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Existence and multiplicity of solutions to boundary value problems for fourth-order impulsive differential equations

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Dedicated to Professor Ivan Kiguradze for his merits in the mathematical sciences

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Abstract
In this paper we study the existence and the multiplicity of solutions for an impulsive boundary value problem for fourth-order differential equations. The notions of classical and weak solutions are introduced. Then the existence of at least one and infinitely many nonzero solutions is proved, using the minimization, the mountain-pass, and Clarke's theorems.
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1 Introduction

The theory of impulsive boundary value problems (IBVPs) became an important area of studies in recent years. IBVPs appear in mathematical models of processes with sudden changes in their states. Such processes arise in population dynamics, optimal control, pharmacology, industrial robotics, etc. For an introduction to theory of IBVPs one is referred to [1]. Some classical tools used in the study of impulsive differential equations are topological methods as fixed point theorems, monotone iterations, upper and lower solutions (see [2–4]). Recently, some authors have studied the existence of solutions of IBVPs using variational methods. The pioneering work in this direction is the paper of Nieto and O'Regan [5], where the second-order impulsive problem

$$\begin{cases} -u'' + \lambda u = f(t,u), & t \neq t_j, \text{a.e. } t \in [0,T], \\ u(0) = u(T), \\ \Delta u'(t_j) = I_j(u(t_j)), & j = 1, 2, \ldots, n \end{cases}$$

(with $\Delta u'(t_j) := u'(t_j^+) - u'(t_j^-)$) is studied, using the minimization and the mountain-pass theorem. We mention also other papers for second-order impulsive equations as [6, 7]. In several recent papers [8–10], fourth-order impulsive problems are considered via variational methods.

In this paper, we consider the boundary value problem for fourth-order differential equation with impulsive effects

$$\text{(P)}\qquad \begin{cases} u^{(4)} - au'' + b(t)u = c(t)|u|^{p-2}u, & t \neq t_j, \text{a.e. } t \in [0,T], \\ u(0) = u(T) = u''(0) = u''(T) = 0, \\ \Delta u'''(t_j) = g_j(u'(t_j)), \qquad \Delta u''(t_j) = -h_j(u(t_j)), & j = 1, 2, \ldots, n. \end{cases}$$

Here, $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = T$, the limits $u''(t_j^{\pm}) = \lim_{t\to t_j^{\pm}} u''(t)$ and $u'''(t_j^{\pm}) = \lim_{t\to t_j^{\pm}} u'''(t)$ exist and $\Delta u'''(t_j) = u'''(t_j^+) - u'''(t_j^-)$, $\Delta u''(t_j) = u''(t_j^+) - u''(t_j^-)$.

We look for solutions in the classical sense, as given in the next definition.

Definition 1 A function $u \in C^1([0,T])$ and $u|_{(t_j,t_{j+1})} \in H^2(t_j, t_{j+1})$, $j = 0, \ldots, n$ is said to be a classical solution of the problem (P), if u satisfies the equation a.e. on $[0,T]\backslash\{t_1, t_2, \ldots, t_n\}$, the limits $u''(t_j^{\pm}) = \lim_{t\to t_j^{\pm}} u''(t)$ and $u'''(t_j^{\pm}) = \lim_{t\to t_j^{\pm}} u'''(t)$ exist and satisfy the impulsive conditions $\Delta u'''(t_j) = g(u'(t_j))$, $\Delta u''(t_j) = -h(u(t_j))$, $j = 1, \ldots, n$, and boundary conditions $u(0) = u(T) = u''(0) = u''(T) = 0$.

Moreover, we introduce, for every $j \in \{1, \ldots, n\}$, the following real functions:

$$G_j(u) = \int_0^u g_j(t)\,dt, \qquad H_j(u) = \int_0^u h_j(t)\,dt. \tag{1}$$

To deduce the existence of solutions, we assume the following conditions:

(H1) The constant a is positive, b and c are continuous functions on $[0,T]$ and there exist positive constants b_1, b_2, c_1, and c_2 such that $0 < b_1 \leq b(t) \leq b_2$ and $0 < c_1 \leq c(t) \leq c_2$. The functions $g_j : \mathbb{R} \to \mathbb{R}$, $h_j : \mathbb{R} \to \mathbb{R}$, $j = 1, 2, \ldots, n$, are continuous functions.

(H2) There exist $\gamma_j, \sigma_j \in (2, p)$ such that functions g_j, h_j, $j = 1, \ldots, n$, satisfy the conditions

$$\gamma_j G_j(t) \geq t g_j(t) > 0, \qquad \sigma_j H_j(t) \geq t h_j(t) > 0, \quad \forall t \in \mathbb{R}\backslash\{0\}. \tag{2}$$

A simple example of functions fulfilling the last condition is given by

$$g_j(t) = d_j |t|^{\gamma_j - 2} t, \qquad h_j(t) = e_j |t|^{\sigma_j - 2} t,$$

where d_j and e_j, $j = 1, \ldots, n$, are positive constants.

Note that (2) implies that there exist positive constants D_j, E_j such that

$$G_j(t) \leq D_j\bigl(1 + |t|^{\gamma_j}\bigr), \qquad H_j(t) \leq E_j\bigl(1 + |t|^{\sigma_j}\bigr). \tag{3}$$

In the next section we will prove the following existence result for $p > 2$.

Theorem 2 *Suppose that* $p > 2$ *and conditions* (H1) *and* (H2) *hold*. *Then the problem* (P) *has at least one nonzero classical solution*.

Having in mind the case $1 < p < 2$, we introduce the following condition:

(H3) There exist positive constants A_j, B_j, $j = 1, \dots, n$ such that the functions G_j, H_j, defined in (3), satisfy the conditions

$$0 \le G_j(t) \le A_j t^2, \qquad 0 \le H_j(t) \le B_j t^2, \quad \forall t \in \mathbb{R}.$$

A simple example of this new situation is given by the functions $g_j(t) = 2A_j t$ and $h_j(t) = 2B_j t$.

The result to be proven is the following.

Theorem 3 *Suppose that* $1 < p < 2$, *the functions* g_j, h_j, $j = 1, \dots, n$, *are odd and conditions* (H1) *and* (H3) *hold. Then the problem* (P) *has infinitely many nonzero classical solutions.*

If we consider the problem

$$(\mathrm{P}_1) \qquad \begin{cases} u^{(4)} - au'' - b(t)u + c(t)|u|^{p-2}u = 0, \quad t \ne t_j, \text{a.e. } t \in [0,T], \\ u(0) = u(T) = u''(0) = u''(T) = 0, \\ \Delta u'''(t_j) = g(u'(t_j)), \qquad \Delta u''(t_j) = -h(u(t_j)), \quad j = 1, 2, \dots, n, \end{cases}$$

we introduce the following condition.

(H2′) There exist $\gamma_j, \sigma_j \in (2, p)$ and positive constants d_j, e_j such that functions g_j, h_j, G_j and H_j, $j = 1, \dots, n$, satisfy the conditions

$$\begin{aligned} &0 \le G_j(t) \le d_j |t|^{\gamma_j}, \qquad 0 \le H_j(t) \le e_j |t|^{\sigma_j}, \quad \forall t \in \mathbb{R}, \\ &0 \le t g_j(t), \qquad 0 \le t h_j(t), \quad \forall t \in \mathbb{R}. \end{aligned} \tag{4}$$

A simple example now is $g_j(t) = \gamma_j d_j |t|^{\gamma_j - 2} t$, $h_j(t) = \sigma_j e_j |t|^{\sigma_j - 2} t$.

The obtained result is the following.

Theorem 4 *Suppose that* $p > 2$ *and conditions* (H1) *and* (H2′) *hold. If* $0 < T \le T_2 = \pi \sqrt{\frac{a + \sqrt{a^2 + 4b_2}}{2b_2}}$, *the problem* (P_1) *has only the zero solution. If* $T > T_1 = \pi \sqrt{\frac{a + \sqrt{a^2 + 4b_1}}{2b_1}}$, *the problem* (P_1) *has at least one nonzero classical solution.*

The proofs of the main results are given in Section 3.

2 Preliminaries

Denote by $L^p(0,T)$ for $p \ge 1$, the Lebesgue space of p-integrable functions over the interval $(0,T)$, endowed with the usual norm $\|u\|_p^p = \int_a^b |u(t)|^p \, dt$, and by $\|\cdot\|$ and $\|\cdot\|_\infty$ the corresponding norms in $L^2(0,T)$ and $C([0,T])$,

$$\begin{aligned} &\|u\|^2 = \int_0^T \left|u(t)\right|^2 dt, \quad u \in L^2(0,T), \\ &\|u\|_\infty = \max_{t \in [0,T]} \left|u(t)\right|, \quad u \in C\bigl([0,T]\bigr). \end{aligned}$$

Denote by $H_0^1(0,T)$ and $H^2(0,T)$ the Sobolev spaces

$$H_0^1(0,T) = \bigl\{ u \in L^2(0,T) : u' \in L^2(0,T), u(0) = u(T) = 0 \bigr\},$$

and

$$H^2(0,T) = \{u \in L^2(0,T) : u' \in L^2(0,T), u'' \in L^2(0,T)\}.$$

Let $X = H_0^1(0,T) \cap H^2(0,T)$ be the Hilbert space endowed with the usual scalar product

$$(u,v) = \int_0^T (u''v'' + u'v' + uv)\,dt$$

and the corresponding norm.

By assumption (H_1) an equivalent scalar product and norm in X are given by

$$\langle u,v\rangle = \int_0^T (u''v'' + au'v' + b(t)uv)\,dt, \tag{5}$$

and

$$\|u\|_X^2 = \int_0^T (u''^2 + au'^2 + b(t)u^2)\,dt. \tag{6}$$

It is well known (see [9, Lemma 2.2], [11]) that the following Poincaré and imbedding inequalities hold for all $u \in X$:

$$\int_0^T u^2\,dt \le \frac{T^2}{\pi^2}\int_0^T u'^2\,dt, \qquad \int_0^T u^2\,dt \le \frac{T^4}{\pi^4}\int_0^T u''^2\,dt, \tag{7}$$

$$\|u\|_{C^1} = \max\{\|u\|_\infty, \|u'\|_\infty\} \le M\|u\|_X, \tag{8}$$

where M is a positive constant depending on T, a and $b(t)$.

We have the following compactness embedding, which can be proved in the standard way.

Proposition 5 *The inclusion $X \subset C^1([0,T])$ is compact.*

We define the functional $\phi : X \to \mathbb{R}$, as follows:

$$\phi(u) = \frac{1}{2}\|u\|_X^2 + \sum_{j=1}^n \bigl(G_j(u'(t_j)) + H_j(u(t_j))\bigr) - \frac{1}{p}\int_0^T c(t)|u|^p\,dt. \tag{9}$$

By assumption (H1), we find that $\phi : X \to \mathbb{R}$ is continuously differentiable and, for $v \in X$, the following identity holds:

$$\begin{aligned}\langle \phi'(u), v\rangle &= \int_0^T (u''v'' + au'v' + b(t)uv)\,dt \\ &\quad + \sum_{j=1}^n \bigl(g_j(u'(t_j))v'(t_j) + h_j(u(t_j))v(t_j)\bigr) - \int_0^T c(t)|u|^{p-2}uv\,dt.\end{aligned} \tag{10}$$

In the sequel we introduce the concept of a weak solution of our problem.

Definition 6 A function $u \in X$ is said to be a weak solution of the problem (P), if for every $v \in X$, the following identity holds:

$$\begin{aligned} &\int_0^T \big(u''v'' + au'v' + b(t)uv\big)\,dt + \sum_{j=1}^n \big(g_j\big(u'(t_j)\big)v'(t_j) + h_j\big(u(t_j)\big)v(t_j)\big) \\ &\quad = \int_0^T c(t)|u|^{p-2}uv\,dt. \end{aligned} \tag{11}$$

As a consequence, the critical points of ϕ are the weak solutions of the problem (P). Let us see that they are, actually, strong solutions too.

Lemma 7 *If u is a weak solution of* (P) *then $u \in X$ is classical solution of* (P).

Proof Let $u \in X$ be a weak solution of (P), *i.e.* (11) holds for any $v \in X$. For a fixed $j \in \{0, 1, \dots, n\}$ we take a test function w_j, such that $w_j(t) = 0$ for $t \in [0, t_j] \cup [t_{j+1}, T]$. We have by (11)

$$\int_{t_j}^{t_{j+1}} \big(u''w_j'' + au'w_j' + b(t)uw_j\big)\,dt = \int_{t_j}^{t_{j+1}} c(t)|u|^{p-2}uw_j\,dt.$$

This means that for every $w \in X_j = H^2(t_j, t_{j+1}) \cap H_0^1(t_j, t_{j+1}) \subset C^1([t_j, t_{j+1}])$

$$\int_{t_j}^{t_{j+1}} \big(u''w'' + au'w' + b(t)uw\big)\,dt = \int_{t_j}^{t_{j+1}} c(t)|u|^{p-2}uw\,dt$$

and $u_j = u|_{(t_j, t_{j+1})}$ satisfies the equation

$$u^{(4)} - au'' + b(t)u = c(t)|u|^{p-2}u, \quad \text{a.e. } t \in (t_j, t_{j+1}).$$

By a standard regularity argument (see [9, 11]) the weak derivative $u_j^{(4)} \in L^2(t_j, t_{j+1})$ and therefore the limits $u''(t_j^{\pm}) = \lim_{t \to t_j^{\pm}} u''(t)$ and $u'''(t_j^{\pm}) = \lim_{t \to t_j^{\pm}} u'''(t)$ exist.

We have for $v \in X$

$$\int_{t_j}^{t_{j+1}} \big(u''v'' + au'v'\big)\,dt = u''v'\big|_{t_j}^{t_{j+1}} - u'''v\big|_{t_j}^{t_{j+1}} + \int_{t_j}^{t_{j+1}} \big(u^{(4)} - au''\big)v\,dt.$$

Summing the last identities for $j = 0, \dots, n$ we obtain

$$\begin{aligned} &\int_0^T \big(u''v'' + au'v'\big)\,dt - \int_0^T \big(u^{(4)} - au''\big)v\,dt \\ &\quad = -u''(0)v'(0) + u''(T)v'(T) - \sum_{j=1}^n \Delta u''(t_j)v'(t_j) + \sum_{j=1}^n \Delta u'''(t_j)v(t_j). \end{aligned} \tag{12}$$

Therefore, by (11) and (12), we have

$$\begin{aligned} 0 &= -u''(0)v'(0) + u''(T)v'(T) + \sum_{j=1}^n \big(g_j\big(u'(t_j)\big) - \Delta u''(t_j)\big)v'(t_j) \\ &\quad + \sum_{j=1}^n \big(h_j\big(u(t_j)\big) + \Delta u'''(t_j)\big)v(t_j). \end{aligned} \tag{13}$$

Now, take a test function $v = v_j$, $j = 0,\dots,n+1$, such that

$$v_j(t_k) = 0, \quad k = 0,1,\dots,n+1,$$
$$v_j'(t_k) = 0, \quad k = 0,1,\dots,j-1,j+1,\dots n+1,$$
$$v_j'(t_j) = 1.$$

Then we obtain $g_j(u'(t_j)) = \Delta u''(t_j)$ and $u''(0) = u''(T) = 0$. Similarly, we prove that $h_j(u'(t_j)) = -\Delta u'''(t_j)$, which shows that u is a classical solution of the problem (P). The lemma is proved. □

In the proofs of the theorems, we will use three critical point theorems which are the main tools to obtain weak solutions of the considered problems.

To this end, we introduce classical notations and results. Let E be a reflexive real Banach space. Recall that a functional $I : E \to \mathbb{R}$ is lower semi-continuous (resp. weakly lower semi-continuous (w.l.s.c.)) if $u_k \to u$ (resp. $u_k \rightharpoonup u$) in E implies $\liminf_{k\to\infty} I(u_k) \geq I(u)$ (see [12], pp.3-5).

We have the following well-known minimization result.

Theorem 8 *Let I be a weakly lower semi-continuous operator that has a bounded minimizing sequence on a reflexive real Banach space E. Then I has a minimum $c = \min_{u\in E} I(u) = I(u_0)$. If $I : E \to \mathbb{R}$ is a differentiable functional, u_0 is a critical point of I.*

Note that a functional $I : E \to \mathbb{R}$ is w.l.s.c. on I if $I(u) = I_1(u) + I_2(u)$, I_1 is convex and continuous and I_2 is sequentially weakly continuous (*i.e.* $u_k \rightharpoonup u$ in E implies $\lim_{k\to\infty} I_2(u_k) = I_2(u)$) (see [13], pp.301-303). The existence of a bounded minimizing sequence appears, when the functional I is coercive, *i.e.* $I(u) \to +\infty$ as $\|u\| \to +\infty$.

Next, recall the notion of the Palais-Smale (PS) condition, the mountain-pass theorem and Clarke's theorem.

We say that I satisfies condition (PS) if any sequence $(u_k) \subset E$ for which $I(u_k)$ is bounded and $I'(u_k) \to 0$ as $k \to \infty$ possesses a convergent subsequence.

Theorem 9 ([14, p.4]) *Let E be a real Banach space and $I \in C^1(E,\mathbb{R})$ satisfying condition* (PS). *Suppose $I(0) = 0$ and*

(i) *there are constants $\rho, \alpha > 0$ such that $I(u) \geq \alpha$ if $\|u\| = \rho$,*
(ii) *there is an $e \in E$, $\|e\| > \rho$ such that $I(e) \leq 0$.*

Then I possesses a critical value $c \geq \alpha$. Moreover, c can be characterized as $c = \inf\{\max\{I(u) : u \in \gamma([0,1])\} : \gamma \in \Gamma\}$ where $\Gamma = \{\gamma \in C([0,1],E) : \gamma(0) = 0, \gamma(1) = e\}$.

Theorem 10 ([14, p.53]) *Let E be a real Banach space and $I \in C^1(E,\mathbb{R})$ with I even, bounded from below, and satisfying condition* (PS). *Suppose that $I(0) = 0$, there is a set $K \subset E$ such that K is homeomorphic to $\mathbb{S}^{m-1}$ by an odd map, and $\sup\{I(u) : u \in K\} < 0$.*

Then I possesses, at least, m distinct pairs of critical points.

3 Proofs of main results

This section is devoted to the proof of the three theorems enunciated in the introduction of this work.

First consider the case $p > 2$ for which we prove that the functional ϕ satisfies the Palais-Smale condition.

Lemma 11 *Suppose that* $p > 2$ *and conditions* (H1) *and* (H2) *hold. Then the functional* $\phi : X \to \mathbb{R}$ *satisfies condition* (PS).

Proof Let $(u_k) \subset X$ and $C > 0$ be such that

$$\left|\phi(u_k)\right| \leq C, \qquad \phi'(u_k) \to 0, \quad k \to \infty.$$

Then we have

$$C \geq \frac{1}{2}\|u_k\|_X^2 + \sum_{j=1}^{n}\bigl(G_j\bigl(u_k'(t_j)\bigr) + H_j\bigl(u_k(t_j)\bigr)\bigr) - \frac{1}{p}\int_0^T c(t)\left|u_k(t)\right|^p dt \tag{14}$$

and

$$\left|\bigl\langle\phi'(u_k), v\bigr\rangle\right| \leq \|v\|_X, \quad \forall v \in X, \tag{15}$$

for all sufficiently large k, $k > N$. Taking $v = u_k$ in (15), we have for $k > N$

$$\left|\|u_k\|_X^2 + \sum_{j=1}^{n}\bigl(g_j\bigl(u_k'(t_j)\bigr)u_k'(t_j) + h_j\bigl(u_k(t_j)\bigr)u_k(t_j)\bigr) - \int_0^T c(t)\left|u_k(t)\right|^p dt\right| \leq \|u_k\|_X.$$

In particular,

$$\frac{\|u_k\|_X}{p} \geq -\frac{\|u_k\|_X^2}{p} - \frac{1}{p}\sum_{j=1}^{n}\bigl(g_j\bigl(u_k'(t_j)\bigr)u_k'(t_j) + h_j\bigl(u_k(t_j)\bigr)u_k(t_j)\bigr) + \frac{1}{p}\int_0^T c(t)\left|u_k(t)\right|^p dt.$$

Adding the last inequality with (14), by assumption (H2), we obtain

$$C + \frac{\|u_k\|_X}{p} \geq \frac{p-2}{2p}\|u_k\|_X^2,$$

which implies that (u_k) is a bounded sequence in X.

Then, by the compact inclusion $X \subset C^1([0,T])$, it follows that, up to a subsequence, $u_k \rightharpoonup u$ weakly in X and $u_k \to u$ strongly in $C^1([0,T])$. As a consequence, from the inequality

$$\left|\bigl\langle\phi'(u_k) - \phi'(u), u_k - u\bigr\rangle\right| \leq \left\|\phi'(u_k)\right\|\|u_k - u\| + \left|\bigl\langle\phi'(u), u_k - u\bigr\rangle\right|$$

it follows that

$$\bigl\langle\phi'(u_k) - \phi'(u), u_k - u\bigr\rangle \to 0 \tag{16}$$

and

$$\sum_{j=1}^{n}\bigl(h_j\bigl(u_k(t_j)\bigr) - h_j\bigl(u(t_j)\bigr)\bigr)\bigl(u_k(t_j) - u(t_j)\bigr) \to 0,$$

$$\sum_{j=1}^{n}\bigl(g_j\bigl(u_k'(t_j)\bigr)-g_j\bigl(u'(t_j)\bigr)\bigr)\bigl(u_k'(t_j)-u'(t_j)\bigr)\to 0,$$

$$\int_0^T c(t)\bigl(|u_k|^{p-2}u_k-|u|^{p-2}u\bigr)(u_k-u)\,dt\to 0.$$

Then by (16) it follows that

$$\|u_k-u\|_X^2\to 0,$$

i.e., $u_k\to u$ strongly in X, which completes the proof. □

Now, we are in a position to prove the main results of this paper.

Proof of Theorem 2 We find by (H1) and (8) that the following inequalities are valid for every $u\in X$:

$$\begin{aligned}\phi(u)&\geq\frac{1}{2}\|u\|_X^2-\frac{c_2T}{p}\|u\|_\infty^p\\&\geq\frac{1}{2}\|u\|_X^2-\frac{c_2TM^p}{p}\|u\|_X^p.\end{aligned}$$

It is evident that this last expression is strictly positive when $\|u\|_X=\rho$, with ρ small enough. Next, let $u_0(t)=\sin(\frac{\pi t}{T})\in X$ and $u_\lambda(t)=\lambda u_0(t)$, with $\lambda>0$. Then, by (H2) and (3), we have

$$\phi(u_\lambda)\leq\frac{\lambda^2}{2}\|u_0\|_X^2+C\left(2n+\sum_{j=1}^{n}\bigl(\lambda^{\gamma_j}+\lambda^{\sigma_j}\bigr)\right)-\frac{\lambda^p}{p}\int_0^T c(t)|u_0|^p\,dt,$$

where $C=\max\{C_j,K_j:1\leq j\leq n\}$.

Since $p>2$, we conclude that $\phi(u_\lambda)<0$ for sufficiently large λ. According to the mountain-pass Theorem 9, together with Lemmas 11 and 7, we deduce that there exists a nonzero classical solution of the problem (P). □

Now consider the case $1<p<2$. In the next result we prove that the Palais-Smale condition is also valid.

Lemma 12 *Suppose that* $1<p<2$ *and conditions* (H1) *and* (H3) *hold*. *Then the functional* $\phi:X\to\mathbb{R}$ *is bounded from below and satisfies condition* (PS).

Proof By $1<p<2$, conditions (H1), (H3), and inequality (8), it follows that the functional ϕ is bounded from below:

$$\begin{aligned}\phi(u)&\geq\frac{1}{2}\|u\|_X^2-\frac{c_2T}{p}\|u\|_\infty^p\geq\frac{1}{2}\|u\|_X^2-\frac{c_2TM^p}{p}\|u\|_X^p\\&\geq\frac{p-2}{2p}\bigl(c_2TM^p\bigr)^{2/(2-p)}.\end{aligned}\tag{17}$$

Further, if (u_k) is a (PS) sequence, by (17) it follows that (u_k) is a bounded sequence in X. Then, as in Lemma 11, we conclude that (u_k) has a convergent subsequence. □

Now we are in a position to prove the next existence result for the problem (P).

Proof of Theorem 3 By assumption, we know that g_j and h_j are odd functions. So G_j are H_j are even functions and the functional ϕ is even. By Lemma 12 we know that ϕ is bounded from below and satisfies condition (PS). Let $m \in \mathbb{N}$, $m \geq 3$ be a natural number and define, for any $\rho > 0$ fixed, the set

$$K_\rho^m = \left\{ \sum_{j=1}^m \lambda_j \sin\left(\frac{j\pi t}{T}\right) : \sum_{j=1}^m \lambda_j^2 = \rho^2 \right\} \subset X.$$

K_ρ^m is homeomorphic to $\mathbb{S}^{m-1}$ by the odd mapping defined as $H : K_\rho^m \to \mathbb{S}^{m-1}$

$$H\left(\sum_{j=1}^m \lambda_j \sin\left(\frac{j\pi t}{T}\right)\right) = \left(-\frac{\lambda_1}{\rho}, -\frac{\lambda_2}{\rho}, \dots, -\frac{\lambda_m}{\rho}\right).$$

Moreover, for $w = \sum_{j=1}^m \lambda_j \sin(\frac{j\pi t}{T}) \in K_\rho^m$, the following inequalities hold:

$$\rho\sqrt{\frac{Tb_1}{2}} \leq \|w\|_X \leq \rho m \sqrt{\frac{T}{2}\left(\left(\frac{m\pi}{T}\right)^4 + a\left(\frac{m\pi}{T}\right)^2 + b_2\right)}. \tag{18}$$

Clearly K_ρ^m is a subset of the m-dimensional subspace

$$X_m = sp\left\{ \sin\left(\frac{\pi t}{T}\right), \dots, \sin\left(\frac{m\pi t}{T}\right) \right\} \subset X,$$

and there exist positive constants $C_1(m)$ and $C_2(m)$, such that

$$C_1(m)\|w\|_{X_m} \leq \|w\|_{L^p} \leq C_2(m)\|w\|_{X_m}, \tag{19}$$

where $\|\cdot\|_{X_m}$ is the induced norm of $\|\cdot\|_X$ on X_m.

Arguing as in [15, pp.16-18], one can prove that there exists $\varepsilon = \varepsilon(m) > 0$, such that

$$\operatorname{meas}\left\{ t \in [0,T] : c(t)\left|u(t)\right|^p \geq \varepsilon\|u\|_X^p, u \in X_m \setminus \{0\} \right\} \geq \varepsilon. \tag{20}$$

Denote

$$\Omega_u = \left\{ t \in [0,T] : c(t)\left|u(t)\right|^p \geq \varepsilon\|u\|_X^p \right\}.$$

By (H3) we see that for every $w \in K_\rho^m$, $w = \sum_{k=1}^m \lambda_k \sin(\frac{k\pi t}{T})$, the following inequalities are fulfilled:

$$\begin{aligned} \left|G_j\left(w'(t_j)\right)\right| &\leq A_j\left|w'(t_j)\right|^2 \leq A_j\left(\sum_{k=1}^m \frac{k\pi|\lambda_k|}{T}\left|\cos\left(\frac{k\pi t_j}{T}\right)\right|\right)^2 \\ &\leq A_j\left(\frac{m\pi}{T}\right)^2\left(\sum_{k=1}^m |\lambda_k|\right)^2 \leq A_j m^3\left(\frac{\pi}{T}\right)^2\left(\sum_{k=1}^m \lambda_k^2\right) \end{aligned} \tag{21}$$

and

$$\begin{aligned} H_j\bigl(w(t_j)\bigr) &\le B_j\bigl|w(t_j)\bigr|^2 \le B_j\left(\sum_{k=1}^m |\lambda_k|\left|\sin\left(\frac{k\pi t_j}{T}\right)\right|\right)^2 \\ &\le B_j\left(\sum_{k=1}^m |\lambda_k|\right)^2 \le B_j m\left(\sum_{k=1}^m \lambda_k^2\right). \end{aligned}$$

Denote $C = \max\{A_j m^3(\frac{\pi}{T})^2, B_j m : 1 \le j \le n\}$. Then by (18)-(21) we have

$$\begin{aligned} \phi(w) &= \frac{1}{2}\|w\|_X^2 + \sum_{j=1}^n \bigl(G_j\bigl(w'(t_j)\bigr) + H_j\bigl(w(t_j)\bigr)\bigr) - \frac{1}{p}\int_0^T c(t)|w|^p\,dt \\ &\le \frac{1}{2}\|w\|_{X_m}^2 + 2nC\sum_{j=1}^m \lambda_j^2 - \frac{\varepsilon}{p}\|w\|_{L^p}^p \operatorname{meas}\Omega_w \\ &= \frac{1}{2}\|w\|_{X_m}^2 + 2nC\rho^2 - \frac{\varepsilon}{p}\|w\|_{L^p}^p \operatorname{meas}\Omega_w \\ &\le \left(\frac{1}{2} + \frac{4nC}{Tb_1}\right)\|w\|_{X_m}^2 - \frac{\varepsilon^2}{p}\|w\|_{L^p}^p \\ &\le K\|w\|_{X_m}^2 - \frac{\varepsilon^2 C_1^p(m)}{p}\|w\|_{X_m}^p, \end{aligned}$$

where $K = \frac{1}{2} + \frac{4nC}{Tb_1}$.

By the last inequality, it follows that $\phi(w) < 0$ if $\|w\|_{X_m} < (\frac{\varepsilon^2 C_1^p(m)}{pK})^{1/(2-p)}$. Then, by (18), choosing

$$\rho < \left(\frac{\varepsilon^2 C_1^p(m)}{pK}\right)^{1/(2-p)}\left(m\sqrt{\frac{T}{2}\left(\left(\frac{m\pi}{T}\right)^4 + a\left(\frac{m\pi}{T}\right)^2 + b_2\right)}\right)^{-1}$$

we obtain $\phi(w) < 0$ for any $w \in K_\rho^m$.

By Clarke's Theorem 10, there exist at least m pairs of different critical points of the functional ϕ. Since m is arbitrary, there exist infinitely many solutions of the problem (P), which concludes the proof. □

Concerning the problem (P_1), one can introduce similarly the notions of classical and weak solutions. In this case it is not difficult to verify that the weak solutions are critical points of the functional $\phi_1 : X \to \mathbb{R}$ defined as

$$\begin{aligned} \phi_1(u) &= \frac{1}{2}\int_0^T \bigl(u''^2 + au'^2\bigr)\,dt - \frac{1}{2}\int_0^T b(t)u^2\,dt \\ &\quad + \sum_{j=1}^n \bigl(G_j\bigl(u'(t_j)\bigr) + H_j\bigl(u(t_j)\bigr)\bigr) + \frac{1}{p}\int_0^T c(t)|u|^p\,dt. \end{aligned} \tag{22}$$

Proof of Theorem 4 By the Poincaré inequalities (7) we find that $|||u|||^2 = \int_0^T (u''^2 + au'^2)\,dt$ is an equivalent norm to $\|\cdot\|_X$ in X and the functional $I_1(u) = \frac{1}{2}|||u|||^2$ is convex.

Since the functional

$$I_2(u) = -\frac{1}{2}\int_0^T b(t)u^2\,dt + \sum_{j=1}^n \bigl(G_j\bigl(u'(t_j)\bigr) + H_j\bigl(u(t_j)\bigr)\bigr) + \frac{1}{p}\int_0^T c(t)|u|^p\,dt$$

is sequentially weakly continuous, from the fact that the inclusion $X \subset C^1([0,T])$ is compact, we deduce that the functional $\phi_1 : X \to \mathbb{R}$ is weakly lower semi-continuous.

Next, let us see that $\phi_1 : X \to \mathbb{R}$ is bounded from below:

$$\begin{aligned}\phi_1(u) &\geq \frac{1}{2}|||u|||^2 + \int_0^T \frac{1}{p}c(t)|u|^p - \frac{1}{2}b(t)u^2\,dt \\ &\geq \frac{1}{2}|||u|||^2 + \int_0^T \left(\frac{1}{p}c_1|u|^p - \frac{1}{2}b_2u^2\right)dt \geq CT,\end{aligned} \tag{23}$$

where

$$C = \min\left\{\frac{1}{p}c_1t^p - \frac{1}{2}b_2t^2 : t \geq 0\right\} = \frac{2-p}{p}b_2\left(\frac{b_2}{c_1}\right)^{2/(p-2)}.$$

Then, by Theorem 8, there exists a minimizer of ϕ_1, which is a critical point of ϕ_1.

Let u be a weak solution of (P_1), *i.e.*, a critical point of ϕ_1. Then

$$\begin{aligned}&\int_0^T \bigl(u''^2 + au'^2 - b(t)u^2\bigr)\,dt \\ &\quad + \sum_{j=1}^n \bigl(g_j\bigl(u'(t_j)\bigr)u'(t_j) + h_j\bigl(u(t_j)\bigr)u(t_j)\bigr) + \int_0^T c(t)|u|^p\,dt = 0.\end{aligned} \tag{24}$$

If $0 < T \leq T_2 = \pi\sqrt{\frac{a+\sqrt{a^2+4b_2}}{2b_2}}$ then $(\frac{\pi}{T})^4 + a(\frac{\pi}{T})^2 - b_2 \geq 0$. Suppose that u is a nonzero solution and $0 < T \leq T_2$. By (H2′), (7), and (24) it follows that

$$\begin{aligned}0 > -\int_0^T c(t)|u|^p\,dt &= \int_0^T \bigl(u''^2 + au'^2 - b(t)u^2\bigr)\,dt \\ &\quad + \sum_{j=1}^n \bigl(g_j\bigl(u'(t_j)\bigr)u'(t_j) + h_j\bigl(u(t_j)\bigr)u(t_j)\bigr) \\ &\geq \int_0^T \left(\left(\frac{\pi}{T}\right)^4 + a\left(\frac{\pi}{T}\right)^2 - b_2\right)u^2\,dt \geq 0,\end{aligned}$$

which is a contradiction. Then, for $0 < T \leq T_2$, the problem (P_1) has only the zero solution.

Suppose now that $T \geq T_1 = \pi\sqrt{\frac{a+\sqrt{a^2+4b_1}}{2b_1}}$.

Take $u_\varepsilon(t) = \varepsilon\sin(\frac{\pi t}{T}) \in X$, $\varepsilon > 0$. Then

$$\phi_1(u_\varepsilon) \leq \frac{\varepsilon^2}{2}\left(\left(\frac{\pi}{T}\right)^4 + a\left(\frac{\pi}{T}\right)^2 - b_1\right) + D\left(\sum_{j=1}^n \bigl(\varepsilon^{\gamma_j} + \varepsilon^{\sigma_j}\bigr)\right) + \frac{c_2T}{p}\varepsilon^p, \tag{25}$$

where $D = \max\{d_j, e_j : 1 \leq j \leq n\}$. For $T > T_1 = \pi\sqrt{\frac{a+\sqrt{a^2+4b_1}}{2b_1}}$ it follows that $(\frac{\pi}{T})^4 + a(\frac{\pi}{T})^2 - b_1 < 0$. Then, since $\gamma_j, \sigma_j \in (2,p)$, by (25) it follows that $\phi_1(u_\varepsilon) < 0$ for sufficiently small

$\varepsilon > 0$. In consequence we show that $\min\{\phi_1(u) : u \in X\} < 0$. So we ensure the existence of a nonzero minimizer of ϕ_1, which completes the proof of Theorem 4. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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