sequence in short) for Φ_λ if $\Phi_\lambda(v_n) \to c$ and $\Phi'_\lambda(v_n) \to 0$ in $(H_G^\lambda)^*$, the dual space of H_G^λ. We say that the functional Φ_λ satisfies the $(\mathrm{PS})_c$ condition if any of $(\mathrm{PS})_c$ sequence (up to a subsequence, if necessary) $\{v_n\}$ converges strongly in H_G^λ.

Lemma 3.4 *Any of* $(PS)_c$ *sequence* $\{v_n\}$ *for* Φ_λ *is bounded.*

Proof Suppose that $\{v_n\}$ is a $(\mathrm{PS})_c$ sequence of Φ_λ. We have $\Phi_\lambda(v_n) \to c \in \mathbb{R}$ and $\Phi'_\lambda(v_n) \to 0$ in the space $(H_G^\lambda)^*$.

Taking $w_n = \frac{g(v_n)}{g'(v_n)}$, then $|\nabla w_n| = (1 + \frac{g^2(v_n)}{1+g^2(v_n)})|\nabla v_n| \le 2|\nabla v_n|$, we have $\|w_n\|_\lambda \le C\|v_n\|_\lambda$, thus

$$\begin{aligned} o(\|v_n\|_\lambda) &= \left\langle \Phi'_\lambda(v_n), \frac{g(v_n)}{g'(v_n)} \right\rangle \\ &= \int_{\mathbb{R}^N} \nabla v_n \nabla w_n\,dx + \int_{\mathbb{R}^N} V_\lambda g(v_n) g'(v_n) w_n\,dx - \int_{\mathbb{R}^N} f(g((v_n)_+)) g'((v_n)_+) w_n\,dx \\ &= \int_{\mathbb{R}^N} \left(1 + \frac{g^2(v_n)}{1+g^2(v_n)}\right) |\nabla v_n|^2\,dx + \int_{\mathbb{R}^N} V_\lambda g^2(v_n)\,dx \\ &\quad - \int_{\mathbb{R}^N} f(g((v_n)_+)) g((v_n)_+)\,dx \end{aligned} \tag{3.3}$$

and

$$\Phi_\lambda(v_n) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_n|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^N} V_\lambda g^2(v_n)\,dx - \int_{\mathbb{R}^N} F(g((v_n)_+))\,dx = c + o(1). \tag{3.4}$$

Taking (3.4) – $\frac{1}{\theta}$(3.3) yields

$$\int_{\mathbb{R}^N}\left(\frac{1}{2}-\frac{1}{\theta}\left(1+\frac{g^2(v_n)}{1+g^2(v_n)}\right)\right)|\nabla v_n|^2\,dx+\left(\frac{1}{2}-\frac{1}{\theta}\right)\int_{\mathbb{R}^N}V_\lambda g^2(v_n)\,dx$$
$$+\frac{1}{\theta}\int_{\mathbb{R}^N}\bigl(f\bigl(g\bigl((v_n)_+\bigr)\bigr)g\bigl((v_n)_+\bigr)-\theta F\bigl(g\bigl((v_n)_+\bigr)\bigr)\bigr)\,dx=c+o(1)+o\bigl(\|v_n\|_\lambda\bigr).$$

Note that

$$\frac{1}{2}-\frac{1}{\theta}\left(1+\frac{g^2(v_n)}{1+g^2(v_n)}\right)>\frac{1}{2}-\frac{2}{\theta}=\frac{\theta-4}{2\theta},$$
$$\frac{1}{2}-\frac{1}{\theta}=\frac{\theta-2}{2\theta},\qquad f(u)\cdot u>\theta F(u),\quad \theta>4,$$

we have

$$\int_{\mathbb{R}^N}|\nabla v_n|^2\,dx+\int_{\mathbb{R}^N}V_\lambda g^2(v_n)\,dx\le\frac{2\theta}{\theta-4}\bigl(c+o(1)+o\bigl(\|v_n\|_\lambda\bigr)\bigr).$$

It follows from Lemma 3.1 that

$$C_1\min\bigl\{\|v\|_\lambda,\|v\|_\lambda^2\bigr\}\le\frac{2\theta}{\theta-4}\bigl(c+o(1)+o\bigl(\|v_n\|_\lambda\bigr)\bigr),\tag{3.5}$$

thus $\{v_n\}$ is bounded in H_G^λ.

Let $K_\lambda=\{v\in H_G^\lambda|\Phi_\lambda'(v)=0\}$ be the critical set of Φ_λ. Suppose $v\in K_\lambda$, then it is easy to check that either $v>0$ or $v\equiv 0$ in $\mathbb{R}^N$ by the definition of Φ_λ and the strong maximum principle. □

Lemma 3.5 *There exists* $0<\sigma<1$ *which is independent of* λ *such that* $\|v\|_\lambda\ge\|v\|_1>\sigma$ *for all* $v\in K_\lambda\setminus\{0\}$ *and* $\lambda\ge 1$.

Proof Assume that $\|v\|_\lambda\le 1$ for any $v\in K_\lambda\setminus\{0\}$ (otherwise, the conclusion is true). From (f_1), (f_2), we see that for any $\varepsilon>0$, there is a constant $C_\varepsilon>0$ such that $f(x,u)\le\varepsilon u+C_\varepsilon u^p$ for $u>0$. We have

$$\begin{aligned}0&=\left\langle\Phi_\lambda'(v),\frac{g(v)}{g'(v)}\right\rangle\\&=\int_{\mathbb{R}^N}\left(\left(1+\frac{g^2(v)}{1+g^2(v)}\right)|\nabla v|^2+V_\lambda g^2(v)\right)dx-\int_{\mathbb{R}^N}f\bigl(g(v_+)\bigr)g(v_+)\,dx\\&\ge\int_{\mathbb{R}^N}\bigl(|\nabla v|^2+V_\lambda g^2(v)\bigr)\,dx-\int_{\mathbb{R}^N}f\bigl(g(v_+)\bigr)g(v_+)\,dx\\&\ge\int_{\mathbb{R}^N}\bigl(|\nabla v|^2+V_\lambda g^2(v)\bigr)\,dx-\int_{\mathbb{R}^N}\bigl(\varepsilon g^2(v_+)+C_\varepsilon g^{p+1}(v_+)\bigr)\,dx\\&\ge\frac{1}{2}\int_{\mathbb{R}^N}\bigl(|\nabla v|^2+V_\lambda g^2(v)\bigr)\,dx-C_\varepsilon\int_{\mathbb{R}^N}g^{p+1}(v_+)\,dx\\&\ge C_1\min\bigl\{\|v\|_\lambda,\|v\|_\lambda^2\bigr\}-C_2\bigl(\max\bigl\{\|v\|_\lambda,\|v\|_\lambda^2\bigr\}\bigr)^{\frac{p+1}{2}}\\&=C_1\|v\|_\lambda^2-C_2\|v\|_\lambda^{\frac{p+1}{2}},\end{aligned}$$

and we can easily deduce the desired result. □

Lemma 3.6 *There exists a positive constant* $c_0 > 0$ *such that*

$$\limsup_{n\to\infty} \|v_n\|_\lambda \le \max\left\{ \frac{2\theta}{(\theta-4)C_1} c, \sqrt{\frac{2\theta}{(\theta-4)C_1} c} \right\}$$

and either $c \ge c_0$ *or* $c = 0$ *if* $\{v_n\}$ *is a* $(PS)_c$ *sequence for* Φ_λ, *where* C_1 *is the constant in Lemma* 3.1.

Proof Since $\{v_n\}$ is a $(\mathrm{PS})_c$ sequence, we have

$$\begin{aligned}
&\Phi_\lambda(v_n) - \frac{1}{\theta}\left\langle \Phi'_\lambda(v_n), \frac{g(v_n)}{g'(v_n)} \right\rangle \\
&\quad = \int_{\mathbb{R}^N} \left(\frac{1}{2} - \frac{1}{\theta}\left(1 + \frac{g^2(v_n)}{1+g^2(v_n)} \right) \right) |\nabla v_n|^2 \, dx + \left(\frac{1}{2} - \frac{1}{\theta} \right) \int_{\mathbb{R}^N} V_\lambda g^2(v_n) \, dx \\
&\qquad + \frac{1}{\theta} \int_{\mathbb{R}^N} \left(f\left(g\left((v_n)_+\right)\right) g\left((v_n)_+\right) - \theta F\left(g\left((v_n)_+\right)\right) \right) dx \\
&\quad = c + o(1) + o\left(\|v_n\|_\lambda\right).
\end{aligned}$$

It follows from (3.5) that

$$\limsup_{n\to\infty} \|v_n\|_\lambda \le \max\left\{ \frac{2\theta}{(\theta-4)C_1} c, \sqrt{\frac{2\theta}{(\theta-4)C_1} c} \right\}.$$

On the other hand, for $\|v_n\|_\lambda \le 1$, we have

$$\begin{aligned}
o\left(\|v_n\|_\lambda\right) &= \left\langle \Phi'_\lambda(v_n), \frac{g(v_n)}{g'(v_n)} \right\rangle \\
&\ge C_1 \min\left\{ \|v_n\|_\lambda, \|v_n\|_\lambda^2 \right\} - C_2 \left(\max\left\{ \|v_n\|_\lambda, \|v_n\|_\lambda^2 \right\} \right)^{\frac{p+1}{2}} \\
&= C_1 \|v_n\|_\lambda^2 - C_2 \|v_n\|_\lambda^{\frac{p+1}{2}}.
\end{aligned} \tag{3.6}$$

Thus, there exists $\sigma_1 > 0$ ($\sigma_1 < 1$) such that

$$\left\langle \Phi'_\lambda(v_n), \frac{g(v_n)}{g'(v_n)} \right\rangle \ge \frac{1}{4} C_1 \|v_n\|_\lambda^2 \quad \text{for } \|v_n\|_\lambda \le \sigma_1. \tag{3.7}$$

Taking $c_0 = \frac{\sigma_1}{\max\{\frac{2\theta}{(\theta-4)C_1}, \sqrt{\frac{2\theta}{(\theta-4)C_1}}\}}$, then we have

$$\max\left\{ \frac{2\theta}{(\theta-4)C_1} c, \sqrt{\frac{2\theta}{(\theta-4)C_1} c} \right\} < \sigma_1 < 1$$

if $c < c_0$. It follows from (3.6) and (3.7) that

$$\frac{1}{4} \|v_n\|_\lambda^2 \le o\left(\|v_n\|_\lambda\right),$$

hence, $\|v_n\|_\lambda \to 0$ and $c = 0$. Therefore, we have proved that there exists a constant c_0 such that either $c \ge c_0$ or $c = 0$. □

Proposition 3.7 *Let $M > 0$ be a constant. Then for any $\varepsilon > 0$, there exist $\Lambda_\varepsilon > 0$, $R_\varepsilon > 0$ such that*

$$\limsup_{n\to\infty}\int_{B^c_{R_\varepsilon}}\left(\frac{1}{2}f\big(g\big((v_n)_+\big)\big)g\big((v_n)_+\big)-F\big(g\big((v_n)_+\big)\big)\right)dx\le\varepsilon$$

if $\{v_n\}$ is a $(PS)_c$ sequence of Φ_λ with $\lambda > \Lambda_\varepsilon$, $c \le M$, where $B^c_{R_\varepsilon} = \{x \in \mathbb{R}^N \mid |x| \ge R_\varepsilon\}$.

Proof For all $R > 0$, let

$$A(R) := \{x\in\mathbb{R}^N \mid |x|\ge R, a(x)\ge M_0\},$$
$$B(R) := \{x\in\mathbb{R}^N \mid |x|\ge R, a(x)\le M_0\}.$$

We have

$$\begin{aligned}\int_{A(R)}g^2(v_n)\,dx &\le \frac{1}{\lambda M_0+1}\int_{A(R)}\big(\lambda a(x)+1\big)g^2(v_n)\,dx\\ &\le \frac{1}{\lambda M_0+1}\int_{A(R)}\big(|\nabla v_n|^2+\big(\lambda a(x)+1\big)g^2(v_n)\big)\,dx\\ &\le \frac{1}{\lambda M_0+1}\left(\frac{2\theta}{\theta-4}c+o\big(\|v_n\|_\lambda\big)\right)\\ &= \frac{1}{\lambda M_0+1}\left(\frac{2\theta}{\theta-4}c+o(1)\right)\\ &\to 0 \quad (\lambda\to\infty).\end{aligned}\tag{3.8}$$

On the other hand, by the Hölder inequality and interpolation inequality, we have

$$\begin{aligned}\int_{B(R)}g^2(v_n)\,dx &\le \left(\int_{B(R)}\big(g^2(v_n)\big)^q\,dx\right)^{\frac{1}{q}}\left(\int_{B(R)}1\,dx\right)^{\frac{q-1}{q}} \quad \big(1<q<2^*\big)\\ &\le \mu\big(B(R)\big)^{\frac{q-1}{q}}\left(\int_{\mathbb{R}^N}g^2(v_n)\,dx\right)^{\beta}\left(\int_{\mathbb{R}^N}g^{2\cdot 2^*}(v_n)\,dx\right)^{\frac{1-\beta}{2^*}} \quad (0<\beta<1)\\ &\le C\mu\big(B(R)\big)^{\frac{q-1}{q}}\big(\max\{\|v_n\|_\lambda,\|v_n\|_\lambda^2\}\big)^\beta\|v_n\|_\lambda^{1-\beta}\\ &\le C\mu\big(B(R)\big)^{\frac{q-1}{q}} \quad \big(\|v_n\|_\lambda \text{ is bounded}\big).\\ &\to 0 \quad (R\to\infty).\end{aligned}\tag{3.9}$$

By using the Gagliardo-Nirenberg inequality, we obtain

$$\begin{aligned}&\int_{B^c_{R_\varepsilon}}\left(\frac{1}{2}f\big(g\big((v_n)_+\big)\big)g\big((v_n)_+\big)-F\big(g\big((v_n)_+\big)\big)\right)dx\\ &\quad\le\int_{B^c_{R_\varepsilon}}\left(\frac{1}{2}f\big(g\big((v_n)_+\big)\big)g\big((v_n)_+\big)-C\big(g\big((v_n)_+\big)\big)^\theta\right)dx \quad \big(\text{by } (f_3)\big)\\ &\quad\le C\int_{B^c_R}\big|g(v_n)\big|^{p+1}\,dx\end{aligned}$$

$$\begin{aligned}
&\le C\left(\int_{B_R^c} \left|\nabla g(v_n)\right|^2 dx\right)^{\frac{(p+1)\beta}{2}} \left(\int_{B_R^c} g^2(v_n)\,dx\right)^{\frac{(p+1)(1-\beta)}{2}} \quad \left(\beta = \frac{N(p-1)}{2(p+1)}\right) \\
&\le C\|v_n\|_\lambda^{(p+1)\beta}\left(\int_{A(R)} g^2(v_n)\,dx + \int_{B(R)} g^2(v_n)\,dx\right)^{\frac{(p+1)(1-\beta)}{2}} \\
&\le C\left(\int_{A(R)} g^2(v_n)\,dx + \int_{B(R)} g^2(v_n)\,dx\right)^{\frac{(p+1)(1-\beta)}{2}} \quad \bigl(\|v_n\|_\lambda \text{ is bounded}\bigr).
\end{aligned}$$

Let λ and R be large enough, from (3.8) and (3.9), we get the desired result. □

Lemma 3.8 $c_\lambda = \inf_{N_\lambda} \Phi_\lambda(v)$ *is achieved by some* $v > 0$.

Proof By the definition of c_λ and the Ekeland variational principle, there exists a $(PS)_c$ sequence $\{v_n\}$, by Lemma 3.4, we know that $\{v_n\}$ is bounded. Hence (up to a subsequence) we have $v_n \rightharpoonup v$ in L^{2^*}, $\nabla v_n \rightharpoonup \nabla v$ in L^2, $v_n \to v$ a.e. in $\mathbb{R}^N$, $g(v_n) \rightharpoonup g(v)$ in L^q for $2 \le q \le 2 \cdot 2^*$.

It is sufficient to prove that $v \neq 0$ and $v \in N_\lambda$. In fact,

$$\begin{aligned}
&\Phi_\lambda(v_n) - \frac{1}{2}\left\langle \Phi_\lambda'(v_n), \frac{g(v_n)}{g'(v_n)}\right\rangle \\
&\quad = -\frac{1}{2}\int_{\mathbb{R}^N} \frac{g^2(v_n)}{1+g^2(v_n)}|\nabla v_n|^2\,dx + \int_{\mathbb{R}^N}\left(\frac{1}{2}f\bigl(g\bigl((v_n)_+\bigr)\bigr)g\bigl((v_n)_+\bigr) - F\bigl(g\bigl((v_n)_+\bigr)\bigr)\right)dx \\
&\quad = c + o(1) + o\bigl(\|v_n\|_\lambda\bigr),
\end{aligned} \tag{3.10}$$

it follows that

$$\int_{\mathbb{R}^N}\left(\frac{1}{2}f\bigl(g\bigl((v_n)_+\bigr)\bigr)g\bigl((v_n)_+\bigr) - F\bigl(g\bigl((v_n)_+\bigr)\bigr)\right) \ge c + o(1) + o\bigl(\|v_n\|_\lambda\bigr).$$

Let $\varepsilon_0 = \frac{1}{4}c$, since $g(v_n) \to g(v)$ strongly in $L^{p+1}(B_R)$ for $R > 0$, by Proposition 3.7, there exist $\Lambda_0 > 0$, $R_0 > 0$ such that for $\lambda \ge \Lambda_0$, $R \ge R_0$,

$$\int_{B_R^c}\left|\left(\frac{1}{2}f\bigl(g\bigl((v_n)_+\bigr)\bigr)g\bigl((v_n)_+\bigr) - F\bigl(g\bigl((v_n)_+\bigr)\bigr)\right) - \left(\frac{1}{2}f\bigl(g(v_+)\bigr)g(v_+) - F\bigl(g(v_+)\bigr)\right)\right|dx < \varepsilon_0,$$

thus

$$\int_{B_R}\left(\frac{1}{2}f\bigl(g(v_+)\bigr)g(v_+) - F\bigl(g(v_+)\bigr)\right)dx \ge \frac{1}{2}c > 0.$$

Hence $v \neq 0$.

Now we prove $v \in N_\lambda$. Indeed, since $\{v_n\}$ is a $(PS)_c$ sequence, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} \nabla v_n \nabla v\,dx + \int_{\mathbb{R}^N} V_\lambda g(v_n)g'(v_n)v\,dx \\
&\quad = \int_{\mathbb{R}^N} f\bigl(g\bigl((v_n)_+\bigr)\bigr)g'\bigl((v_n)_+\bigr)v\,dx + o\bigl(\|v_n\|_\lambda\bigr),
\end{aligned} \tag{3.11}$$

where $o(\|v_n\|_\lambda) \to 0$ as $n \to \infty$.

Let $l_n := \frac{g(v_n)}{\sqrt{1+g^2(v_n)}}$, then $\{l_n\}$ is bounded in $L^q(\mathbb{R}^N)$ for $2 \le q \le 2 \cdot 2^*$, by the continuity of g, we have, up to a subsequence, $l_n \rightharpoonup l = \frac{g(v)}{\sqrt{1+g^2(v)}}$ in $L^q(\mathbb{R}^N)$.

Similarly, we have $s_n := \frac{f(g(v_n))}{\sqrt{1+g^2(v_n)}} \le f(g(v_n))$ is bounded in $L^{\frac{p+1}{p}}(\mathbb{R}^N)$. Again, by the continuity of g, we have $s_n \rightharpoonup s = \frac{f(g(v))}{\sqrt{1+g^2(v)}}$ in $L^{\frac{p+1}{p}}(\mathbb{R}^N)$. Passing to the limits in (3.11), we get

$$\int_{\mathbb{R}^N} |\nabla v|^2\,dx + \int_{\mathbb{R}^N} V_\lambda g(v)g'(v)v\,dx = \int_{\mathbb{R}^N} f\bigl(g(v_+)\bigr)g'(v_+)v\,dx,$$

which is equivalent to $\langle \Phi'_\lambda(v), v\rangle = 0$, that is, $v \in N_\lambda$. □

4 Proof of the main result

Consider the following quasilinear Schrödinger equation in $\Omega \subset \mathbb{R}^N$ ($N \ge 3$):

$$\begin{cases} -\triangle u + u - \frac{1}{2}(\triangle |u|^2)u = f(u), & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{D}$$

We have the same change of variables and the same notation as in the previous sections. Define the corresponding Orlicz space $E_G(\Omega)$ by

$$E_G(\Omega) = \left\{ v \,\middle|\, \int_\Omega g^2(v)\,dx < +\infty \right\}$$

with the norm

$$|v|_{G(\Omega)} := \inf_{\xi>0} \xi \left(1 + \int_\Omega G\bigl(\xi^{-1}v\bigr)\,dx \right).$$

The space $H_G(\Omega)$ is defined by

$$H_G(\Omega) := \left\{ v \,\middle|\, \int_\Omega |\nabla v|^2\,dx < +\infty, \int_\Omega g^2(v)\,dx < +\infty \right\}$$

with the norm

$$\|v\|_\Omega = \|v\|_{L^2} + |v|_{G(\Omega)}.$$

The following Lemma 4.1 is a counterpart of Lemma 3.1.

Lemma 4.1 *There exist two constants $C_1 > 0$, $C_2 > 0$ such that*

$$C_1 \min\bigl\{\|v\|_\Omega, \|v\|_\Omega^2\bigr\} \le \int_\Omega |\nabla v|^2\,dx + \int_\Omega g^2(v)\,dx \le C_2 \max\bigl\{\|v\|_\Omega, \|v\|_\Omega^2\bigr\}$$

for any $v \in H_G(\Omega)$.

We denote by $H_G^0(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $H_G(\Omega)$. We define the functional Φ_Ω on $H_G^0(\Omega)$ by

$$\Phi_\Omega(v) = \frac{1}{2}\int_\Omega \bigl(|\nabla v|^2 + g^2(v)\bigr)\,dx - \int_\Omega F\bigl(g(v_+)\bigr)\,dx, \tag{4.1}$$

and we define the Nehari manifold N_Ω by

$$N_\Omega := \bigl\{v \in H_G(\Omega)\backslash\{0\} | \langle \Phi'_\Omega(v), v\rangle = 0\bigr\}.$$

Let

$$c(\Omega) = \inf_{N_\Omega} \Phi_\Omega(v).$$

We recall that $u = g(v)$ is a least energy solution of (D) if $v \in N_\Omega$ such that $c(\Omega)$ is achieved.

Lemma 4.2 *Suppose* $c_\lambda = \inf_{N_\lambda} \Phi_\lambda$. *Then* $\lim_{\lambda\to+\infty} c_\lambda = c(\Omega)$.

Proof It is easy to see that $c_\lambda \le c(\Omega)$ for $\lambda \ge 1$. We claim that c_λ is monotone increasing with respect to λ. In fact, for $\lambda_1 < \lambda_2$, we assume that c_{λ_1}, c_{λ_2} are achieved for $v_{\lambda_1} > 0$, $v_{\lambda_2} > 0$. Obviously,

$$\int_{\mathbb{R}^N} |\nabla v_{\lambda_2}|^2\,dx + \int_{\mathbb{R}^N} (\lambda_1 a + 1) g(v_{\lambda_2}) g'(v_{\lambda_2}) v_{\lambda_2}\,dx < \int_{\mathbb{R}^N} f\bigl(g(v_{\lambda_2})\bigr) g'(v_{\lambda_2}) v_{\lambda_2}\,dx. \tag{4.2}$$

We first prove that there exists $0 < \alpha < 1$ such that $\alpha v_{\lambda_2} \in N_{\lambda_1}$. This is sufficient to prove that

$$\begin{aligned}&\alpha^2 \int_{\mathbb{R}^N} |\nabla v_{\lambda_2}|^2\,dx + \int_{\mathbb{R}^N} (\lambda_1 a + 1)\frac{g(\alpha v_{\lambda_2})}{\sqrt{1+g^2(\alpha v_{\lambda_2})}}\alpha v_{\lambda_2}\,dx \\ &\quad= \int_{\mathbb{R}^N} f\bigl(g(\alpha v_{\lambda_2})\bigr) g'(\alpha v_{\lambda_2})\alpha v_{\lambda_2}\,dx.\end{aligned}$$

That is,

$$\alpha \int_{\mathbb{R}^N} |\nabla v_{\lambda_2}|^2\,dx + \int_{\mathbb{R}^N} (\lambda_1 a + 1)\frac{g(\alpha v_{\lambda_2})}{\sqrt{1+g^2(\alpha v_{\lambda_2})}} v_{\lambda_2}\,dx = \int_{\mathbb{R}^N} f\bigl(g(\alpha v_{\lambda_2})\bigr) g'(\alpha v_{\lambda_2}) v_{\lambda_2}\,dx.$$

Let

$$I(\alpha) = \int_{\mathbb{R}^N} \frac{(f(g(\alpha v_{\lambda_2})) - (\lambda_1 a + 1) g(\alpha v_{\lambda_2})) v_{\lambda_2}}{\alpha\sqrt{1+g^2(\alpha v_{\lambda_2})}}\,dx.$$

Then by (f_1), we can obtain $\lim_{\alpha\to 0} I(\alpha) \le 0$ and

$$0 < \int_{\mathbb{R}^N} |\nabla v_{\lambda_2}|^2\,dx < I(1).$$

Hence, there exists $0 < \alpha_0 < 1$ such that $\int_{\mathbb{R}^N} |\nabla v_{\lambda_2}|^2\,dx = I(\alpha_0)$, *i.e.*, $\alpha_0 v_{\lambda_2} \in N_{\lambda_1}$. Thus

$$c_{\lambda_1} \le \Phi_{\lambda_1}(\alpha_0 v_{\lambda_2}).$$

In the following, we will prove that

$$\Phi_{\lambda_1}(\alpha_0 v_{\lambda_2}) \le \Phi_{\lambda_2}(v_{\lambda_2}) = c_{\lambda_2}.$$

In fact, we consider the function $\rho(\alpha)$ defined by

$$\rho(\alpha) = \frac{(f(g(\alpha v_{\lambda_2})) - (\lambda_2 a + 1)g(\alpha v_{\lambda_2}))v_{\lambda_2}}{\alpha\sqrt{1 + g^2(\alpha v_{\lambda_2})}}.$$

By $g(t)\sqrt{1+g^2(t)} \le 2t$ for $t \ge 0$, we have $\frac{g(\alpha v_{\lambda_2})}{g'(\alpha v_{\lambda_2})} = g(\alpha v_{\lambda_2})\sqrt{1+g^2(\alpha v_{\lambda_2})} \le 2\alpha v_{\lambda_2} \le (\theta - 2)\alpha v_{\lambda_2}$. It follows that

$$\frac{\alpha v_{\lambda_2} g(\alpha v_{\lambda_2})}{g(\alpha v_{\lambda_2})/g'(\alpha v_{\lambda_2}) + \alpha v_{\lambda_2}} \ge \frac{\alpha v_{\lambda_2} g(\alpha v_{\lambda_2})}{(\theta - 2)\alpha v_{\lambda_2} + \alpha v_{\lambda_2}}.$$

Obviously,

$$f'\big(g(\alpha v_{\lambda_2})\big)g'(\alpha v_{\lambda_2})v_{\lambda_2}^2 \alpha g(\alpha v_{\lambda_2}) - f\big(g(\alpha v_{\lambda_2})\big)v_{\lambda_2}\big(g(\alpha v_{\lambda_2}) + \alpha g'(\alpha v_{\lambda_2})v_{\lambda_2}\big) \ge 0$$

and hence it is easy to check that

$$\begin{aligned}
&\frac{d}{d\alpha}\left(\frac{f(g(\alpha v_{\lambda_2}))v_{\lambda_2}}{\alpha\sqrt{1+g^2(\alpha v_{\lambda_2})}}\right) \\
&\quad = \frac{d}{d\alpha}\left(\frac{g(\alpha v_{\lambda_2})}{\sqrt{1+g^2(\alpha v_{\lambda_2})}}\right)\frac{f(g(\alpha v_{\lambda_2}))v_{\lambda_2}}{\alpha g(\alpha v_{\lambda_2})} + \frac{d}{d\alpha}\left(\frac{f(g(\alpha v_{\lambda_2}))v_{\lambda_2}}{\alpha g(\alpha v_{\lambda_2})}\right)\frac{g(\alpha v_{\lambda_2})}{\sqrt{1+g^2(\alpha v_{\lambda_2})}} \\
&\quad = \frac{v_{\lambda_2}^2}{(1+g^2(\alpha v_{\lambda_2}))^2}\cdot\frac{f(g(\alpha v_{\lambda_2}))v_{\lambda_2}}{\alpha g(\alpha v_{\lambda_2})} \\
&\qquad + \frac{f'(g(\alpha v_{\lambda_2}))g'(\alpha v_{\lambda_2})v_{\lambda_2}^2\alpha g(\alpha v_{\lambda_2}) - f(g(\alpha v_{\lambda_2}))v_{\lambda_2}(g(\alpha v_{\lambda_2}) + \alpha g'(\alpha v_{\lambda_2})v_{\lambda_2})}{\alpha^2 g^2(\alpha v_{\lambda_2})} \\
&\qquad \times \frac{g(\alpha v_{\lambda_2})}{\sqrt{1+g^2(\alpha v_{\lambda_2})}} \\
&\quad \ge \frac{f'(g(\alpha v_{\lambda_2}))g'(\alpha v_{\lambda_2})v_{\lambda_2}^2\alpha g(\alpha v_{\lambda_2}) - f(g(\alpha v_{\lambda_2}))v_{\lambda_2}(g(\alpha v_{\lambda_2}) + \alpha g'(\alpha v_{\lambda_2})v_{\lambda_2})}{\alpha^2 g(\alpha v_{\lambda_2})\sqrt{1+g^2(\alpha v_{\lambda_2})}} \\
&\quad \ge 0.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\frac{d}{d\alpha}\left((\lambda_2 a + 1)\frac{g(\alpha v_{\lambda_2})v_{\lambda_2}}{\alpha\sqrt{1+g^2(\alpha v_{\lambda_2})}}\right) \\
&\quad = (\lambda_2 a + 1)\frac{\alpha v_{\lambda_2}^2 - g(\alpha v_{\lambda_2})\sqrt{1+g^2(\alpha v_{\lambda_2})}(1+g^2(\alpha v_{\lambda_2}))}{\alpha^2(1+g^2(\alpha v_{\lambda_2}))^2}v_{\lambda_2},
\end{aligned}$$

by $v = \frac{1}{2}g(v)\sqrt{1+g^2(v)} + \frac{1}{2}\ln(g(v) + \sqrt{1+g^2(v)})$, it is easy to check that for any $t \ge 0$,

$$t - g(t)\sqrt{1+g^2(t)}\big(1+g^2(t)\big) \le 0,$$

which implies

$$\frac{d}{d\alpha}\left((\lambda_2 a+1)\frac{g(\alpha v_{\lambda_2})v_{\lambda_2}}{\alpha\sqrt{1+g^2(\alpha v_{\lambda_2})}}\right)\le 0$$

for any $\alpha > 0$, thus we have proved that $\rho(\alpha)$ is monotone increasing for $\alpha > 0$.

Now we consider the function $\gamma(\alpha)$ defined by

$$\gamma(\alpha)=\frac{1}{2}\int_{\mathbb{R}^N}\left(\alpha^2|\nabla v_{\lambda_2}|^2+\left(\lambda_2 a(x)+1\right)g^2(\alpha v_{\lambda_2})\right)dx-\int_{\mathbb{R}^N}F\left(g(\alpha v_{\lambda_2})\right)dx.$$

Then

$$\begin{aligned}\frac{d\gamma(\alpha)}{d\alpha} &= \int_{\mathbb{R}^N}\left(\alpha|\nabla v_{\lambda_2}|^2+\left(\lambda_2 a(x)+1\right)g(\alpha v_{\lambda_2})g'(\alpha v_{\lambda_2})v_{\lambda_2}\right)dx \\ &\quad -\int_{\mathbb{R}^N}f\left(g(\alpha v_{\lambda_2})\right)g'(\alpha v_{\lambda_2})v_{\lambda_2}\,dx \\ &= \alpha\left(\int_{\mathbb{R}^N}f\left(g(v_{\lambda_2})\right)g'(v_{\lambda_2})v_{\lambda_2}\,dx-\int_{\mathbb{R}^N}\left(\lambda_2 a(x)+1\right)g(v_{\lambda_2})g'(v_{\lambda_2})v_{\lambda_2}\,dx\right) \\ &\quad +\left(\int_{\mathbb{R}^N}\left(\lambda_2 a(x)+1\right)g(\alpha v_{\lambda_2})g'(\alpha v_{\lambda_2})v_{\lambda_2}\,dx-\int_{\mathbb{R}^N}f\left(g(\alpha v_{\lambda_2})\right)g'(\alpha v_{\lambda_2})v_{\lambda_2}\,dx\right) \\ &= \alpha\int_{\mathbb{R}^N}\frac{(f(g(v_{\lambda_2}))-(\lambda_2 a(x)+1)g(v_{\lambda_2}))v_{\lambda_2}}{\sqrt{1+g^2(v_{\lambda_2})}}\,dx \\ &\quad -\alpha\int_{\mathbb{R}^N}\frac{(f(g(\alpha v_{\lambda_2}))-(\lambda_2 a(x)+1)g(\alpha v_{\lambda_2}))v_{\lambda_2}}{\alpha\sqrt{1+g^2(\alpha v_{\lambda_2})}}\,dx \\ &\ge 0 \quad \left(\text{by the monotonicity of } \rho(\alpha)\right)\end{aligned}$$

for $0 < \alpha < 1$. Therefore, $\gamma(\alpha)$ is monotone increasing with respect to $\alpha < 1$. Thus, we deduce that

$$\begin{aligned}c_{\lambda_1} &\le \Phi_{\lambda_1}(\alpha v_{\lambda_2}) \\ &= \frac{1}{2}\int_{\mathbb{R}^N}\alpha^2|\nabla v_{\lambda_2}|^2\,dx+\int_{\mathbb{R}^N}\left(\lambda_1 a(x)+1\right)g^2(\alpha v_{\lambda_2})\,dx-\int_{\mathbb{R}^N}F\left(g(\alpha v_{\lambda_2})\right)dx \\ &\le \frac{1}{2}\int_{\mathbb{R}^N}|\nabla v_{\lambda_2}|^2\,dx+\int_{\mathbb{R}^N}\left(\lambda_2 a(x)+1\right)g^2(v_{\lambda_2})\,dx-\int_{\mathbb{R}^N}F\left(g(v_{\lambda_2})\right)dx \\ &= \Phi_{\lambda_2}(v_{\lambda_2})=c_{\lambda_2}.\end{aligned}$$

Assume that $\lim_{\lambda\to\infty} c_\lambda = k \le c(\Omega)$. If $k < c(\Omega)$, then for any sequence $\{\lambda_n\}$ ($\lambda_n \to +\infty$), we have $c_{\lambda_n} \to k < c(\Omega)$.

We assume that v_n is such that c_{λ_n} is achieved, by Lemma 3.4, $\{v_n\}$ is bounded in $H_G^{\lambda_n}$. Since $\|v_n\|_{H_G^0} \le \|v_n\|_{H_G^{\lambda_n}}$, $\{v_n\}$ is bounded in H_G^0, as a result, we have $\nabla v_n \rightharpoonup \nabla v$ in $L^2(\mathbb{R}^N)$, $g(v_n) \to g(v)$ in $L^q_{\text{loc}}(\mathbb{R}^N)$ for $2 \le q \le 2\cdot 2^*$, $v_n \rightharpoonup v$ in $L^q(\mathbb{R}^N)$ for $2 \le q \le 2\cdot 2^*$, $v_n \to v$ a.e. in $\mathbb{R}^N$.

We claim that $v|_{\Omega^c} = 0$, where $\Omega^c := \{x | x \in \mathbb{R}^N\setminus\Omega\}$. Indeed, it is sufficient to prove $g(v)|_{\Omega^c} = 0$. If not, then there exists a compact subset $\Sigma \subset \Omega^c$ with $\text{dist}\{\Sigma, \partial\Omega\} > 0$ such that $g(v)|_\Sigma \not\equiv 0$ and

$$\int_\Sigma g^2(v_n)\,dx \to \int_\Sigma g^2(v)\,dx > 0.$$

Moreover, there exists $\varepsilon_0 > 0$ such that $a(x) \geq \varepsilon_0$ for any $x \in \Sigma$.

By the choice of v_n, we have

$$\begin{aligned} 0 &= \left\langle \Phi'_{\lambda_n}(v_n), \frac{g(v_n)}{g'(v_n)} \right\rangle \\ &= \int_{\mathbb{R}^N} \left(1 + \frac{g^2(v_n)}{1+g^2(v_n)}\right)|\nabla v_n|^2\,dx + \int_{\mathbb{R}^N} V_{\lambda_n} g^2(v_n)\,dx - \int_{\mathbb{R}^N} f\big(g((v_n)_+)\big)g\big((v_n)_+\big)\,dx, \end{aligned}$$

hence,

$$\begin{aligned} \Phi_{\lambda_n}(v_n) &= \frac{1}{2}\int_{\mathbb{R}^N} \big(|\nabla v_n|^2 + V_{\lambda_n} g^2(v_n)\big)\,dx - \int_{\mathbb{R}^N} F\big(g((v_n)_+)\big)\,dx \\ &\geq \frac{1}{2}\int_{\mathbb{R}^N} \big(|\nabla v_n|^2 + V_{\lambda_n} g^2(v_n)\big)\,dx - \frac{1}{\theta}\int_{\mathbb{R}^N} f\big(g((v_n)_+)\big)g\big((v_n)_+\big)\,dx, \\ &= \frac{1}{2}\int_{\mathbb{R}^N} \big(|\nabla v_n|^2 + V_{\lambda_n} g^2(v_n)\big)\,dx \\ &\quad - \frac{1}{\theta}\int_{\mathbb{R}^N} \left(\left(1 + \frac{g^2(v_n)}{1+g^2(v_n)}\right)|\nabla v_n|^2 + V_{\lambda_n} g^2(v_n)\right)dx \\ &= \int_{\mathbb{R}^N} \left(\frac{1}{2} - \frac{1}{\theta}\left(1 + \frac{g^2(v_n)}{1+g^2(v_n)}\right)\right)|\nabla v_n|^2\,dx + \left(\frac{1}{2} - \frac{1}{\theta}\right)\int_{\mathbb{R}^N} V_{\lambda_n} g^2(v_n)\,dx \\ &\geq \left(\frac{1}{2} - \frac{1}{\theta}\right)\int_\Sigma (\lambda_n \varepsilon_0 + 1) g^2(v_n)\,dx \\ &\to +\infty \quad (n \to \infty). \end{aligned}$$

This contradiction shows that $g(v)|_{\Omega^c} = 0$ and so does v.

Now we show that

$$g(v_n) \to g(v) \quad \text{in } L^q\big(\mathbb{R}^N\big) \text{ for } 2 < q < 2^*. \tag{4.3}$$

Suppose that (4.3) is not true, then by the concentration compactness principle of Lions (see [12]), there exist $\delta > 0$, $\varrho > 0$ and $x_n \in \mathbb{R}^N$ with $|x_n| \to +\infty$ such that

$$\limsup_{n\to\infty} \int_{B_\varrho(x_n)} \big|g(v_n) - g(v)\big|^2 \geq \delta > 0.$$

On the other hand, by the choice of $\{v_n\}$, we have

$$\begin{aligned} \Phi_{\lambda_n}(v_n) &= \frac{1}{2}\int_{\mathbb{R}^N} \big(|\nabla v_n|^2 + V_{\lambda_n} g^2(v_n)\big)\,dx - \int_{\mathbb{R}^N} F\big(g((v_n)_+)\big)\,dx \\ &\geq \left(\frac{1}{2} - \frac{1}{\theta}\right)\int_{B_\varrho(x_n)\cap\{x|a(x)\geq M_0\}} \big(\lambda_n a(x) + 1\big) g^2(v_n)\,dx \\ &\geq \left(\frac{1}{2} - \frac{1}{\theta}\right)\int_{B_\varrho(x_n)\cap\{x|a(x)\geq M_0\}} \lambda_n a(x)\big|g(v_n) - g(v)\big|^2\,dx \end{aligned}$$

$$
\begin{aligned}
&\geq \left(\frac{1}{2}-\frac{1}{\theta}\right)\lambda_n M_0 \\
&\quad \times \left(\int_{B_\varrho(x_n)} \left|g(v_n)-g(v)\right|^2 dx - \int_{B_\varrho(x_n)\cap\{x|a(x)\leq M_0\}} \left|g(v_n)-g(v)\right|^2 dx\right) \\
&= \left(\frac{1}{2}-\frac{1}{\theta}\right)\lambda_n M_0 \left(\int_{B_\varrho(x_n)} \left|g(v_n)-g(v)\right|^2 dx - o(1)\right) \\
&\to +\infty \quad (n\to\infty),
\end{aligned}
$$

which shows that $g(v_n)\to g(v)$ in $L^q(\mathbb{R}^N)$ for $2<q<2^*$. In the above proof, we have used the fact that $\mu\{B_\varrho(x_n)\cap\{x|a(x)\leq M_0\}\}\to 0$ as $n\to\infty$ and the L^2 bounded property of $g(v_n)$.

Now, since $\{v_n\}$ is bounded, by the Fatou lemma, we obtain

$$
\begin{aligned}
&\int_\Omega |\nabla v|^2\,dx \leq \lim_{n\to\infty}\int_{\mathbb{R}^N} |\nabla v_n|^2\,dx, \\
&\int_\Omega g'(v)g(v)v\,dx \leq \lim_{n\to\infty}\int_{\mathbb{R}^N} g'(v_n)g(v_n)v_n\,dx.
\end{aligned}
$$

But, by the choice of v_n, we have

$$
\int_{\mathbb{R}^N} \left(|\nabla v_n|^2 + (\lambda_n a+1)g(v_n)g'(v_n)v_n\right) dx = \int_{\mathbb{R}^N} f\left(g\left((v_n)_+\right)\right)g'\left((v_n)_+\right)v_n\,dx,
$$

hence,

$$
\begin{aligned}
\int_\Omega \left(|\nabla v|^2 + g(v)g'(v)v\right) dx &\leq \lim_{n\to\infty}\int_{\mathbb{R}^N} |\nabla v_n|^2\,dx + \lim_{n\to\infty}\int_{\mathbb{R}^N} g(v_n)g'(v_n)v_n\,dx \\
&\leq \lim_{n\to\infty}\int_{\mathbb{R}^N} f\left(g\left((v_n)_+\right)\right)g'\left((v_n)_+\right)v_n\,dx. \qquad (4.4)
\end{aligned}
$$

In the following, we will prove that

$$
\int_{\mathbb{R}^N} \left|f\left(g\left((v_n)_+\right)\right)g'\left((v_n)_+\right)v_n - f\left(g(v_+)\right)g'(v_+)v\right| dx \to 0.
$$

Indeed,

$$
\begin{aligned}
&\int_{\mathbb{R}^N} \left(f\left(g\left((v_n)_+\right)\right)g'\left((v_n)_+\right)v_n - f\left(g(v_+)\right)g'(v_+)v\right) dx \\
&\quad = \int_{\mathbb{R}^N} \left(f\left(g\left((v_n)_+\right)\right)g'\left((v_n)_+\right)v_n - f\left(g(v_+)\right)g'\left((v_n)_+\right)v_n\right) dx \\
&\qquad + \int_{\mathbb{R}^N} \left(f\left(g(v_+)\right)g'\left((v_n)_+\right)v_n - f\left(g(v_+)\right)g'(v_+)v\right) dx \\
&\quad := I_1 + I_2.
\end{aligned}
$$

Since $f(g(v_+))g'((v_n)_+)v_n \rightharpoonup f(g(v_+))g'(v_+)v$, one can easily see that $I_2\to 0$ as $n\to\infty$, and

$$
\begin{aligned}
I_1 &= \int_{\mathbb{R}^N} \left(f\left(g\left((v_n)_+\right)\right)g'\left((v_n)_+\right)v_n - f\left(g(v_+)\right)g'\left((v_n)_+\right)v_n\right) dx \\
&= \int_{\mathbb{R}^N} \left(f\left(g\left((v_n)_+\right)\right) - f\left(g(v_+)\right)\right)g'\left((v_n)_+\right)v_n\,dx
\end{aligned}
$$

$$\begin{aligned}&\leq \int_{\mathbb{R}^N} \bigl|f\bigl(g\bigl((v_n)_+\bigr)\bigr) - f\bigl(g(v_+)\bigr)\bigr| \cdot |v_n|\,dx \\ &\leq \bigl\|f\bigl(g\bigl((v_n)_+\bigr)\bigr) - f\bigl(g(v_+)\bigr)\bigr\|_q \|v_n\|_{q'} \quad \bigl(1/q + 1/q' = 1\bigr) \\ &\to 0 \quad (n \to \infty)\end{aligned}$$

by using $g(v_n) \to g(v)$ in $L^q(\mathbb{R}^N)$ for $2 < q < 2^*$. It follows from (4.4) that

$$\int_\Omega \bigl(|\nabla v|^2 + g(v)g'(v)v\bigr)\,dx \leq \int_{\mathbb{R}^N} f\bigl(g(v_+)\bigr)g'(v_+)v\,dx,$$

thus, there exists $0 < \alpha_0 \leq 1$ such that $\alpha_0 v \in N_\Omega$ and

$$\Phi_\Omega(\alpha_0 v) \leq \Phi_\Omega(v),$$

hence $c(\Omega) \leq \Phi_\Omega(\alpha_0 v) < \Phi_\Omega(v) \leq \lim_{n\to\infty} \Phi_{\lambda_n} = k < c(\Omega)$. A contradiction. Thus we have proved that $\lim_{\lambda\to\infty} c_\lambda \to c(\Omega)$ as $\lambda \to +\infty$. □

Proof of Theorem 2.1 Suppose that $\{v_n\}$ is a sequence such that $v_n \in N_\lambda$, $\Phi_{\lambda_n}(v_n) = c_{\lambda_n}$, by the proof of Lemma 3.2, we have $\nabla v_n \rightharpoonup \nabla v$ in $L^2(\mathbb{R}^N)$, $g(v_n) \to g(v)$ in $L^q(\mathbb{R}^N)$ for $2 < q < 2^*$ and $v|_{\Omega^c} = 0$. Moreover, $\Phi_\Omega(v) \leq \lim_{n\to\infty} \Phi_{\lambda_n}(v_n) \leq c(\Omega)$, and if $v \in N_\Omega$, then $\Phi_\Omega(v) = c(\Omega)$. Hence, in the following, we need only to prove that $v \in N_\Omega$. To do this, it is sufficient to prove that

$$\begin{aligned}&\lim_{n\to\infty} \int_{\mathbb{R}^N} \lambda_n a(x) g^2(v_n)\,dx = 0, \\ &\lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla v_n|^2\,dx = \int_{\mathbb{R}^N} |\nabla v|^2\,dx\end{aligned}$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} g(v_n)g'(v_n)v_n\,dx = \int_{\mathbb{R}^N} g(v)g'(v)v\,dx.$$

In fact, if one of the above three limits does not hold, by the Fatou lemma, we have

$$\int_{\mathbb{R}^N} |\nabla v|^2\,dx + \int_{\mathbb{R}^N} g(v)g'(v)v\,dx < \int_{\mathbb{R}^N} f\bigl(g(v)\bigr)g'(v)v\,dx.$$

Similar to above, there exists $\alpha_0 \in (0,1)$ such that $\alpha_0 v \in N_\Omega$ and $c(\Omega) \leq \Phi_\Omega(\alpha_0 v) < \Phi_\Omega(v) \leq \lim_{n\to\infty} \Phi_{\lambda_n}(v_n) \leq c(\Omega)$. A contradiction, and thus we complete the proof of Theorem 2.1. □

Competing interests
The author declares that she has no competing interests.

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