Lian and Yang *Boundary Value Problems* 2013, **2013**:90
http://www.boundaryvalueproblems.com/content/2013/1/90

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Global solution to the exterior problem for spherically symmetric compressible Navier-Stokes equations with density-dependent viscosity and discontinuous initial data

Ruxu Lian* and Jianwei Yang

*Correspondence: ruxu.lian.math@gmail.com
College of Mathematics and Information Science, North China University of Water Resources and Electric Power, Zhengzhou, 450011, P.R. China

Abstract
In this paper, we consider the exterior problem for spherically symmetric isentropic compressible Navier-Stokes equations with density-dependent viscosity coefficients and discontinuous initial data. We prove that there exists a unique global piecewise regular solution for piecewise regular initial density with bounded jump discontinuity. In particular, the jump of density decays exponentially in time and the piecewise regular solution tends to the equilibrium state exponentially as $t \to +\infty$.

Keywords: spherically symmetric Navier-Stokes equations; exterior problem; discontinuous initial data

1 Introduction

In the present paper, we consider the exterior problem to N-dimensional isentropic compressible Navier-Stokes equations with density-dependent viscosity coefficients. In general, the N-dimensional isentropic compressible Navier-Stokes equations with density-dependent viscosity coefficients reads

$$\begin{cases} \rho_t + \operatorname{div}(\rho \mathbf{U}) = 0, \\ (\rho \mathbf{U})_t + \operatorname{div}(\rho \mathbf{U} \otimes \mathbf{U}) + \nabla P(\rho) - \operatorname{div}(\mu(\rho) D(\mathbf{U})) - \nabla(\lambda(\rho) \operatorname{div} \mathbf{U}) = 0, \end{cases} \tag{1.1}$$

where $t \in (0, +\infty)$ is the time and $\mathbf{x} \in R^N$, N is the spatial coordinate, $\rho > 0$ and u denote the density and velocity, respectively. Pressure function is taken as $P(\rho) = \rho^\gamma$ with $\gamma > 1$, and

$$D(\mathbf{U}) = \frac{\nabla(\mathbf{U}) +^{\mathbb{T}} \nabla(\mathbf{U})}{2} \tag{1.2}$$

is the strain tensor and $\mu(\rho)$, $\lambda(\rho)$ are the Lamé viscosity coefficients satisfying

$$\mu(\rho) > 0, \qquad \mu(\rho) + N\lambda(\rho) \geq 0. \tag{1.3}$$

There are many significant progresses achieved on the global existence of weak solutions and dynamical behaviors of jump discontinuity for the compressible Navier-Stokes equations with discontinuous initial data, for example, as the viscosity coefficients $\mu(\rho)$ and $\lambda(\rho)$ are both constants, Hoff investigated the global existence of discontinuous solutions of one-dimension Navier-Stokes equations [1–3]. The construction of global spherically symmetric weak solutions of compressible Navier-Stokes equations for isothermal flow with large and discontinuous initial data was derived by Hoff [4], therein it is also showed that the discontinuities in the density and pressure persist for all time, convecting along particle trajectories, and decaying at a rate inversely proportional to the viscosity coefficient. Hoff also proved the global existence theorems for the multidimensional Navier-Stokes equations of isothermal compressible flows with the polytropic equation of state $p(\rho) = \rho^{\gamma}$ $(\gamma \geq 1)$ [5, 6]. The global existence of weak solutions was obtained for the Navier-Stokes equations for compressible, heat-conducting flow in one space dimension with large, discontinuous initial data by Chen-Hoff-Trivisa [7]. The global existence of weak solutions of the Navier-Stokes equations for compressible, heat-conducting fluids in two- and three-space dimensions was proved by Hoff, when the initial data may be discontinuous across a hypersurface of R^n [8]. Hoff also showed the global existence of solutions of the Navier-Stokes equations for compressible, barotropic flow in two space dimensions which exhibit convecting singularity curves [9].

If the viscosity coefficients $\mu(\rho) = \rho^{\alpha}$, $\lambda(\rho) = 0$, for the case of one space dimension, the global existence of unique piecewise smooth solution to the free boundary value problem was obtained by Fang-Zhang for (1.1) with $0 < \alpha < 1$, where the initial density is piecewise smooth with possibly large jump discontinuities [10]. Lian-Guo-Li addressed the initial boundary value problem for (1.1) with $0 < \alpha \leq 1$ subject to piecewise regular initial data with initial vacuum state included [11], Lian-Liu-Li-Xiao also consider the Cauchy problem for one-dimensional isentropic compressible Navier-Stokes equations with adensity-dependent viscosity coefficient [12]; in these two cases above, they proved the global existence of unique piecewise regular solution and proved the finite time vanishing of vacuum state in [11]. In particular, they got that the jump discontinuity of density decays exponentially, but never vanish in any finite time and the piecewise regular solution tends to the equilibrium state as $t \to +\infty$.

Recently, there is also huge literature on the studies of the compressible Navier-Stokes equations with density-dependent viscosity coefficients. For instance, the mathematical derivations are obtained in the simulation of flow surface in shallow region [13, 14]. The prototype model is the viscous Saint-Venant (corresponding to (1.1) with $P(\rho) = \rho^2$, $\mu(\rho) = \rho$ and $\lambda(\rho) = 0$). Many authors considered the well-posedness of solutions to the free boundary value problem with initial finite mass and the flow density being connected with the infinite vacuum either continuously or *via* jump discontinuity; refer to [15–22] and references therein. The global existence of classical solutions is shown by Mellet and Vasseur [23]. The qualitative behaviors of global solutions and dynamical asymptotics of vacuum states are also made, such as the finite time vanishing of finite vacuum or asymptotical formation of vacuum in long time, the dynamical behaviors of vacuum boundary, the long time convergence to rarefaction wave with vacuum and the stability of shock profile with large shock strength; refer to [24–27] and references therein.

In this present paper, we consider the exterior problem for the spherically symmetric isentropic compressible Navier-Stokes equations with density-dependent viscosity coefficients and discontinuous initial data and focus on the existence, regularity and dynamical behaviors of global weak solutions, *etc.* We show that the exterior problem with piecewise regular initial data admits a unique global piecewise regular solution, where the discontinuity in piecewise regular initial density is bounded. In particular, the jump discontinuity of density decays exponentially and the piecewise regular solution tends to the equilibrium state as $t \to +\infty$.

The rest part of the paper is arranged as follows. In Section 2, the main results about the existence, regularity and dynamical behaviors of global piecewise regular solution for compressible Navier-Stokes equations are stated. Then some important *a-priori* estimates will be given in Section 3. Finally, the theorem is proved in Section 4.

2 Notations and main results

For simplicity, we will take $\mu(\rho) = \rho^\alpha$ and $\lambda(\rho) = (\alpha - 1)\rho^\alpha$ and $D(\mathbf{U}) = \nabla\mathbf{U}$ in (1.1). The isentropic compressible Navier-Stokes equations become

$$\begin{cases} \rho_t + \operatorname{div}(\rho\mathbf{U}) = 0, \\ (\rho\mathbf{U})_t + \operatorname{div}(\rho\mathbf{U}\otimes\mathbf{U}) + \nabla P(\rho) - \operatorname{div}(\rho^\alpha\nabla\mathbf{U}) - (\alpha-1)\nabla(\rho^\alpha \operatorname{div}\mathbf{U}) = 0. \end{cases} \tag{2.1}$$

The initial data and boundary conditions of (2.1) are imposed as:

$$\begin{cases} (\rho,\mathbf{U})(\mathbf{x},0) = (\rho_0,\mathbf{U}_0)(\mathbf{x}), \quad \mathbf{x}\in\Omega, \\ \mathbf{U} = 0, \quad \text{on } \partial\Omega, \qquad \lim_{|\mathbf{x}|\to+\infty}(\rho,u)(\mathbf{x},t) = (\bar\rho,0), \quad t\in[0,T], \end{cases} \tag{2.2}$$

where $\Omega := R^3/\Omega_{r_-}$, Ω_{r_-} is a ball of radius r_- centered at the origin in R^3, and $\bar\rho > 0$ is a constant.

We are concerned with the spherically symmetric solutions of the system (2.1) in the spherically symmetric exterior domain Ω. To this end, we denote that

$$|\mathbf{x}| = r, \qquad \rho(\mathbf{x},t) = \rho(r,t), \qquad \mathbf{U}(\mathbf{x},t) = u(r,t)\frac{\mathbf{x}}{r}, \tag{2.3}$$

which leads to the following system of equations for $r > 0$:

$$\begin{cases} \rho_t + (\rho u)_r + \frac{2\rho u}{r} = 0, \\ (\rho u)_t + (\rho u^2 + \rho^\gamma)_r + \frac{2\rho u^2}{r} - \alpha(\rho^\alpha u_r)_r - \alpha\rho^\alpha(\frac{2u}{r})_r - 2(\alpha-1)\frac{(\rho^\alpha)_r u}{r} = 0, \end{cases} \tag{2.4}$$

with the initial data and boundary conditions

$$\begin{cases} (\rho,u)(r,0) = (\rho_0,u_0)(r), \quad r\in[r_-,+\infty), \\ u(r_-,t) = 0, \qquad \lim_{r\to+\infty}(\rho,u)(r,t) = (\bar\rho,0), \quad t\in[0,T]. \end{cases} \tag{2.5}$$

Next, we give the definition of weak solution to the exterior problem (2.1)-(2.2).

Definition 2.1 (weak solution) For any $T > 0$, (ρ, u) is said to be a weak solution of the exterior problem (2.1)-(2.2), if (ρ, u) has the following regularities:

$$\begin{cases} \rho \geq 0 \quad \text{a.e.}, 0 < \rho - \bar{\rho} \in L^\infty([0,T]; L^1(\Omega) \cap L^2(\Omega)), \\ \sqrt{\rho}\mathbf{U} \in L^\infty([0,T]; L^2(\Omega)), \\ \nabla(\rho^{\alpha - \frac{1}{2}}) \in L^\infty([0,T]; L^2(\Omega)), \qquad \rho^\alpha \nabla \mathbf{U} \in L^2([0,T]; L^2(\Omega)), \end{cases} \tag{2.6}$$

and the equations (2.1) are satisfied in the sense of distribution. Namely, it holds for all $\varphi \in C_0^\infty(\bar{\Omega} \times [0,T])$ that

$$\int_\Omega \rho_0 \varphi(\mathbf{x}, 0)\, d\mathbf{x} + \int_0^T \int_\Omega \rho \varphi_t \, d\mathbf{x}\, dt + \int_0^T \int_\Omega \rho \mathbf{U} \cdot \nabla \varphi \, d\mathbf{x}\, dt = 0, \tag{2.7}$$

and for all $\psi = (\psi_1, \psi_2, \psi_3) \in C_0^\infty(\bar{\Omega} \times [0,T])$ that

$$\begin{aligned} &\int_\Omega \rho_0 \mathbf{U}_0 \cdot \psi(\mathbf{x}, 0)\, d\mathbf{x} + \int_0^T \int_\Omega \big(\sqrt{\rho}(\sqrt{\rho}\mathbf{U}) \cdot \psi_t + \sqrt{\rho}\mathbf{U} \otimes \sqrt{\rho}\mathbf{U} : \nabla \psi\big)\, d\mathbf{x}\, dt \\ &\quad + \int_0^T \int_\Omega \rho^\gamma \operatorname{div} \psi \, d\mathbf{x}\, dt - \langle \rho^\alpha \nabla \mathbf{U}, \nabla \psi \rangle = 0, \end{aligned} \tag{2.8}$$

where the diffusion term makes sense as

$$\begin{aligned} \langle \rho^\alpha \nabla \mathbf{U}, \nabla \psi \rangle = &-\int_0^T \int_\Omega \rho^{\alpha - \frac{1}{2}} (\sqrt{\rho}\mathbf{U}) \cdot \triangle \psi \, d\mathbf{x}\, dt \\ &- \frac{2\alpha}{2\alpha - 1} \int_0^T \int_\Omega (\sqrt{\rho}\mathbf{U}) \cdot \big(\nabla\big(\rho^{\alpha - \frac{1}{2}}\big) \cdot \nabla\big)\psi \, d\mathbf{x}\, dt. \end{aligned} \tag{2.9}$$

For simplicity, we consider the initial data in exterior problem (2.4)-(2.5) with one discontinuous point $\xi_0 \in (r_-, +\infty)$, namely,

$$\begin{cases} \rho_0 - \bar{\rho} \in L^1 \cap L^2([r_-, \xi_0) \cup (\xi_0, +\infty)), \qquad \inf_{\mathbf{x} \in [r_-, \xi_0) \cup (\xi_0, +\infty)} \rho_0 > \rho_- > 0, \\ \rho_0(\xi_0 - 0) \neq \rho_0(\xi_0 + 0), \qquad |\rho_0(\xi_0 + 0) - \rho_0(\xi_0 - 0)| < \delta, \\ r(\rho_0^{\alpha - \frac{1}{2}})_r \in L^2([r_-, \xi_0) \cup (\xi_0, +\infty)), \qquad r^2 u_0 \in H^2([r_-, \xi_0) \cup (\xi_0, +\infty)), \end{cases} \tag{2.10}$$

where ρ_- and $\delta > 0$ are positive constants, and $\delta > 0$ is bounded. Next, we define that

$$E_0 := \frac{1}{2} \fint_{r_-}^{+\infty} \rho_0 u_0^2 r^2 \, dr + \fint_{r_-}^{+\infty} \rho_0 \left(\frac{1}{\gamma - 1} \big(\rho_0^{\gamma - 1} - \bar{\rho}^{\gamma - 1}\big) + \bar{\rho}^\gamma \big(\rho_0^{-1} - \bar{\rho}^{-1}\big) \right) r^2 \, dr, \tag{2.11}$$

$$\begin{aligned} E_1 := &\fint_{r_-}^{+\infty} \frac{\rho_0 (u_0 + \rho_0^{-1} (\rho_0^\alpha)_r)^2}{2} r^2 \, dr \\ &+ \fint_{r_-}^{+\infty} \rho_0 \left(\frac{1}{\gamma - 1} \big(\rho_0^{\gamma - 1} - \bar{\rho}^{\gamma - 1}\big) + \bar{\rho}^\gamma \big(\rho_0^{-1} - \bar{\rho}^{-1}\big) r^2 \right) dr, \end{aligned} \tag{2.12}$$

and denote that $\fint_{r_-}^{+\infty} := \int_{r_-}^{\xi_0} + \int_{\xi_0}^{+\infty}$ in this paper. Then we can give the main results as follows.

Theorem 2.1 *Let $\alpha > \frac{2}{3}$ and $\gamma > 1$. Assume that the initial data satisfies* (2.10) *and $E_0 + E_1 < \nu\alpha r_-^2 \bar{\rho}^{\frac{\gamma}{2}+\alpha-\frac{1}{2}}$, ν is a positive constant. Then there exist two positive constants ρ_*, ρ^* and a unique global piecewise regular solution (ρ, u) to the exterior problem* (2.4)-(2.5), *namely, satisfying*

$$\begin{cases} 0 < \rho_* \le \rho(r,t) \le \rho^*, & (r,t) \in [r_-, \xi(t)) \cup (\xi(t), +\infty) \times [0,T], \\ \rho \in L^\infty([0,T]; H^1([r_-, \xi(t)) \cup (\xi(t), +\infty))), \\ u \in L^\infty([0,T]; H^1([r_-, \xi(t)) \cup (\xi(t), +\infty))), \\ u_r \in L^2([0,T]; H^1([r_-, \xi(t)) \cup (\xi(t), +\infty))), \end{cases} \tag{2.13}$$

where $r = \xi(t)$ is a curve defined by

$$\frac{d\xi(t)}{dt} = u\big(\xi(t), t\big), \qquad \xi(0) = \xi_0, \quad t > 0, \tag{2.14}$$

along which the Rankine-Hugoniot conditions hold

$$\big[u\big(\xi(t), t\big)\big] = 0, \qquad \big[\rho^\gamma\big(\xi(t), t\big)\big] = \left[\alpha\left(\rho^\alpha u_r + \frac{2\rho^\alpha u}{r}\right)\big(\xi(t), t\big)\right], \tag{2.15}$$

where $[f(\xi(t), t)] := f(\xi(t) + 0, t) - f(\xi(t) - 0, t)$, and along the discontinuity $r = \xi(t)$ the jump decays exponentially

$$\big|\big[\rho_0^\alpha(\xi_0)\big]\big| e^{-C_0 t} \le \big|\big[\rho^\alpha\big(\xi(t), t\big)\big]\big| \le \big|\big[\rho_0^\alpha(\xi_0)\big]\big| e^{-C_1 t}, \tag{2.16}$$

where C_0, C_1 are positive constants independent of time. Furthermore, the solution tends to equilibrium state $(\bar{\rho}, 0)$

$$\big\|(\rho - \bar{\rho}, u)(\cdot, t)\big\|_{L^\infty([r_-, \xi(t)) \cup (\xi(t), +\infty))} \to 0, \quad t \to +\infty. \tag{2.17}$$

Remark 2.1 Theorems 2.1 holds for the Saint-Venant model for shallow water, *i.e.*, $\gamma = 2$, $\alpha = 1$.

Remark 2.2 (2.16) shows that the discontinuity in the density decays exponentially in time.

Remark 2.3 For piecewise regular initial data, it can be shown in terms of the difference scheme made by Hoff [4], there is a (piecewise regular) weak solution to the exterior problem (2.4) and (2.5). In addition, there is a curve $r = \xi(t)$ starting from $\xi(0) = \xi_0$, so that the density is discontinuous cross $r = \xi(t)$, and the Rankine-Hugoniot conditions hold

$$\big[u\big(\xi(t), t\big)\big] = 0, \qquad \big[\rho^\gamma\big(\xi(t), t\big)\big] = \left[\alpha\left(\rho^\alpha u_r + \frac{2\rho^\alpha u}{r}\right)\big(\xi(t), t\big)\right]. \tag{2.18}$$

To extend the local solution globally in time, we need to establish the uniformly *a-priori* estimates.

3 The *a-priori* estimates

According to the analysis made in [28], there is a curve $r = \xi(t)$ defined by

$$\frac{d\xi(t)}{dt} = u\big(\xi(t), t\big), \qquad \xi(0) = \xi_0, \quad t > 0, \tag{3.1}$$

along which the Rankine-Hugoniot conditions hold

$$\big[u\big(\xi(t), t\big)\big] = 0, \qquad \big[\rho^\gamma\big(\xi(t), t\big)\big] = \left[\alpha\left(\rho^\alpha u_r + \frac{2\rho^\alpha u}{r}\right)\big(\xi(t), t\big)\right], \tag{3.2}$$

where $[f(\xi(t), t)] := f(\xi(t) + 0, t) - f(\xi(t) - 0, t)$. It is convenient to make use of the Lagrange coordinates in order to establish the uniformly *a-priori* estimates, denote the Lagrange coordinates transform

$$x = \int_{r_-}^{r} \rho(r, t) r^2 \, dr, \qquad \tau = t, \tag{3.3}$$

which maps $(r, t) \in [r_-, +\infty) \times R^+$ into $(x, \tau) \in [0, +\infty) \times R^+$. The relations between Lagrangian and Eulerian coordinates are satisfied as

$$\frac{\partial x}{\partial r} = \rho r^2, \qquad \frac{\partial x}{\partial t} = -\rho u r^2. \tag{3.4}$$

Since the conservation of total mass holds, the curve $r = \xi(t)$ in Eulerian coordinates is changed to a line $x = x_0$ in Lagrangian coordinates, where

$$x_0 = \int_{r_-}^{\xi(t)} \rho(r, t) r^2 \, dr = \int_{r_-}^{\xi_0} \rho_0(r) r^2 \, dr \tag{3.5}$$

in Lagrangian coordinates, and the jump conditions become

$$\big[u(x_0, \tau)\big] = 0, \qquad \big[\rho^\gamma(x_0, \tau)\big] = \big[\alpha\rho^{1+\alpha}\big(r^2 u\big)_x(x_0, \tau)\big]. \tag{3.6}$$

Meanwhile, the exterior problem (2.4)-(2.5) is reformulated to

$$\begin{cases} \rho_\tau + \rho^2(r^2 u)_x = 0, \\ r^{-2} u_\tau + (\rho^\gamma)_x = \alpha(\rho^{1+\alpha}(r^2 u)_x)_x - \frac{2(\rho^\alpha)_x u}{r}, \\ (\rho, u)(x, 0) = (\rho_0, u_0)(x), \quad x \in [0, +\infty), \\ u(0, t) = 0, \qquad \lim_{x \to +\infty}(\rho, u) = (\bar{\rho}, 0), \quad \tau \in [0, +\infty), \end{cases} \tag{3.7}$$

where the initial data satisfies

$$\begin{cases} \rho_0 - \bar{\rho} \in L^1 \cap L^2([0, x_0) \cup (x_0, +\infty)), \qquad \inf_{x \in [0, x_0) \cup (x_0, +\infty)} \rho_0 > \rho_- > 0, \\ \rho_0(x_0 - 0) \neq \rho_0(x_0 + 0), \qquad |\rho_0(x_0 + 0) - \rho_0(x_0 - 0)| < \delta, \\ r^2(\rho_0^\alpha)_x \in L^2([0, x_0) \cup (x_0, +\infty)), \qquad \frac{1}{\sqrt{r^2 \rho_0}}(r^2 u_0) \in L^2([0, x_0) \cup (x_0, +\infty)), \\ \sqrt{r^2 \rho_0}(r^2 u_0)_x \in L^2([0, x_0) \cup (x_0, +\infty)), \\ \frac{1}{\sqrt{r^2 \rho_0}}(r^2 \rho_0(r^2 \rho_0(r^2 u_0)_x)_x) \in L^2([0, x_0) \cup (x_0, +\infty)). \end{cases} \tag{3.8}$$

First, we are ready to establish the *a-priori* estimates for the solution (ρ, u) to the exterior problem (3.7). To obtain the *a-priori* estimates, we assume *a-priori* that there are constants $\rho_\pm > 0$ so that

$$0 < \rho_- \le \rho(x,\tau) \le \rho_+, \qquad (x,\tau) \in \big([0,x_0) \cup (x_0,+\infty)\big) \times [0,T], \quad T > 0. \tag{3.9}$$

Lemma 3.1 *Let $T > 0$. Under the conditions in Theorem 2.1, it holds for any solution (ρ, u) to the exterior problem (3.7) that*

$$\begin{aligned} &\frac{1}{4}\fint_0^{+\infty} u^2\,dx + \fint_0^{+\infty}\left(\frac{1}{\gamma-1}\big(\rho^{\gamma-1}-\bar\rho^{\gamma-1}\big) + \bar\rho^\gamma\big(\rho^{-1}-\bar\rho^{-1}\big)\right)dx \\ &\quad + \alpha\int_0^\tau\fint_0^{+\infty}\left(\frac{2\rho^{\alpha-1}u^2}{r^2} + \frac{1}{2}\rho^{1+\alpha}u_x^2r^4\right)dx\,ds \le E_0, \quad \tau \in [0,T], \end{aligned} \tag{3.10}$$

where $\fint_0^{+\infty} := \int_0^{x_0} + \int_{x_0}^{+\infty}$.

Proof Multiplying $(3.7)_2$ by r^2u and integrating the result with respect to x over $[0,+\infty)$, making use of (3.6), $(3.7)_1$ and (3.8), we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{d\tau}\fint_0^{+\infty} u^2\,dx + \frac{d}{d\tau}\fint_0^{+\infty}\left(\frac{1}{\gamma-1}\big(\rho^{\gamma-1}-\bar\rho^{\gamma-1}\big) + \bar\rho^\gamma\big(\rho^{-1}-\bar\rho^{-1}\big)\right)dx \\ &\quad + \alpha\fint_0^{+\infty}\rho^{1+\alpha}\big(r^2u\big)_x^2\,dx \\ &\quad = 2\fint_0^{+\infty}\rho^\alpha\big(ru^2\big)_x\,dx + 2\big[\rho^\alpha\big]\big(ru^2\big)|_{x=x_0}, \end{aligned} \tag{3.11}$$

integrating (3.11) with respect to τ, we obtain

$$\begin{aligned} &\frac{1}{2}\fint_0^{+\infty} u^2\,dx + \fint_0^{+\infty}\left(\frac{1}{\gamma-1}\big(\rho^{\gamma-1}-\bar\rho^{\gamma-1}\big) + \bar\rho^\gamma\big(\rho^{-1}-\bar\rho^{-1}\big)\right)dx \\ &\quad + \alpha\fint_0^{+\infty}\left(\frac{2\rho^{\alpha-1}u^2}{r^2} + \rho^{1+\alpha}u_x^2r^4\right)dx \\ &\quad = \frac{1}{2}\fint_0^{+\infty} u_0^2\,dx + \fint_0^{+\infty}\left(\frac{1}{\gamma-1}\big(\rho_0^{\gamma-1}-\bar\rho^{\gamma-1}\big) + \bar\rho^\gamma\big(\rho_0^{-1}-\bar\rho^{-1}\big)\right)dx \\ &\quad\quad + \int_0^\tau 2\big[\rho^\alpha\big]\big(ru^2\big)|_{x=x_0}\,ds. \end{aligned} \tag{3.12}$$

Applying equations (3.6) and $(3.7)_1$, it holds that

$$\big[\rho^\alpha\big]_\tau + \alpha\big[\rho^\gamma\big] = 0, \tag{3.13}$$

which implies

$$\big[\rho^\alpha\big] = \big[\rho_0^\alpha\big]\exp\left(-\alpha\int_0^\tau\frac{[\rho^\gamma]}{[\rho^\alpha]}\,ds\right). \tag{3.14}$$

From (3.9) and (3.14), we can find

$$\frac{\gamma}{\alpha}\rho_-^{\gamma-\alpha} \le \frac{[\rho^\gamma]}{[\rho^\alpha]} \le \frac{\gamma}{\alpha}\rho_+^{\gamma-\alpha}. \tag{3.15}$$

It holds from (3.8), (3.14) and (3.15) that

$$\begin{aligned}
&\left|\int_0^\tau 2\big[\rho^\alpha\big](ru^2)|_{x=x_0}\,ds\right| \\
&\quad \le \big(\bar C+\rho_-^{-1}\big)\int_0^\tau \big|\big[\rho_0^\alpha\big]\big| e^{-\gamma\rho_-^{\gamma-\alpha}s}\left(\int_0^{+\infty} u^2\,dx+\int_0^{+\infty} u_x^2\,dx\right)ds \\
&\quad \le \big(\bar C+\rho_-^{-1}\big)\big|\big[\rho_0^\alpha\big]\big|\sup_{\tau\in[0,T]}\int_0^{+\infty} u^2\,dx\int_0^\tau e^{-\gamma\rho_-^{\gamma-\alpha}s}\,ds \\
&\qquad + \big(\bar C+\rho_-^{-1}\big)\big|\big[\rho_0^\alpha\big]\big|\rho_-^{-(1+\alpha)}r_-^{-4}\int_0^\tau\int_0^{+\infty}\rho^{1+\alpha}u_x^2 r^4\,dx\,ds \\
&\quad \le \frac14\sup_{\tau\in[0,T]}\int_0^{+\infty} u^2\,dx+\frac{\alpha}{2}\int_0^\tau\int_0^{+\infty}\rho^{1+\alpha}u_x^2r^4\,dx\,ds,
\end{aligned} \tag{3.16}$$

where $\bar C$ is a positive constant independent of time and we assume that

$$0<\big|\big[\rho_0^\alpha\big]\big|\le\min\left\{\frac{\gamma}{4(\bar C+\rho_-^{-1})}\rho_-^{\gamma-\alpha},\frac{\alpha}{2(\bar C+\rho_-^{-1})}\rho_-^{1+\alpha}r_-^4\right\}:=M_1. \tag{3.17}$$

From (3.12) and (3.16), Lemma 3.1 can be obtained. □

Lemma 3.2 *Let* $T>0$. *Under the conditions in Theorem* 2.1, *it holds for any solution* (ρ,u) *to the exterior problem* (3.7) *that*

$$\begin{aligned}
&\frac12\int_0^{+\infty}\big(u+r^2(\rho^\alpha)_x\big)^2\,dx+\int_0^{+\infty}\left(\frac{1}{\gamma-1}\big(\rho^{\gamma-1}-\bar\rho^{\gamma-1}\big)+\bar\rho^\gamma\big(\rho^{-1}-\bar\rho^{-1}\big)\right)dx \\
&\quad+\alpha\gamma\int_0^\tau\int_0^{+\infty}\rho^{\gamma+\alpha-2}\rho_x^2r^4\,dx\,ds\le 2E_1,\quad \tau\in[0,T].
\end{aligned} \tag{3.18}$$

Proof Multiplying $(3.7)_1$ by $\rho^{\alpha-1}$ and differentiating the equation with respect to x, we have

$$\big(\rho^\alpha\big)_{x\tau}+\alpha\big(\rho^{1+\alpha}\big(r^2u\big)_x\big)_x=0. \tag{3.19}$$

Summing (3.19) and $(3.7)_2$, we get

$$\big(r^{-2}u+\big(\rho^\alpha\big)_x\big)_\tau+\big(\rho^\gamma\big)_x=\big(r^{-2}\big)_\tau u-\frac{2(\rho^\alpha)_xu}{r}. \tag{3.20}$$

Note that

$$r^3(x,\tau)=r_-^3+3\int_0^x\frac{1}{\rho(z,\tau)}\,dz, \tag{3.21}$$

and so

$$\frac{\partial r}{\partial\tau}=\frac{1}{r^2}\int_0^x\left(\frac1\rho\right)_t(z,t)\,dz=\frac{1}{r^2}\int_0^x\big(r^2u\big)_z(z,t)\,dz=u(x,\tau), \tag{3.22}$$

which together with (3.20) yields

$$\big(r^{-2}u+\big(\rho^\alpha\big)_x\big)_\tau+\big(\rho^\gamma\big)_x=-2r^{-3}u^2-\frac{2(\rho^\alpha)_xu}{r}. \tag{3.23}$$

Multiplying (3.23) by $(u + r^2(\rho^\alpha)_x)r^2$, and integrating the result with respect to x and τ, we have

$$\begin{aligned}&\frac{1}{2}\int_0^{+\infty}\big(u + r^2\big(\rho^\alpha\big)_x\big)^2\,dx + \frac{1}{\gamma-1}\int_0^{+\infty}\rho^{\gamma-1}\,dx + \alpha\gamma\int_0^\tau\int_0^{+\infty}\rho^{\gamma+\alpha-2}\rho_x^2 r^4\,dx\,ds \\ &\quad = \frac{1}{2}\int_0^{+\infty}\big(u_0 + r^2\big(\rho_0^\alpha\big)_x\big)^2\,dx + \frac{1}{\gamma-1}\int_0^{+\infty}\rho_0^{\gamma-1}\,dx + \int_0^\tau\big[\rho^\gamma\big]\big(r^2u\big)|_{x=x_0}\,ds. \qquad (3.24)\end{aligned}$$

It holds from (3.8), (3.10), (3.14) and (3.15) that

$$\begin{aligned}&\left|\int_0^\tau\big[\rho^\gamma\big]\big(r^2u\big)|_{x=x_0}\,ds\right| \\ &\quad \le \big(\hat{C} + \rho_-^{-2}\big)\int_0^\tau \rho_+^{\gamma-\alpha}\big|\big[\rho^\alpha\big]\big|\left(\int_0^1 u^2\,dx + \int_0^1 u_x^2\,dx\right)^{\frac{1}{2}}ds \\ &\quad \le \big(\hat{C} + \rho_-^{-2}\big)\int_0^\tau \big|\big[\rho_0^\alpha\big]\big|e^{-\gamma\rho_-^{\gamma-\alpha}s}\left(\rho_+^{2\gamma-2\alpha} + \int_0^1 u^2\,dx + \rho_-^{-(1+\alpha)}r_-^{-4}\int_0^1\rho^{1+\alpha}u_x^2r^4\,dx\right)ds \\ &\quad \le \big(\hat{C} + \rho_-^{-2}\big)\big|\big[\rho_0^\alpha\big]\big|\big(\rho_-^{\alpha-\gamma}\rho_+^{2\gamma-2\alpha} + \rho_-^{\alpha-\gamma} + \rho_-^{-(1+\alpha)}r_-^{-4}\big) \le E_1, \qquad (3.25)\end{aligned}$$

where $\hat{C}$ is a positive constant independent of time and we assume that

$$0 < \big|\big[\rho_0^\alpha\big]\big| < \min\left\{\frac{\rho_-^{\gamma-\alpha}\rho_+^{2\alpha-2\gamma}}{(\hat{C} + \rho_-^{-2})}E_1, \frac{\rho_-^{\gamma-\alpha}}{(\hat{C} + \rho_-^{-2})}E_1, \frac{\rho_-^{1+\alpha}r_-^4}{(\hat{C} + \rho_-^{-2})}E_1, M_1\right\} := \delta. \qquad (3.26)$$

The proof of (3.18) is completed. □

Lemma 3.3 *Let* $T > 0$. *Under the conditions in Theorem* 2.1, *there exist two constants* $0 < \rho_* < \rho^*$ *such that*

$$0 < \rho_* \le \rho(x,\tau) \le \rho^*, \qquad (x,\tau) \in [0,+\infty) \times [0,T]. \qquad (3.27)$$

Proof Let

$$\varphi(\rho) := \frac{1}{\gamma-1}\big(\rho^{\gamma-1} - \bar{\rho}^{\gamma-1}\big) + \bar{\rho}^\gamma\big(\rho^{-1} - \bar{\rho}^{-1}\big), \qquad (3.28)$$

and

$$\psi(\rho) := \int_{\bar{\rho}}^\rho \varphi(\eta)^{\frac{1}{2}}\eta^{\alpha-1}\,d\eta. \qquad (3.29)$$

It follows from (3.10) and (3.18) that

$$\begin{aligned}\big|\psi(\rho)\big| &\le \left|\int_0^{x_0}\partial_x\psi(\rho)\,dx\right| + \left|\int_{x_0}^{+\infty}\partial_x\psi(\rho)\,dx\right| \\ &\le r_-^{-2}\left|\int_0^{x_0}\varphi(\rho)^{\frac{1}{2}}\rho^{\alpha-1}\rho_x r^2\,dx\right| + r_-^{-2}\left|\int_{x_0}^{+\infty}\varphi(\rho)^{\frac{1}{2}}\rho^{\alpha-1}\rho_x r^2\,dx\right| \\ &\le \frac{2}{\alpha}r_-^{-2}\left|\int_0^{+\infty}\varphi(\rho)\,dx\int_0^{+\infty}r^4\big(\rho^\alpha\big)_x^2\,dx\right|^{\frac{1}{2}} \le \frac{4}{\alpha}r_-^{-2}(E_0 + E_1). \qquad (3.30)\end{aligned}$$

We can verify that

1° As $\rho \to +\infty$, we have for some $\theta \in (0,1)$ that if $1 < \gamma \le 3$, it holds

$$\begin{aligned}
&\lim_{\rho\to+\infty} \psi(\rho) \\
&= \lim_{\rho\to+\infty} \int_{\bar\rho}^{\rho} \bigl((\gamma-2)\bigl(\theta\bar\rho+(1-\theta)\eta\bigr)^{\gamma-3} + 2\bar\rho^{\gamma}\bigl(\theta\bar\rho+(1-\theta)\eta\bigr)^{-3}\bigr)^{\frac12}(\eta-\bar\rho)\eta^{\alpha-1}\,d\eta \\
&\ge \lim_{\rho\to+\infty} \bigl((\gamma-2)\bigl(\theta\bar\rho+(1-\theta)\rho\bigr)^{\gamma-3} + 2\bar\rho^{\gamma}\bigl(\theta\bar\rho+(1-\theta)\rho\bigr)^{-3}\bigr)^{\frac12}\int_{\bar\rho}^{\rho}(\eta-\bar\rho)\eta^{\alpha-1}\,d\eta \\
&= \lim_{\rho\to+\infty} \bigl((\gamma-2)\bigl(\theta\bar\rho+(1-\theta)\rho\bigr)^{\gamma-3} + 2\bar\rho^{\gamma}\bigl(\theta\bar\rho+(1-\theta)\rho\bigr)^{-3}\bigr)^{\frac12} \\
&\quad\cdot\left(\frac{1}{\alpha+1}\bigl(\rho^{\alpha+1}-\bar\rho^{\alpha+1}\bigr) - \frac{1}{\alpha}\bar\rho\bigl(\rho^{\alpha}-\bar\rho^{\alpha}\bigr)\right) \\
&\to +\infty,
\end{aligned} \tag{3.31}$$

and if $\gamma > 3$, it holds

$$\begin{aligned}
&\lim_{\rho\to+\infty} \psi(\rho) \\
&= \lim_{\rho\to+\infty} \int_{\bar\rho}^{\rho} \bigl((\gamma-2)\bigl(\theta\bar\rho+(1-\theta)\eta\bigr)^{\gamma-3} + 2\bar\rho^{\gamma}\bigl(\theta\bar\rho+(1-\theta)\eta\bigr)^{-3}\bigr)^{\frac12}(\eta-\bar\rho)\eta^{\alpha-1}\,d\eta \\
&\ge \lim_{\rho\to+\infty} \bigl((\gamma-2)\bar\rho^{\gamma-3}\bigr)^{\frac12}\int_{\bar\rho}^{\rho}(\eta-\bar\rho)\eta^{\alpha-1}\,d\eta \\
&= \lim_{\rho\to+\infty} \bigl((\gamma-2)\bar\rho^{\gamma-3}\bigr)^{\frac12}\left(\frac{1}{\alpha+1}\bigl(\rho^{\alpha+1}-\bar\rho^{\alpha+1}\bigr) - \frac{1}{\alpha}\bar\rho\bigl(\rho^{\alpha}-\bar\rho^{\alpha}\bigr)\right) \\
&\to +\infty.
\end{aligned} \tag{3.32}$$

2° As $\rho \to 0$, we have for some $\theta \in (0,1)$ that if $1 < \gamma \le 3$, it holds

$$\begin{aligned}
&\lim_{\rho\to0} \psi(\rho) \\
&= -\lim_{\rho\to0} \int_{\bar\rho}^{\rho} \bigl((\gamma-2)\bigl(\theta\bar\rho+(1-\theta)\eta\bigr)^{\gamma-3} + 2\bar\rho^{\gamma}\bigl(\theta\bar\rho+(1-\theta)\eta\bigr)^{-3}\bigr)^{\frac12}(\eta-\bar\rho)\eta^{\alpha-1}\,d\eta \\
&\le -\lim_{\rho\to0} \bigl((\gamma-2)\bigl(\theta\bar\rho+(1-\theta)\rho\bigr)^{\gamma-3} + 2\bar\rho^{\gamma}\bigl(\theta\bar\rho+(1-\theta)\rho\bigr)^{-3}\bigr)^{\frac12}\int_{\bar\rho}^{\rho}(\eta-\bar\rho)\eta^{\alpha-1}\,d\eta \\
&= -\lim_{\rho\to0} \bigl(\gamma\bar\rho^{\gamma-3}\bigr)^{\frac12}\cdot\left(\frac{1}{\alpha+1}\bigl(\rho^{\alpha+1}-\bar\rho^{\alpha+1}\bigr) - \frac{1}{\alpha}\bar\rho\bigl(\rho^{\alpha}-\bar\rho^{\alpha}\bigr)\right) \\
&:= -4\nu_1\bar\rho^{\frac{\gamma}{2}+\alpha-\frac12},
\end{aligned} \tag{3.33}$$

where $\nu_1 > 0$ is a positive constant, and if $\gamma > 3$, it holds

$$\begin{aligned}
&\lim_{\rho\to0} \psi(\rho) \\
&= -\lim_{\rho\to0} \int_{\bar\rho}^{\rho} \bigl((\gamma-2)\bigl(\theta\bar\rho+(1-\theta)\eta\bigr)^{\gamma-3} + 2\bar\rho^{\gamma}\bigl(\theta\bar\rho+(1-\theta)\eta\bigr)^{-3}\bigr)^{\frac12}(\eta-\bar\rho)\eta^{\alpha-1}\,d\eta \\
&\le -\lim_{\rho\to0} \bigl(2\bar\rho^{\gamma-3}\bigr)^{\frac12}\int_{\bar\rho}^{\rho}(\eta-\bar\rho)\eta^{\alpha-1}\,d\eta
\end{aligned}$$

$$= -\lim_{\rho\to 0}\big(2\bar\rho^{\gamma-3}\big)^{\frac{1}{2}}\left(\frac{1}{\alpha+1}\big(\rho^{\alpha+1}-\bar\rho^{\alpha+1}\big)-\frac{1}{\alpha}\bar\rho\big(\rho^{\alpha}-\bar\rho^{\alpha}\big)\right)$$
$$:= -4\nu_2\bar\rho^{\frac{\gamma}{2}+\alpha-\frac{1}{2}}, \tag{3.34}$$

where $\nu_2 > 0$ is a positive constant, and we denote the positive constant $\nu := \min\{\nu_1,\nu_2\}$. Applying (3.31)-(3.34) and $E_0 + E_1 < \nu\alpha r_-^2\bar\rho^{\frac{\gamma}{2}+\alpha-\frac{1}{2}}$, let

$$\rho_- = \frac{1}{2}\rho_*, \qquad \rho_+ = 2\rho^*, \tag{3.35}$$

we can prove (3.27). □

Lemma 3.4 *Let $T > 0$. Under the conditions in Theorem 2.1, it holds for any solution (ρ,u) to the exterior problem (3.7) that*

$$\big|\big[\rho_0^\alpha(x_0)\big]\big|e^{-C_0\tau} \le \big|\big[\rho^\alpha(x_0,\tau)\big]\big| \le \big|\big[\rho_0^\alpha(x_0)\big]\big|e^{-C_1\tau}, \quad \tau\in[0,T], \tag{3.36}$$

where C_0, C_1 are positive constants independent of time.

Proof From (3.14), (3.15) and (3.27), we can get (3.36). □

Lemma 3.5 *Let $T > 0$. Under the conditions in Theorem 2.1, it holds for any solution (ρ,u) to the exterior problem (3.7) that*

$$\begin{aligned}&⨍_0^{+\infty}\big(r^2u\big)_x^2\,dx + ⨍_0^{+\infty}\big(r^2u\big)_\tau^2 r^{-4}\,dx + \int_0^\tau ⨍_0^{+\infty}\big(r^2u\big)_{xx}^2\,dx\,ds\\&\quad+\int_0^\tau ⨍_0^{+\infty}\big(r^2u\big)_s^2 r^{-4}\,dx\,ds + \int_0^\tau ⨍_0^{+\infty}\rho^{1+\alpha}\big(r^2u\big)_{xs}^2\,dx\,ds \le C, \quad \tau\in[0,T],\end{aligned} \tag{3.37}$$

where $C > 0$ denotes a constant independent of time.

Proof Multiplying $(3.7)_2$ by $\rho^{-(1+\alpha)}(r^2u)_\tau$ and integrating the result with respect to x over $[0,+\infty)$, making use of (3.6) and (3.8), we obtain

$$\begin{aligned}&\frac{d}{d\tau}⨍_0^{+\infty}\left(\frac{\alpha}{2}\big(r^2u\big)_x^2-\rho^{\gamma-(1+\alpha)}\big(r^2u\big)_x\right)dx + ⨍_0^{+\infty}\rho^{-(1+\alpha)}\big(r^2u\big)_\tau^2 r^{-4}\,dx\\&\quad=\big(\gamma-(1+\alpha)\big)⨍_0^{+\infty}\rho^{\gamma-\alpha}\big(r^2u\big)_x^2\,dx-(1+\alpha)⨍_0^{+\infty}\rho^{\gamma-(2+\alpha)}\rho_x\big(r^2u\big)_\tau\,dx\\&\qquad+\alpha(1+\alpha)⨍_0^{+\infty}\rho^{-1}\rho_x\big(r^2u\big)_x\big(r^2u\big)_\tau\,dx\\&\qquad+2⨍_0^{+\infty}\rho^{-(1+\alpha)}u^2\big(r^2u\big)_\tau r^{-3}\,dx-2\alpha⨍_0^{+\infty}\rho^{-2}\rho_x u\big(r^2u\big)_\tau r^{-1}\,dx\\&\qquad+\big[\rho^{-(1+\alpha)}\big]\big(\big(\rho^\gamma-\alpha\rho^{1+\alpha}\big(r^2u\big)_x\big)\big(r^2u\big)_\tau\big)\big|_{x=x_0^+},\end{aligned} \tag{3.38}$$

which implies

$$\begin{aligned}&⨍_0^{+\infty}\big(r^2u\big)_x^2\,dx + \int_0^\tau ⨍_0^{+\infty}\big(r^2u\big)_s^2 r^{-4}\,dx\,ds\\&\quad\le C + C⨍_0^{+\infty}\big(\rho^{\gamma-(1+\alpha)}-\bar\rho^{\gamma-(1+\alpha)}\big)^2\,dx + C\int_0^\tau ⨍_0^{+\infty}\left(\frac{u^2}{r^2}+u_x^2r^4\right)dx\,ds\end{aligned}$$

$$
\begin{aligned}
&+C\int_0^\tau\int_0^{+\infty}\rho_x^2r^4\,dx\,ds+C\int_0^\tau\int_0^{+\infty}\rho_x^2\big(r^2u\big)_x^2r^4\,dx\,ds\\
&+C\int_0^\tau\int_0^{+\infty}u^4r^{-2}\,dx\,ds+C\int_0^\tau\int_0^{+\infty}\rho_x^2u^2r^2\,dx\,ds\\
&+\int_0^\tau\big|\big[\rho^{-(1+\alpha)}\big]\big(\big(\rho^\gamma-\alpha\rho^{1+\alpha}\big(r^2u\big)_x\big)\big(r^2u\big)_s\big)|_{x=x_0^+}\big|\,ds\\
\le{}& C+C\int_0^\tau\int_0^{+\infty}\rho_x^2\big(r^2u\big)_x^2r^4\,dx\,ds+\sup_{\tau\in[0,T]}\|u\|_{L^\infty}^2\\
&+\int_0^\tau\big|\big[\rho^{-(1+\alpha)}\big]\big(\big(\rho^\gamma-\alpha\rho^{1+\alpha}\big(r^2u\big)_x\big)\big(r^2u\big)_s\big)|_{x=x_0^+}\big|\,ds. \qquad (3.39)
\end{aligned}
$$

From $(3.7)_2$, (3.10), (3.18) and (3.27), we can deduce that

$$
\begin{aligned}
&C\int_0^\tau\int_0^{+\infty}\rho_x^2\big(r^2u\big)_x^2r^4\,dx\,ds\\
&\quad\le\frac{C}{3}\int_0^\tau\int_0^{+\infty}\rho_x^2\big(r^2u\big)_x^2r^4\,dx\,ds+\frac{1}{6}\int_0^\tau\int_0^{+\infty}\big(r^2u\big)_s^2r^{-4}\,dx\,ds\\
&\qquad+C\int_0^\tau\int_0^{+\infty}\rho_x^2\,dx\,ds+C\int_0^\tau\int_0^{+\infty}\left(\frac{u^2}{r^2}+u_x^2r^4\right)dx\,ds, \qquad (3.40)
\end{aligned}
$$

$$
C\sup_{\tau\in[0,T]}\|u\|_{L^\infty}^2\le\frac{1}{4}\sup_{\tau\in[0,T]}\int_0^{+\infty}\big(r^2u\big)_x^2\,dx+C\sup_{\tau\in[0,T]}\int_0^{+\infty}u^2\,dx, \qquad (3.41)
$$

and

$$
\begin{aligned}
&\int_0^\tau\big|\big[\rho^{-(1+\alpha)}\big]\big(\big(\rho^\gamma-\alpha\rho^{1+\alpha}\big(r^2u\big)_x\big)\big(r^2u\big)_s\big)|_{x=x_0^+}\big|\,ds\\
&\quad\le C\int_0^\tau\big|\big[\rho_0^\alpha\big]\big|e^{-Cs}\big\|\rho^\gamma-\alpha\rho^{1+\alpha}\big(r^2u\big)_x\big\|_{L^\infty}\big\|\big(r^2u\big)_s\big\|_{L^\infty}\,ds\\
&\quad\le C\int_0^\tau e^{-Cs}\big\|\rho^\gamma\big\|_{L^\infty}^2\,ds+C\int_0^\tau\int_0^{+\infty}\rho_x^2\,dx\,ds+C\int_0^\tau\int_0^{+\infty}\left(\frac{u^2}{r^2}+u_x^2r^4\right)dx\,ds\\
&\qquad+\frac{1}{4}\int_0^\tau\int_0^{+\infty}\big(r^2u\big)_s^2r^{-4}\,dx\,ds+C\epsilon\int_0^\tau\int_0^{+\infty}\big(r^2u\big)_{xs}^2r^{-4}\,dx\,ds, \qquad (3.42)
\end{aligned}
$$

where C denotes a constant independent of time and $\epsilon\in(0,1)$ is a small constant which will be determined later. Use (3.39)-(3.42), and we can obtain that

$$
\int_0^{+\infty}\big(r^2u\big)_x^2\,dx+\int_0^\tau\int_0^{+\infty}\big(r^2u\big)_s^2r^{-4}\,dx\,ds\le C+C\epsilon\int_0^\tau\int_0^{+\infty}\big(r^2u\big)_{xs}^2r^{-4}\,dx\,ds. \qquad (3.43)
$$

Differentiating $(3.7)_2$ with respect to τ, multiplying the result by $(r^2u)_\tau$ and integrating the result with respect to x over $[0,+\infty)$, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{d\tau}\int_0^{+\infty}\big(r^2u\big)_\tau^2r^{-4}\,dx+\alpha\int_0^{+\infty}\rho^{1+\alpha}\big(r^2u\big)_{x\tau}^2\,dx\\
&\quad=2\int_0^{+\infty}uu_\tau\big(r^2u\big)_\tau r^{-3}\,dx-\frac{1}{2}\int_0^{+\infty}\big(r^{-4}\big)_\tau\big(r^2u\big)_\tau^2\,dx+2\int_0^{+\infty}u\big(r^{-1}u\big)_\tau\big(r^2u\big)_\tau r^{-2}\,dx\\
&\qquad+\int_0^{+\infty}\big(\rho^\gamma\big)_\tau\big(r^2u\big)_{x\tau}\,dx-\alpha\int_0^{+\infty}\big(\rho^{1+\alpha}\big)_\tau\big(r^2u\big)_x\big(r^2u\big)_{x\tau}\,dx
\end{aligned}
$$

$$
\begin{aligned}
&- \int_0^{+\infty} \left(\frac{2(\rho^\alpha)_x u}{r} \right)_\tau (r^2 u)_\tau \, dx \\
&+ 2 \int_0^{+\infty} (r^{-2})_\tau r^{-1} u^2 (r^2 u)_\tau \, dx. \qquad (3.44)
\end{aligned}
$$

A complicated computation gives

$$
\begin{aligned}
&\frac{d}{d\tau} \int_0^{+\infty} (r^2 u)_\tau^2 r^{-4} \, dx + \int_0^{+\infty} \rho^{1+\alpha} (r^2 u)_{x\tau}^2 \, dx \\
&\quad \le C \int_0^{+\infty} (r^2 u)_\tau^2 r^{-4} \, dx + C \int_0^{+\infty} \left(\frac{u^2}{r^2} + u_x^2 r^4 \right) dx \int_0^{+\infty} (r^2 u)_\tau^2 r^{-4} \, dx \\
&\qquad + C \sup_{\tau \in [0,T]} \left\| (r^2 u)_x^2 \right\|_{L^\infty} \int_0^{+\infty} (r^2 u)_x^2 \, dx + C \int_0^{+\infty} \left(\frac{u^2}{r^2} + u_x^2 r^4 \right) dx, \qquad (3.45)
\end{aligned}
$$

and by means of Gronwall's inequality, $(3.7)_2$, (3.10), (3.18), (3.27) and (3.43), it holds that

$$
\begin{aligned}
&\int_0^{+\infty} (r^2 u)_\tau^2 r^{-4} \, dx + \int_0^\tau \int_0^{+\infty} \rho^{1+\alpha} (r^2 u)_{xs}^2 \, dx \, ds \\
&\quad \le C + C\epsilon \int_0^\tau \int_0^{+\infty} \rho^{1+\alpha} (r^2 u)_{xs}^2 \, dx \, ds + C \sup_{\tau \in [0,T]} \left\| (r^2 u)_x^2 \right\|_{L^\infty} \\
&\quad \le C + C\epsilon \int_0^\tau \int_0^{+\infty} \rho^{1+\alpha} (r^2 u)_{xs}^2 \, dx \, ds \\
&\qquad + C \sup_{\tau \in [0,T]} \left(\int_0^{+\infty} (r^2 u)_x^2 \, dx \right)^{\frac{1}{2}} \left(\int_0^{+\infty} (r^2 u)_{xx}^2 \, dx \right)^{\frac{1}{2}} \\
&\quad \le C + C\epsilon \int_0^\tau \int_0^{+\infty} \rho^{1+\alpha} (r^2 u)_{xs}^2 \, dx \, ds + \frac{1}{2} \int_0^{+\infty} (r^2 u)_\tau^2 r^{-4} \, dx, \qquad (3.46)
\end{aligned}
$$

where C denotes a constant independent of time, choosing the constant ϵ small sufficiently, we can complete the proof of Lemma 3.5. □

Remark 3.1 By Lemmas 3.1-3.5, the following inequality holds:

$$
\begin{aligned}
&\int_0^{+\infty} (\rho - \bar{\rho})^2 \, dx + \int_0^{+\infty} \rho_x^2 \, dx + \int_0^{+\infty} u^2 \, dx + \int_0^{+\infty} u_x^2 \, dx + \int_0^{+\infty} u_\tau^2 \, dx \\
&\quad + \int_0^\tau \int_0^{+\infty} \rho_x^2 \, dx \, ds + \int_0^\tau \int_0^{+\infty} u_x^2 \, dx \, ds + \int_0^\tau \int_0^{+\infty} u_s^2 \, dx \, ds \\
&\quad + \int_0^\tau \int_0^{+\infty} u_{xx}^2 \, dx \, ds + \int_0^\tau \int_0^{+\infty} u_{xs}^2 \, dx \, ds \le C. \qquad (3.47)
\end{aligned}
$$

Lemma 3.6 *Under the conditions in Theorem* 2.1, *it holds for any solution* (ρ, u) *to the exterior problem* (3.7) *that*

$$
\left\| (\rho - \bar{\rho}, u)(\cdot, \tau) \right\|_{L^\infty([0,x_0) \cup (x_0,+\infty))} \to 0, \quad \tau \to +\infty. \qquad (3.48)
$$

Proof From Lemmas 3.1-3.5, we can obtain

$$\begin{aligned}&\int_0^{+\infty}\left\|(\rho-\bar{\rho},u)_x\right\|^2_{L^2([0,x_0)\cup(x_0,+\infty))}\,d\tau\\&\quad\leq r_-^{-4}\int_0^{+\infty}\left\|(\rho-\bar{\rho},u)_x r^2\right\|^2_{L^2([0,x_0)\cup(x_0,+\infty))}\,d\tau\leq C,\end{aligned}\tag{3.49}$$

and

$$\begin{aligned}&\int_0^{+\infty}\left|\frac{d}{d\tau}\left\|(\rho-\bar{\rho},u)_x\right\|^2_{L^2([0,x_0)\cup(x_0,+\infty))}\right|\,d\tau\\&\quad=\int_0^{+\infty}\left|\fint_0^{+\infty}\left(-4\rho\rho_x^2\left(r^2u\right)_x-2\rho^2\rho_x\left(r^2u\right)_{xx}\right)dx+2\fint_0^{+\infty}u_xu_{x\tau}\,dx\right|\,d\tau\\&\quad\leq C\int_0^{+\infty}\fint_0^{+\infty}\rho_x^2\,dx\,d\tau+C\int_0^{+\infty}\fint_0^{+\infty}\left(\frac{u^2}{r^2}+u_x^2r^4+\left(r^2u\right)_{xx}^2+u_{x\tau}^2\right)dx\,d\tau\\&\quad\leq C,\end{aligned}\tag{3.50}$$

which together with (3.49) implies

$$\left\|(\rho-\bar{\rho},u)_x\right\|^2_{L^2([0,x_0)\cup(x_0,+\infty))}\in W^{1,1}\left(R^+\right).\tag{3.51}$$

It holds from Gagliardo-Nirenberg-Sobolev inequality that

$$\begin{aligned}&\left\|(\rho-\bar{\rho},u)\right\|_{L^\infty([0,x_0)\cup(x_0,+\infty))}\\&\quad\leq\left\|(\rho-\bar{\rho},u)\right\|^{\frac{1}{2}}_{L^2([0,x_0)\cup(x_0,+\infty))}\left\|(\rho-\bar{\rho},u)_x\right\|^{\frac{1}{2}}_{L^2([0,x_0)\cup(x_0,+\infty))},\end{aligned}\tag{3.52}$$

which together with (3.10), (3.27) and (3.51) implies this lemma. □

4 Proof of the main results

Proof The global existence of unique piecewise regular solution to the exterior problem (2.4)-(2.5) can be established in terms of the short time existence carried out as in [2, 10], the uniform *a-priori* estimates and the analysis of regularities, which indeed follow from Lemmas 3.1-3.5. In addition, one can show that (ρ,u) is also global weak solution to the exterior problem (2.4)-(2.5) with initial data satisfying (2.10). We omit the details. The large time behaviors follow from Lemma 3.6 directly. The proof of Theorem 2.1 is completed. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed to each part of this work equally.

Acknowledgements
The research of RX Lian is supported by NNSFC No. 11101145.

Received: 5 November 2012 Accepted: 1 April 2013 Published: 15 April 2013

References

1. Hoff, D: Construction of solutions for compressible, isentropic Navier-Stokes equations in one space dimension with nonsmooth initial data. Proc. R. Soc. Edinb. A **103**, 301-315 (1986)
2. Hoff, D: Global existence for 1D, compressible, isentropic Navier-Stokes equations with large initial data. Trans. Am. Math. Soc. **303**, 169-181 (1987)
3. Hoff, D: Global well-posedness of the Cauchy problem for nonisentropic gas dynamics with discontinuous initial data. J. Differ. Equ. **95**, 33-73 (1992)
4. Hoff, D: Spherically symmetric solutions of the Navier-Stokes equations for compressible, isothermal flow with large, discontinuous initial data. Indiana Univ. Math. J. **41**, 1225-1302 (1992)
5. Hoff, D: Global solutions of the Navier-Stokes equations for multidimensional compressible flow with discontinuous initial data. J. Differ. Equ. **120**, 215-254 (1995)
6. Hoff, D: Strong convergence to global solutions for multidimensional flows of compressible, viscous fluids with polytropic equations of state and discontinuous initial data. Arch. Ration. Mech. Anal. **132**, 1-14 (1995)
7. Chen, G, Hoff, D, Trivisa, K: Global solutions of the compressible Navier-Stokes equations with large discontinuous initial data. Commun. Partial Differ. Equ. **25**, 2233-2257 (2000)
8. Hoff, D: Discontinuous solutions of the Navier-Stokes equations for multidimensional flows of heat-conducting fluids. Arch. Ration. Mech. Anal. **139**, 303-354 (1997)
9. Hoff, D: Dynamics of singularity surfaces for compressible, viscous flows in two space dimensions. Commun. Pure Appl. Math. **LV**, 1365-1407 (2002)
10. Fang, D, Zhang, T: Global solutions of the Navier-Stokes equations for compressible flow with density-dependent viscosity and discontinuous initial data. J. Differ. Equ. **222**, 63-94 (2006)
11. Lian, R, Guo, Z, Li, H: Dynamical behaviors of vacuum states for 1D compressible Navier-Stokes equations. J. Differ. Equ. **248**, 1926-1954 (2010)
12. Lian, R, Liu, J, Li, H, Xiao, L: Cauchy problem for the one-dimensional compressible Navier-Stokes equations. Acta Math. Sci. **32**, 315-324 (2012)
13. Gerbeau, JF, Perthame, B: Derivation of viscous Saint-Venant system for laminar shallow water, numerical validation. Discrete Contin. Dyn. Syst., Ser. B **1**, 89-102 (2001)
14. Marche, F: Derivation of a new two-dimensional viscous shallow water model with varying topography, bottom friction and capillary effects. Eur. J. Mech. B, Fluids **26**, 49-63 (2007)
15. Guo, Z, Jiu, Q, Xin, Z: Spherically symmetric isentropic compressible flows with density-dependent viscosity coefficients. SIAM J. Math. Anal. **39**, 1402-1427 (2008)
16. Jiang, S, Xin, Z, Zhang, P: Global weak solutions to 1D compressible isentropy Navier-Stokes with density-dependent viscosity. Methods Appl. Anal. **12**, 239-252 (2005)
17. Liu, T, Xin, Z, Yang, T: Vacuum states for compressible flow. Discrete Contin. Dyn. Syst., Ser. A **4**, 1-32 (1998)
18. Okada, M, Nečasoá, ŠM, Makino, T: Free boundary problem for the equation of one-dimensional motion of compressible gas with density-dependent viscosity. Ann. Univ. Ferrara, Sez. 7: Sci. Mat. **48**, 1-20 (2002)
19. Vong, S, Yang, T, Zhu, C: Compressible Navier-Stokes equations with degenerate viscosity coefficient and vacuum. II. J. Differ. Equ. **192**, 475-501 (2003)
20. Yang, T, Yao, Z, Zhu, C: Compressible Navier-Stokes equations with density-dependent viscosity and vacuum. Commun. Partial Differ. Equ. **26**, 965-981 (2001)
21. Yang, T, Zhao, H: A vacuum problem for the one-dimensional compressible Navier-Stokes equations with density-dependent viscosity. J. Differ. Equ. **184**, 163-184 (2002)
22. Yang, T, Zhu, C: Compressible Navier-Stokes equations with degenerate viscosity coefficient and vacuum. Commun. Math. Phys. **230**, 329-363 (2002)
23. Mellet, A, Vasseur, A: Existence and uniqueness of global strong solutions for one-dimensional compressible Navier-Stokes equations. SIAM J. Math. Anal. **39**, 1344-1365 (2008)
24. Jiu, Q, Wang, Y, Xin, Z: Stability of rarefaction waves to the 1D compressible Navier-Stokes equations with density-dependent viscosity. Commun. Partial Differ. Equ. **36**, 602-634 (2011)
25. Jiu, Q, Xin, Z: The Cauchy problem for 1D compressible flows with density-dependent viscosity coefficients. Kinet. Relat. Models **1**, 313-330 (2008)
26. Li, H, Li, J, Xin, Z: Vanishing of vacuum states and blow-up phenomena of the compressible Navier-Stokes equations. Commun. Math. Phys. **281**, 401-444 (2008)
27. Guo, Z, Li, H, Xin, Z: Lagrange structure and dynamics for solutions to the spherically symmetric compressible Navier-Stokes equations. Commun. Math. Phys. **309**, 371-412 (2012)
28. Hoff, D, Smoller, J: Solutions in the large for certain nonlinear parabolic systems. Ann. Inst. H. Poincaré Anal. Non Linéaire **2**(3), 213-235 (1985)

doi:10.1186/1687-2770-2013-90
Cite this article as: Lian and Yang: **Global solution to the exterior problem for spherically symmetric compressible Navier-Stokes equations with density-dependent viscosity and discontinuous initial data.** *Boundary Value Problems* 2013 **2013**:90.