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On solvability of a boundary value problem for the Poisson equation with the boundary operator of a fractional order

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Abstract
In this work, we investigate the solvability of a boundary value problem for the Poisson equation. The considered problem is a generalization of the known Dirichlet and Neumann problems on operators of a fractional order. We obtain exact conditions for solvability of the studied problem.

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1 Introduction

Let $\Omega = \{x \in R^n : |x| < 1\}$ be the unit ball, $n \geq 3$, $\partial\Omega = \{x \in R^n : |x| = 1\}$ be the unit sphere, $|x|^2 = x_1^2 + x_2^2 + \cdots + x_n^2$. Further, let $u(x)$ be a smooth function in the domain Ω, $r = |x|$, $\theta = x/|x|$. For any $0 < \alpha$, the expression

$$J^\alpha[u](x) = \frac{1}{\Gamma(1-\alpha)} \int_0^r (r-\tau)^{\alpha-1} u(\tau\theta)\, d\tau$$

is called the operator of order α in the sense of Riemann-Liouville [1]. From here on, we denote $J^0[u](x) = u(x)$. Let $m-1 < \alpha \leq m$, $m = 1, 2, \ldots,$

$$\frac{\partial}{\partial r} = \sum_{j=1}^n \frac{x_j}{r} \frac{\partial}{\partial x_j}, \qquad \frac{\partial^k u}{\partial r^k} = \frac{\partial}{\partial r}\left(\frac{\partial^{k-1} u}{\partial r^{k-1}}\right), \quad k = 1, 2, \ldots.$$

The operators

$$D^\alpha[u](x) = \frac{\partial^m}{\partial r^m} J^{m-\alpha}[u](x),$$
$$D_*^\alpha[u](x) = J^{m-\alpha}\left[\frac{\partial^m}{\partial r^m}[u]\right](x)$$

are called the derivative of order α in the sense of Riemann-Liouville and Caputo, respectively [1]. Further, let $0 \leq \beta \leq 1$, $0 < \alpha \leq 1$. Consider the operator

$$D^{\alpha,\beta}[u](x) = J^{\beta(1-\alpha)} \frac{d}{dr} J^{(1-\beta)(1-\alpha)} u(x).$$

$D^{\alpha,\beta}$ is said to be the derivative of the order α in the Riemann-Liouville sense and of type β.

We note that the operator $D^{\alpha,\beta}$ was introduced in [2]. Some questions concerning solvability for differential equations of a fractional order connected with $D^{\alpha,\beta}$ were studied in [3, 4]. Introduce the notations:

$$B^{\alpha,\beta}[u](x) = r^\alpha D^{\alpha,\beta}[u](x),$$

$$B^{-\alpha}[u](x) = \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}s^{-\alpha}u(sx)\,ds.$$

In what follows, we denote

$$B^{\alpha,0} = B^\alpha,$$

$$B^{\alpha,1} = B^\alpha_*.$$

2 Statement of the problem and formulation of the main result

Consider in the domain Ω the following problem:

$$\Delta u(x) = g(x), \quad x \in \Omega, \tag{2.1}$$

$$D^{\alpha,\beta}[u](x) = f(x), \quad x \in \partial\Omega. \tag{2.2}$$

We call a solution of the problem (2.1), (2.2) a function $u(x) \in C^2(\Omega) \cap C(\bar{\Omega})$ such that $B^{\alpha,\beta}[u](x) \in C(\bar{\Omega})$, which satisfies the conditions (2.1) and (2.2) in the classical sense. If $\alpha = 1$, then the equality

$$B^{1,\beta} = r\frac{\partial}{\partial r} = \frac{\partial}{\partial \nu}$$

holds for all $x \in \partial\Omega$, here ν is the vector of the external normal to $\partial\Omega$. Therefore, the problem (2.1), (2.2) is represented the Neumann problem in the case of $\alpha = 1$, and the Dirichlet problem for the equation (2.1) in the case of $\alpha = 0$.

It is known, the Dirichlet problem is undoubtedly solvable, and the Neumann problem is solvable if and only if the following condition is valid [5]:

$$\int_{\partial\Omega} f(x)\,ds_x = \int_\Omega g(x)\,dx. \tag{2.3}$$

Problems with boundary operators of a fractional order for elliptic equations are studied in [6, 7]. The problem (2.1), (2.2) is studied for the Riemann-Liouville and Caputo operators in the case of the Laplace equation, *i.e.*, when $g(x) = 0$, in the same works [8, 9]. It is established that the problem (2.1), (2.2) is undoubtedly solvable for the case of the Riemann-Liouville operator

$$D^{\alpha,0} = D^\alpha, \quad 0 < \alpha < 1,$$

and in the case of the Caputo operator

$$D^{\alpha,1} = D^\alpha_*, \quad 0 < \alpha < 1,$$

the problem (2.1), (2.2) is solvable if and only if the condition

$$\int_{\partial\Omega} f(x)\,ds_x = 0$$

is valid, *i.e.*, in this case, the condition for solvability of the problem (2.1), (2.2) coincides with the condition of the Neumann problem.

Let $v(x)$ be a solution of the Dirichlet problem

$$\begin{cases} \Delta v(x) = g_1(x), & x \in \Omega, \\ v(x) = f(x), & x \in \partial\Omega. \end{cases} \tag{2.4}$$

The main result of the present work is the following.

Theorem 2.1 *Let* $0 < \alpha \le 1$, $0 \le \beta \le 1$, $0 < \lambda < 1$, $f(x) \in C^{\lambda+2}(\partial\Omega)$, $g(x) \in C^{\lambda+1}(\bar{\Omega})$. *Then*:

(1) *If* $0 < \alpha < 1$, $0 \le \beta < 1$, *then a solution of the problem* (2.1), (2.2) *exists, is unique and is represented in the form of*

$$u(x) = B^{-\alpha}[v](x), \tag{2.5}$$

where $v(x)$ *is the solution of the problem* (2.4) *with the function*

$$g_1(x) = |x|^{-2} B^{\alpha,\beta}\big[|x|^2 g\big](x).$$

(2) *If* $0 < \alpha \le 1$, $\beta = 1$, *then the problem* (2.1), (2.2) *is solvable if and only if the condition*

$$\int_{\partial\Omega} f(x)\,ds_x = \int_{\Omega} \frac{|x|^{2-n} - 1}{n-2} |x|^{-2} B^{\alpha,1}\big[|x|^2 g\big](x)\,dx \tag{2.6}$$

is satisfied.

If a solution of the problem exists, then it is unique up to a constant summand and is represented in the form of (2.5), *where* $v(x)$ *is the solution of the problem* (2.4) *with the function*

$$g_1(x) = |x|^{-2} B^{\alpha,1}\big[|x|^2 g\big](x),$$

satisfying to the condition $v(0) = 0$.

(3) *If the solution of the problem exists, then it belongs to the class* $C^{\lambda+2}(\bar{\Omega})$.

3 Properties of the operators $B^{\alpha,\beta}$ and $B^{-\alpha}$

It should be noted that properties and applications of the operators B^{α}, B^{α}_{*} and $B^{-\alpha}$ in the class of harmonic functions in the ball Ω are studied in [10]. Later on, we assume that $u(x)$ is a smooth function in the domain Ω. The following proposition establishes a connection between operators $B^{\alpha,\beta}$ and B^{α}.

Lemma 3.1 *Let* $0 < \alpha \leq 1$, $0 \leq \beta \leq 1$. *Then the equalities*

$$B^{\alpha,\beta}[u](x) = \begin{cases} B^{\alpha}[u](x), & 0 \leq \beta < 1, 0 < \alpha < 1, \\ B^{\alpha}[u](x) - \frac{u(0)}{\Gamma(1-\alpha)}, & \beta = 1, 0 < \alpha \leq 1, \end{cases} \tag{3.1}$$

hold for any $x \in \Omega$.

Proof Denote

$$\delta_1 = \beta(1-\alpha),$$
$$\delta_2 = (1-\beta)(1-\alpha).$$

Let $0 < \alpha < 1$, $0 \leq \beta < 1$. Using definition of the operator $B^{\alpha,\beta}$, we obtain

$$\begin{aligned} B^{\alpha,\beta}[u](x) &= r^{\alpha} J^{\beta(1-\alpha)} \frac{d}{dr} J^{(1-\beta)(1-\alpha)} u(x) \\ &= r^{\alpha} \left\{ \frac{1}{\Gamma(\delta_1)} \int_0^r (r-\tau)^{\delta_1 - 1} \frac{1}{\Gamma(\delta_2)} \frac{d}{d\tau} \int_0^{\tau} (\tau - s)^{\delta_2 - 1} u(s\theta)\, ds\, d\tau \right\} \\ &= r^{\alpha} \left\{ \frac{1}{\Gamma(\delta_1)} \frac{d}{dr} \int_0^r \frac{(r-\tau)^{\delta_1}}{\delta_1} \frac{1}{\Gamma(\delta_2)} \frac{d}{d\tau} \int_0^{\tau} (\tau - s)^{\delta_2 - 1} u(s\theta)\, ds\, d\tau \right\} \\ &= r^{\alpha} \frac{1}{\Gamma(\delta_1)} \frac{1}{\Gamma(\delta_2)} \frac{d}{dr} \left\{ \int_0^r (r-\tau)^{\delta_1 - 1} \int_0^{\tau} (\tau - s)^{\delta_2 - 1} u(s\theta)\, ds\, d\tau \right\} \\ &= r^{\alpha} \frac{1}{\Gamma(\delta_1)} \frac{1}{\Gamma(\delta_2)} \frac{d}{dr} \left\{ \int_0^r u(s\theta) \int_s^r (r-\tau)^{\delta_1 - 1} (\tau - s)^{\delta_2 - 1}\, d\tau\, ds \right\}. \end{aligned}$$

Consider the inner integral. If we change variables $\tau = r + \xi(s - r)$, this integral can be represented in the form of

$$\int_s^r (r-\tau)^{\delta_1 - 1} (\tau - s)^{\delta_2 - 1}\, d\tau = (r-s)^{\delta_1 + \delta_2 - 1} \int_0^1 (1-\xi)^{\delta_1 - 1} \xi^{\delta_2 - 1}\, d\xi.$$

Since

$$\int_0^1 (1-\xi)^{\delta_1 - 1} \xi^{\delta_2 - 1}\, d\xi = \frac{\Gamma(\delta_1)\Gamma(\delta_2)}{\Gamma(\delta_1 + \delta_2)}, \quad \delta_1 + \delta_2 = 1 - \alpha,$$

we have

$$\begin{aligned} B^{\alpha,\beta}[u](x) &= \frac{r^{\alpha}}{\Gamma(\delta_1 + \delta_2)} \frac{d}{dr} \left\{ \int_0^r (r-s)^{\delta_1 + \delta_2 - 1} u(s\theta)\, ds \right\} \\ &= \frac{r^{\alpha}}{\Gamma(1-\alpha)} \frac{d}{dr} \left\{ \int_0^r (r-s)^{-\alpha} u(s\theta)\, ds \right\} = B^{\alpha}[u](x). \end{aligned}$$

The first equality from (3.1) is proved.

If $\beta = 1$, then

$$\begin{aligned} B^{\alpha,1}[u](x) &= \frac{r^{\alpha}}{\Gamma(1-\alpha)} \left\{ \int_0^r (r-\tau)^{-\alpha} \frac{d}{d\tau} u(\tau\theta)\, d\tau \right\} \\ &= \frac{r^{\alpha}}{\Gamma(1-\alpha)} \frac{d}{dr} \left\{ \int_0^r \frac{(r-\tau)^{1-\alpha}}{1-\alpha} \frac{d}{d\tau} u(\tau\theta)\, d\tau \right\} \end{aligned}$$

$$= \frac{r^\alpha}{\Gamma(1-\alpha)} \frac{d}{dr} \left\{ -\frac{r^{1-\alpha}}{1-\alpha} u(0) + \int_0^r (r-\tau)^{-\alpha} u(\tau\theta)\, d\tau \right\}$$
$$= \frac{r^\alpha}{\Gamma(1-\alpha)} \left\{ -r^{-\alpha} u(0) + \frac{d}{dr} \int_0^r (r-\tau)^{-\alpha} u(\tau\theta)\, d\tau \right\}$$
$$= B^\alpha[u](x) - \frac{u(0)}{\Gamma(1-\alpha)}.$$

The lemma is proved. □

This lemma implies that for any $0 \le \beta \le 1$, the problem (2.1), (2.2) can be always reduced to the problems with the boundary Riemann-Liouville or Caputo operators.

Corollary 3.2 *If* $0 < \alpha < 1$, $\beta = 1$, *then the equality*

$$B_*^\alpha[u](x) = B^\alpha[u](x) - \frac{u(0)}{\Gamma(1-\alpha)} \tag{3.2}$$

is correct.

Lemma 3.3 *If* $0 < \alpha < 1$, $\beta = 1$, *then the equality*

$$B_*^\alpha[u](0) = 0$$

holds.

Proof Since

$$B^\alpha[u](x) = \frac{r^\alpha}{\Gamma(1-\alpha)} \frac{d}{dr} \int_0^r (r-s)^{-\alpha} u(s\theta)\, ds$$
$$= \frac{1-\alpha}{\Gamma(1-\alpha)} \int_0^1 (1-\xi)^{-\alpha} u(\xi x)\, d\xi$$
$$+ \frac{r}{\Gamma(1-\alpha)} \frac{d}{dr} \int_0^1 (1-\xi)^{-\alpha} u(\xi x)\, d\xi,$$

by virtue of smoothness of the function $u(x)$ at $x \to 0$, the second integral converges to zero.

Then the equality (3.2) implies

$$\lim_{x\to 0} B_*^\alpha[u](x) = \lim_{x\to 0} \left[B^\alpha[u](x) - \frac{u(0)}{\Gamma(1-\alpha)} \right]$$
$$= \frac{1-\alpha}{\Gamma(1-\alpha)} \lim_{x\to 0} \int_0^1 (1-\xi)^{-\alpha} u(\xi x)\, d\xi - \frac{u(0)}{\Gamma(1-\alpha)}$$
$$= \frac{1-\alpha}{\Gamma(1-\alpha)} u(0) \int_0^1 (1-\xi)^{-\alpha}\, d\xi - \frac{u(0)}{\Gamma(1-\alpha)} = 0.$$

The lemma is proved. □

Lemma 3.4 *Let* $0 < \alpha < 1$. *Then the equality*

$$u(x) = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} \tau^{-\alpha} B^\alpha[u](\tau x)\, d\tau \tag{3.3}$$

holds for any $x \in \Omega$.

Proof Let $x \in \Omega$ and $t \in (0,1]$. Consider the function

$$\Im_t[u](x) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \tau^{-\alpha} B^\alpha[u](\tau x)\, d\tau.$$

Represent $\Im_t[u](x)$ in the form of

$$\Im_t[u](x) = \frac{1}{\Gamma(\alpha)} \frac{d}{dt} \left\{ \int_0^t \frac{(t-\tau)^\alpha}{\alpha} \tau^{-\alpha} B^\alpha[u](\tau x)\, d\tau \right\}.$$

Further, using definition of the operator B^α, we have

$$\begin{aligned}
&\Im_t[u](x) \\
&\quad = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \left\{ \int_0^t \frac{(t-\tau)^\alpha}{\alpha \Gamma(\alpha)} \tau^{-\alpha} \tau^\alpha \frac{d}{d\tau} \int_0^\tau (\tau-\xi)^{-\alpha} u(\xi x)\, d\xi\, d\tau \right\} \\
&\quad = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \frac{d}{dt} \left\{ \frac{(t-\tau)^\alpha}{\alpha} \int_0^\tau (\tau-\xi)^{-\alpha} u(\xi x)\, d\xi \Big|_{\tau=0}^{\tau=t} \right. \\
&\quad\quad \left. + \int_0^t (t-\tau)^{\alpha-1} \int_0^\tau (\tau-\xi)^{-\alpha} u(\xi x)\, d\xi\, d\tau \right\} \\
&\quad = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \frac{d}{dt} \left\{ \int_0^t (t-\tau)^{\alpha-1} \int_0^\tau (\tau-\xi)^{-\alpha} u(\xi x)\, d\xi\, d\tau \right\} \\
&\quad = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \frac{d}{dt} \left\{ \int_0^t u(\xi x) \int_\xi^t (t-\tau)^{\alpha-1} (\tau-\xi)^{-\alpha}\, d\tau\, d\xi \right\}.
\end{aligned}$$

It is easy to show that

$$\int_\xi^t (t-\tau)^{\alpha-1} (\tau-\xi)^{-\alpha}\, d\tau = \Gamma(\alpha)\Gamma(1-\alpha).$$

Then

$$\Im_t[u](x) = \frac{d}{dt} \int_0^t u(\xi x)\, d\xi = u(tx).$$

If now we suppose $t = 1$, then

$$u(x) = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} \tau^{-\alpha} B^\alpha[u](\tau x)\, d\tau.$$

The lemma is proved. □

Using connection between operators B^α and B_*^α, one can prove the following.

Lemma 3.5 *Let* $0 < \alpha < 1$. *Then the representation*

$$u(x) = u(0) + \frac{1}{\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} \tau^{-\alpha} B_*^\alpha[u](\tau x)\, d\tau \tag{3.4}$$

is valid for any $x \in \Omega$.

Proof Using the equality (3.3), taking into account (3.2), we obtain

$$\begin{aligned} u(x) &= \frac{1}{\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} \tau^{-\alpha} B^\alpha[u](\tau x)\, d\tau \\ &= \frac{1}{\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} \tau^{-\alpha} \left[\frac{u(0)}{\Gamma(1-\alpha)} + B_*^\alpha[u](\tau x) \right] d\tau \\ &= u(0) + \frac{1}{\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} \tau^{-\alpha} B_*^\alpha[u](\tau x)\, d\tau. \end{aligned}$$

The lemma is proved. □

Lemma 3.6 *Let* $0 < \alpha < 1$. *Then the equalities*

$$B^{-\alpha}\big[B^\alpha[u]\big](x) = B^\alpha\big[B^{-\alpha}[u]\big](x) = u(x) \tag{3.5}$$

hold for any $x \in \Omega$.

Proof Let us prove the first equality. Apply to the function $B^\alpha[u]$, the operator $B^{-\alpha}$. By definition of $B^{-\alpha}$, we have

$$B^{-\alpha}\big[B^\alpha[u]\big](x) = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} \tau^{-\alpha} B^\alpha[u](\tau x)\, d\tau.$$

But by virtue of the equality (3.3), the last integral is equal to $u(x)$, *i.e.*,

$$B^{-\alpha}\big[B^\alpha[u]\big](x) = u(x).$$

Now let us prove the second equality. Applying the operator B^α to the function $B^{-\alpha}[u](x)$, we obtain

$$\begin{aligned} B^\alpha\big[B^{-\alpha}[u]\big](x) &= \frac{r^\alpha}{\Gamma(1-\alpha)} \frac{d}{dr} \int_0^r (r-\tau)^{-\alpha} B^{-\alpha}[u](\tau x)\, d\tau \\ &= \frac{r^\alpha}{\Gamma(\alpha)\Gamma(1-\alpha)} \frac{d}{dr} \left\{ \int_0^r (r-\tau)^{-\alpha} \int_0^1 (1-s)^{\alpha-1} s^{-\alpha} u(s\tau\theta)\, ds\, d\tau \right\}. \end{aligned}$$

Further, it is not difficult to verify correctness of the following equalities:

$$\begin{aligned} &\frac{r^\alpha}{\Gamma(1-\alpha)} \frac{d}{dr} \int_0^r (r-\tau)^{-\alpha} u(s\tau\theta)\, d\tau \\ &\quad \underset{s\tau=\xi}{=} \frac{r^\alpha}{\Gamma(1-\alpha)} \frac{d}{dr} \int_0^{rs} \left(r - \frac{\xi}{s} \right)^{-\alpha} u(\xi\theta) \frac{d\xi}{s} \end{aligned}$$

$$= \frac{r^\alpha s^{\alpha-1}}{\Gamma(1-\alpha)} \frac{d}{dr} \int_0^{rs} (sr-\xi)^{-\alpha} u(\xi\theta)\, d\xi$$
$$= \frac{(sr)^\alpha}{\Gamma(1-\alpha)} \frac{d}{d(sr)} \int_0^{rs} (sr-\xi)^{-\alpha} u(\xi\theta)\, d\xi = B^\alpha[u](sx).$$

Here, it is taken into account $\theta = \frac{x}{|x|} = \frac{sx}{|sx|}$. Therefore,

$$B^\alpha\big[B^{-\alpha}[u]\big](x) = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} s^{-\alpha} B^\alpha[u](sx)\, ds.$$

Hence, using the equality (3.3), we obtain

$$B^\alpha\big[B^{-\alpha}[u]\big](x) = u(x).$$

The lemma is proved. □

In [11], the following is proved.

Lemma 3.7 *Let $u(x) \in C^1(\Omega)$. Then the equality*

$$u(x) = u(0) + \int_0^1 \left[\sum_{k=1}^n x_k \frac{\partial u(sx)}{\partial x_k}\right] ds \tag{3.6}$$

holds for any $x \in \Omega$.

Since

$$\sum_{k=1}^n x_k \frac{\partial u(x)}{\partial x_k} = r\frac{\partial u(x)}{\partial r} = B^1[u](x) = B^1_*[u](x),$$

the equality (3.5) *can be represented in the form of*

$$u(x) = u(0) + \int_0^1 s^{-1} B^1[u](sx)\, ds. \tag{3.7}$$

Then Lemma 3.5 and the equality (3.6) imply the following.

Corollary 3.8 *Let $0 < \alpha \le 1$. Then for any $x \in \Omega$, the representation*

$$u(x) = u(0) + \frac{1}{\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} \tau^{-\alpha} B^\alpha_*[u](\tau x)\, d\tau$$

is valid. Hence, as the inverse operator to B^1, we can consider the following operator:

$$B^{-1}[u](x) = \int_0^1 \tau^{-1} u(\tau x)\, d\tau.$$

Note that if

$$u(0) \neq 0,$$

then the operator B^{-1} is not defined in such functions. Let $u(x)$ be a smooth function. Obviously,

$$B^1[u](0) = 0.$$

Consider action of the operator B^{-1} to the function $B^1[u](x)$. By definition of the operator B^{-1}, we have

$$B^{-1}\bigl[B^1[u]\bigr](x) = \int_0^1 s^{-1}B^1[u](tx)\,dt.$$

By virtue of (3.7), the value of the last integral is equal to $u(x) - u(0)$. Thus, the equality

$$B^{-1}\bigl[B^1[u]\bigr](x) = u(x) - u(0)$$

holds.

Conversely, let $u(0) = 0$. Then the operator B^{-1} is defined for such functions, and

$$B^1\bigl[B^{-1}[u]\bigr](x) = r\frac{\partial}{\partial r}\left[\int_0^1 s^{-1}u(sx)\,ds\right] = r\frac{\partial}{\partial r}\left[\int_0^r \xi^{-1}u(\xi\theta)\,d\xi\right] = u(x).$$

It means that

$$B^1\bigl[B^{-1}[u]\bigr](x) = u(x).$$

Thus, we prove the following.

Lemma 3.9 *For any $x \in \Omega$, the following equalities are valid*:

(1) $B^{-1}[B^1[u]](x) = u(x) - u(0)$;

(2) *if* $u(0) = 0$, *then*

$$B^1\bigl[B^{-1}[u]\bigr](x) = u(x).$$

Using Lemma 3.9 and connection between operators B^α and B_*^α, we get the following.

Corollary 3.10 *Let* $0 < \alpha \le 1$. *Then for any* $x \in \Omega$, *the following equalities hold*: *if* $0 < \alpha < 1$, *then*

$$B^{-\alpha}\bigl[B_*^\alpha[u]\bigr](x) = u(x) - \frac{u(0)}{\Gamma(1-\alpha)};$$

$$B^{-1}\bigl[B^1[u]\bigr](x) = u(x) - u(0);$$

if $0 < \alpha \le 1$ *and* $u(0) = 0$, *then*

$$B_*^\alpha\bigl[B^{-\alpha}[u]\bigr](x) = u(x).$$

Lemma 3.11 *Let*

$$\Delta u(x) = g(x), \quad x \in \Omega, 0 < \alpha \le 1.$$

Then for any $x \in \Omega$, the equality

$$\Delta B^{\alpha}[u](x) = |x|^{-2} B^{\alpha}\big[|x|^2 g\big](x) \tag{3.8}$$

holds.

Proof Let $0 < \alpha < 1$. After changing of variables, the function $B^{\alpha}[u](x)$ can be represented in the form of

$$B^{\alpha}[u](x) = \frac{1-\alpha}{\Gamma(1-\alpha)} \int_0^1 (1-\xi)^{-\alpha} u(\xi x)\,d\xi + r\frac{d}{dr}\int_0^1 \frac{(1-\xi)^{-\alpha}}{\Gamma(1-\alpha)} u(\xi x)\,d\xi = I_1(x) + I_2(x).$$

Since

$$\Delta u(x) = g(x),$$

it is easy to show that

$$\Delta I_1(x) = \frac{1-\alpha}{\Gamma(1-\alpha)} \int_0^1 (1-\xi)^{-\alpha}\xi^2 g(\xi x)\,d\xi.$$

Further, if $v(x)$ is a smooth function, then obviously,

$$\Delta\left[r\frac{\partial}{\partial r} v(x)\right] = r\frac{\partial}{\partial r}\Delta v(x) + 2\Delta v(x).$$

That is why

$$\Delta I_2(x) = r\frac{d}{dr}\int_0^1 \frac{(1-\xi)^{-\alpha}}{\Gamma(1-\alpha)}\xi^2 g(\xi x)\,d\xi + 2\int_0^1 \frac{(1-\xi)^{-\alpha}}{\Gamma(1-\alpha)}\xi^2 g(\xi x)\,d\xi.$$

Consider the integral

$$\int_0^1 (1-\xi)^{-\alpha}\xi^2 g(\xi x)\,d\xi.$$

After changing of variables $\xi r = \tau$, $\xi = r^{-1}\tau$, the integral can be transformed to the following form:

$$\int_0^1 (1-\xi)^{-\alpha}\xi^2 g(\xi x)\,d\xi = r^{\alpha-3}\int_0^r (r-\tau)^{-\alpha}\tau^2 g(\tau\theta)\,d\tau.$$

Then

$$\begin{aligned} r\frac{d}{dr}\int_0^1 \frac{(1-\xi)^{-\alpha}}{\Gamma(1-\alpha)}\xi^2 g(\xi x)\,d\xi &= r\frac{d}{dr}\left[r^{\alpha-3}\int_0^r (r-\tau)^{-\alpha}\tau^2 g(\tau\theta)\,d\tau\right] \\ &= (\alpha-3)r^{\alpha-3}\int_0^r (r-\tau)^{-\alpha}\tau^2 g(\tau\theta)\,d\tau \\ &\quad + r^{\alpha-2}\frac{d}{dr}\int_0^r (r-\tau)^{-\alpha}\tau^2 g(\tau\theta)\,d\tau. \end{aligned}$$

Hence,

$$\begin{aligned}\Delta I_1(x)+\Delta I_2(x)&=\frac{1-\alpha}{\Gamma(1-\alpha)}r^{\alpha-3}\int_0^r(r-\tau)^{-\alpha}\tau^2g(\tau\theta)\,d\tau\\&\quad+\frac{(\alpha-3)}{\Gamma(1-\alpha)}r^{\alpha-3}\int_0^r(r-\tau)^{-\alpha}\tau^2g(\tau\theta)\,d\tau\\&\quad+\frac{r^{\alpha-2}}{\Gamma(1-\alpha)}\frac{d}{dr}\int_0^r(r-\tau)^{-\alpha}\tau^2g(\tau\theta)\,d\tau\\&\quad+\frac{2}{\Gamma(1-\alpha)}r^{\alpha-3}\int_0^r(r-\tau)^{-\alpha}\tau^2g(\tau\theta)\,d\tau\\&=\frac{r^{\alpha-2}}{\Gamma(1-\alpha)}\frac{d}{dr}\int_0^r(r-\tau)^{-\alpha}\tau^2g(\tau\theta)\,d\tau\\&=r^{-2}\frac{r^{\alpha}}{\Gamma(1-\alpha)}\frac{d}{dr}\int_0^r(r-\tau)^{-\alpha}\tau^2g(\tau\theta)\,d\tau=r^{-2}B^{\alpha}\big[|x|^2g\big](x).\end{aligned}$$

Let now $\alpha=1$. In this case,

$$B^1[u](x)=r\frac{\partial u(x)}{\partial r},$$

and therefore.

$$\Delta B^1[u](x)=\Delta\left[r\frac{\partial u(x)}{\partial r}\right]=r\frac{\partial\Delta u(x)}{\partial r}+2\Delta u(x)=r\frac{\partial g(x)}{\partial r}+2g(x).$$

On the other hand,

$$\left(r\frac{\partial}{\partial r}+2\right)g(x)=|x|^{-2}r\frac{\partial}{\partial r}\big[r^2g(x)\big]=|x|^{-2}B^1\big[|x|^2g\big](x).$$

The lemma is proved. □

4 Some properties of a solution of the Dirichlet problem

Let $v(x)$ be a solution of the problem (2.4). It is known (see [12]), if functions$f(x)$ and $g_1(x)$ are sufficiently smooth, then a solution of the problem (2.4) exists and is represented in the form of

$$v(x)=-\frac{1}{\omega_n}\int_{\Omega}G(x,y)g_1(y)\,dy+\frac{1}{\omega_n}\int_{\partial\Omega}P(x,y)f(y)\,ds_y, \tag{4.1}$$

here ω_n is the area of the unit sphere, $G(x,y)$ is the Green function of the Dirichlet problem for the Laplace equation, and $P(x,y)$ is the Poisson kernel.

In addition, the representations

$$\begin{aligned}G(x,y)&=\frac{1}{n-2}\left[|x-y|^{2-n}-\left||y|x-\frac{y}{|y|}\right|^{2-n}\right],\\P(x,y)&=\frac{1-|x|^2}{|x-y|^n}\end{aligned}$$

take place.

Lemma 4.1 *Let $v(x)$ be a solution of the problem* (2.4).

Then

(1) *if $v(0) = 0$, then*

$$\int_{\partial\Omega} f(y)\, ds_y = \int_{\Omega} \frac{|y|^{2-n} - 1}{n-2} g_1(y)\, dy; \tag{4.2}$$

(2) *if the equality* (4.2) *is valid, then the condition $v(0) = 0$ is fulfilled for a solution of the problem* (2.4).

Proof Let a solution of the problem (2.4) exist. Represent it in the form of (4.1). We have from the representation of the function $G(x,y)$

$$G(0,y) = \frac{1}{n-2}\left[|y|^{2-n} - 1\right]$$

and

$$P(0,y) = 1.$$

Then

$$0 = v(0) = -\frac{1}{\omega_n}\int_{\Omega} G(0,y) g_1(y)\, dy + \frac{1}{\omega_n}\int_{\partial\Omega} P(0,y) f(y)\, ds_y.$$

Hence,

$$\int_{\partial\Omega} f(y)\, ds_y = \int_{\Omega} \frac{|y|^{2-n} - 1}{n-2} g_1(y)\, dy.$$

The equality (4.2) is proved. The second assertion of the lemma is proved in the inverse order. The lemma is proved. □

Lemma 4.2 *Let $v(x)$ be a solution of the problem* (2.4), *and the function $g_1(x)$ be represented in the form of*

$$g_1(y) = \left(\rho\frac{\partial}{\partial\rho} + 2\right) g(y), \qquad \rho = |y|.$$

Then the condition (4.2) *can be represented in the form of*

$$\int_{\partial\Omega} f(y)\, ds_y = \int_{\Omega} g(y)\, dy. \tag{4.3}$$

Proof Using representation of the function $g_1(y)$, we have

$$\begin{aligned}
&\int_{\Omega} \frac{|y|^{2-n} - 1}{n-2} g_1(y)\, dy \\
&\quad = \int_0^1 \rho^{n-1} \int_{|\xi|=1} \frac{\rho^{2-n} - 1}{n-2}\left(\rho\frac{\partial}{\partial\rho} + 2\right) g(\rho\xi)\, d\xi\, d\rho.
\end{aligned}$$

Then

$$\int_0^1 \rho^{n-1}\frac{\rho^{2-n}-1}{n-2}\left(\rho\frac{\partial}{\partial\rho}+2\right)g(\rho\xi)\,d\rho = \frac{1}{n-2}\int_0^1\left[\rho^2-\rho^n\right]\frac{\partial}{\partial\rho}[g](\rho\xi)\,d\rho$$
$$+\frac{2}{n-2}\int_0^1\left[\rho-\rho^{n-1}\right]g(\rho\xi)\,d\rho = I_1+I_2.$$

Consider I_1. After integrating by parts, we get

$$I_1=\frac{1}{n-2}\int_0^1\left[n\rho^{n-1}-2\rho\right]g(\rho\xi)\,d\rho,$$

what follows

$$\begin{aligned}
&I_1+I_2\\
&\quad=\frac{1}{n-2}\left\{\int_0^1(n-2)\rho^{n-1}g(\rho\xi)\,d\rho-2\int_0^1\rho g(\rho\xi)\,d\rho+2\int_0^1\rho g(\rho\xi)\,d\rho\right\}\\
&\quad=\int_0^1\rho^{n-1}g(\rho\xi)\,d\rho.
\end{aligned}$$

Hence,

$$\int_\Omega\frac{|y|^{2-n}-1}{n-2}g_1(y)\,dy=\int_0^1\rho^{n-1}\int_{|\xi|=1}g(\rho\xi)\,d\xi\,d\rho=\int_\Omega g(y)\,dy.$$

The lemma is proved. □

5 The proof of the main proposition

Let $0<\alpha<1$, $0\le\beta<1$, and $u(x)$ be a solution of the problem (2.1), (2.2). In this case, by Lemma 3.1, $B^{\alpha,\beta}=B^\alpha$. Apply to the function $u(x)$ the operator B^α, and denote

$$v(x)=B^\alpha[u](x).$$

Then, using the equality (3.8), we obtain

$$\Delta v(x)=\Delta B^\alpha[u](x)=|x|^{-2}B^\alpha\left[|x|^2g\right](x)\equiv g_1(x).$$

Since $B^{\alpha,\beta}=B^\alpha$, it is obviously,

$$v(x)|_{\partial\Omega}=B^\alpha[u](x)|_{\partial\Omega}=f(x).$$

Thus, if $u(x)$ is a solution of the problem (2.1), (2.2), then we obtain for the function

$$v(x)=B^\alpha[u](x)$$

the problem (2.4) with

$$g_1(x)=|x|^{-2}B^\alpha\left[|x|^2g\right](x).$$

Further, since

$$g_1(x) = |x|^{-2}B^{\alpha}\left[|x|^2 g\right](x) = \left[r\frac{d}{dr} + 3 - \alpha\right]\int_0^1 \frac{(1-\xi)^{-\alpha}}{\Gamma(1-\alpha)}\xi^2 g(\xi x)\,d\xi,$$

for $g(x) \in C^{\lambda+1}(\bar{\Omega})$, we have $g_1(x) \in C^{\lambda}(\bar{\Omega})$.

Then for $g_1(x) \in C^{\lambda}(\bar{\Omega})$, $f(x) \in C^{\lambda+2}(\partial\Omega)$, a solution of the problem (2.4) exists and belongs to the class $C^{\lambda+2}(\bar{\Omega})$ (see, for example, [13]).

Further, applying to the equality

$$v(x) = B^{\alpha}[u](x)$$

the operator $B^{-\alpha}$, by virtue of the first equality of the formula (3.5), we obtain

$$u(x) = B^{-\alpha}[v](x).$$

The last function satisfies to all the conditions of the problem (2.1), (2.2).

Really,

$$\begin{aligned}
\Delta u(x) = \Delta B^{-\alpha}[v](x) &= \frac{1}{\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1}\tau^{2-\alpha}\Delta v(\tau x)\,d\tau \\
&= \frac{1}{\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1}\tau^{2-\alpha}g_1(\tau x)\,d\tau \\
&= \frac{1}{\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1}\tau^{2-\alpha}\tau^{-2}|x|^{-2}B^{\alpha}\left[|x|^2 g\right](\tau x)\,d\tau \\
&= \frac{1}{\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1}\tau^{-\alpha}|x|^{-2}B^{\alpha}\left[|x|^2 g\right](\tau x)\,d\tau \\
&= \frac{|x|^{-2}}{\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1}\tau^{-\alpha}B^{\alpha}\left[|x|^2 g\right](\tau x)\,d\tau = |x|^{-2}|x|^2 g(x) = g(x).
\end{aligned}$$

Now, using the second equality from (3.5), we obtain

$$B^{\alpha}[u](x)|_{\partial\Omega} = B^{\alpha}\left[B^{-\alpha}[v]\right](x)|_{\partial\Omega} = v(x)|_{\partial\Omega} = f(x).$$

So, the function $u(x) = B^{-\alpha}[v](x)$ satisfies equation (2.1) and the boundary condition (2.2). Let now $0 < \alpha < 1$, $\beta = 1$, and $u(x)$ be a solution of the problem (2.1), (2.2). Apply to the function $u(x)$ the operator $B^{\alpha,1} = B_*^{\alpha}$, and denote $v(x) = B_*^{\alpha}[u](x)$. In this case, we obtain for the function $v(x)$ the problem (2.4) with the function

$$g_1(x) = |x|^{-2}B_*^{\alpha}\left[|x|^2 g\right](x).$$

Since $B_*^{\alpha}[u](0) = 0$, the function $v(x)$ must satisfy in addition to the condition $v(0) = 0$.

Arbitrary solution of the problem (2.4) at smooth $f(x)$ and $g(x)$ is represented in the form of (4.1). And in order that this solution satisfies to the condition $v(0) = 0$, according to Lemma 3.11, it is necessary and sufficient fulfillment of the condition (4.2).

In our case, the condition (4.2) has the form

$$\int_{\partial\Omega} f(y)\,ds_y = \int_{\Omega} \frac{|y|^{2-n}-1}{n-2}|y|^{-2}B_*^{\alpha}\bigl[|y|^2 g\bigr](y)\,dy.$$

In this case, $B^{\alpha,1} = B_*^{\alpha}$ and, therefore, the condition (4.2) coincides with the condition (2.6).

Thus, necessity of (2.6) is proved. This condition is also sufficient condition for existence of a solution for the problem (2.1), (2.2).

In fact, if the condition (2.6) holds, then $\nu(0) = 0$, and the function

$$u(x) = B^{-\alpha}[\nu](x) + C$$

satisfies to all conditions of the problem (2.1), (2.2). Let us check these conditions. Fulfillment of the condition

$$\Delta u(x) = g(x)$$

can be checked similarly as in the case of the proof of the first part of the theorem. Further, using the equality (3.5) and connection between operators B^{α} and B_*^{α}, we get

$$\begin{aligned} B_*^{\alpha}[u](x) &= B_*^{\alpha}\bigl[B^{-\alpha}[\nu] + C\bigr](x) \\ &= B_*^{\alpha}\bigl[B^{-\alpha}[\nu]\bigr](x) + B_*^{\alpha}[C] \\ &= B^{\alpha}\bigl[B^{-\alpha}[\nu]\bigr](x) + \frac{B^{-\alpha}[\nu](0)}{\Gamma(1-\alpha)} = \nu(x). \end{aligned}$$

Hence,

$$B^{\alpha,1}[u](x)|_{\partial\Omega} = B_*^{\alpha}[u](x)|_{\partial\Omega} = \nu(x)|_{\partial\Omega} = f(x).$$

If $\alpha = 1$, then

$$D^{\alpha,1} = D_*^{\alpha} = \frac{\partial}{\partial r}$$

and

$$B^{\alpha,1} = r\frac{\partial}{\partial r}.$$

In this case,

$$|x|^{-2}B^{\alpha,1}\bigl[|x|^2 g\bigr](x) = \left(r\frac{\partial}{\partial r} + 2\right)g(x).$$

Then by virtue of Lemma 4.1, the solvability condition of the problem (2.1), (2.2) can be rewritten in the form of

$$\int_{\partial\Omega} f(x)\,ds_x = \int_{\Omega} g(x)\,dx.$$

It is the solvability condition for the Neumann problem. Further, since $v(x) \in C^{\lambda+2}(\bar{\Omega})$, the function $u(x) = B^{-\alpha}[v](x)$ also belongs to the class $C^{\lambda+2}(\bar{\Omega})$. The theorem is proved.

6 Example

Example Let $0 < \alpha < 1$, $\beta = 1$ and

$$g(x) = |x|^{2k}, \quad k = 0, 1, \ldots.$$

Then

$$\begin{aligned} |x|^{-2} B^{\alpha,1}\bigl[|x|^2 g\bigr](x) &= \frac{|x|^{\alpha-2}}{\Gamma(1-\alpha)} \int_0^r (r-\tau)^{-\alpha} \frac{\partial}{\partial \tau} \tau^{2k+2}\, d\tau \\ &= \frac{(2k+2)|x|^{\alpha-2}}{\Gamma(1-\alpha)} \int_0^r (r-\tau)^{-\alpha} \tau^{2k+2-1}\, d\tau \\ &= \frac{(2k+2)|x|^{\alpha-2}}{\Gamma(1-\alpha)} |x|^{2k+2-\alpha} \int_0^1 (1-\xi)^{-\alpha} \xi^{2k+1}\, d\xi \\ &= \frac{(2k+2)|x|^{2k}}{\Gamma(1-\alpha)} \frac{\Gamma(1-\alpha)\Gamma(2k+2)}{\Gamma(2k+3-\alpha)} = \frac{\Gamma(2k+3)}{\Gamma(2k+3-\alpha)} |x|^{2k}. \end{aligned}$$

Since

$$\begin{aligned} \int_0^1 r^{2k+n-1}\bigl(r^{2-n} - 1\bigr)\, dr &= \int_0^1 \bigl(r^{2k+1} - r^{2k+n-1}\bigr)\, dr \\ &= \frac{1}{2k+2} - \frac{1}{2k+n} = \frac{n-2}{(2k+2)(2k+n)}, \end{aligned}$$

we have

$$\begin{aligned} \int_\Omega \frac{|y|^{2-n} - 1}{n-2} |y|^{-2} B_*^{\alpha}\bigl[|y|^2 g\bigr](y)\, dy &= \int_{|\xi|=1} \int_0^1 \frac{r^{2-n} - 1}{n-2} r^{-2} B_*^{\alpha}\bigl[|y|^2 g\bigr](r\xi)\, dr\, d\xi \\ &= \frac{\Gamma(2k+2)}{\Gamma(2k+3-\alpha)} \frac{\omega_n}{(2k+n)}. \end{aligned}$$

Then the solvability condition for the problem (2.1), (2.2) has in this case the form

$$\int_{\partial\Omega} f(y)\, ds_y = \frac{\Gamma(2k+2)}{\Gamma(2k+3-\alpha)} \frac{\omega_n}{(2k+n)}.$$

For example, if $f(x) = 1$, this condition is not fulfilled. If

$$f(x) = \frac{\Gamma(2k+2)}{\Gamma(2k+3-\alpha)} \frac{1}{(2k+n)},$$

then the solvability condition of the problem is carried out. In this case, solving the Dirichlet problem (2.4) with the functions

$$\begin{aligned} g_1(x) &\equiv |x|^{-2} B^{\alpha,1}\bigl[|x|^2 g\bigr](x) = \frac{\Gamma(2k+3)}{\Gamma(2k+3-\alpha)} |x|^{2k}, \\ f(x) &= \frac{\Gamma(2k+2)}{\Gamma(2k+3-\alpha)} \frac{1}{(2k+n)}, \end{aligned}$$

we obtain (see [14])

$$v(x) = \frac{\Gamma(2k+3)}{\Gamma(2k+3-\alpha)} \frac{|x|^{2k+2}}{(2k+2)(2k+n)} = \frac{\Gamma(2k+2)}{\Gamma(2k+3-\alpha)} \frac{|x|^{2k+2}}{2k+n}.$$

Using the formula (2.5), we obtain the solution of the problem (2.1), (2.2)

$$\begin{aligned} u(x) &= B^{-\alpha}[v](x) \\ &= \frac{\Gamma(2k+2)}{\Gamma(2k+3-\alpha)} \frac{|x|^{2k+2}}{2k+n} \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} s^{2k+2-\alpha}\, ds \\ &= \frac{\Gamma(2k+2)}{\Gamma(2k+3-\alpha)} \frac{|x|^{2k+2}}{2k+n} \frac{1}{\Gamma(\alpha)} \frac{\Gamma(\alpha)\Gamma(2k+3-\alpha)}{\Gamma(2k+3)} = \frac{|x|^{2k+2}}{(2k+2)(2k+n)}. \end{aligned}$$

Thus, the solution of the problem (2.1), (2.2) has the form

$$u(x) = \frac{|x|^{2k+2}}{(2k+2)(2k+n)} + C.$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors completed the paper together. All authors read and approved the final manuscript.

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