Bravyi *Boundary Value Problems* 2014, 2014:119
http://www.boundaryvalueproblems.com/content/2014/1/119

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

On estimates of solutions of the periodic boundary value problem for first-order functional differential equations

Eugene Bravyi*

Dedicated to Professor Ivan Kiguradze

*Correspondence: bravyi@perm.ru
Scientific center 'Functional Differential Equations', Perm National Research Polytechnical University, Komsomol'sky pr. 29, Perm, 614990, Russia

Abstract
Inequalities for periodic solutions of first-order functional differential equations are obtained. These inequalities are best possible in a certain sense.
MSC: Primary 34K06; 34K10; 34K13

Keywords: functional differential equations; periodic solutions; periodic boundary value problem; estimates of solutions

1 Introduction

Periodic solutions of functional differential equations are important in different applications (see, for example, [1–4] and the references therein, and also works on the general theory of boundary value problems for functional differential equations [5–11]). Conditions for the solvability of first-order periodic problems are found in [12–23]. In [15, 16] the linear case is considered, and unimprovable sufficient conditions for the solvability of the periodic problem

$$\dot{x}(t) = \big(T^{+}x\big)(t) - \big(T^{-}x\big)(t) + f(t), \quad t \in [a,b], \tag{1}$$

$$x(a) = x(b), \tag{2}$$

are found in terms of the norms $\mathcal{T}^{+}$, $\mathcal{T}^{-}$ of linear positive functional operators T^{+}, T^{-} : $\mathbf{C} \to \mathbf{L}$:

$$\frac{\mathcal{T}^{-}}{1-\mathcal{T}^{-}} < \mathcal{T}^{+} < 2\big(1+\sqrt{1-\mathcal{T}^{-}}\big) \tag{3}$$

or

$$\frac{\mathcal{T}^{+}}{1-\mathcal{T}^{+}} < \mathcal{T}^{-} < 2\big(1+\sqrt{1-\mathcal{T}^{+}}\big). \tag{4}$$

If both of these conditions are not satisfied for some norms $\mathcal{T}^{+}$, $\mathcal{T}^{-}$, there exist linear positive operators T^{+}, T^{-} with these norms such that problem (1)-(2) has no solution. As to our knowledge, similar unimprovable estimates for solutions of (1)-(2) in terms of norms

$\mathcal{T}^+$, $\mathcal{T}^-$ are yet unknown. Here we will fill this gap. Moreover, the estimates obtained here (in Theorems 1, 2, 3) can be expanded to some non-linear functional differential equations (see Remark 1). Theorem 1 gives the best possible estimates of the norm of the Green operator for the periodic boundary value problem. In Theorem 2, we obtain unimprovable estimates of the solutions of (1)-(2) for non-negative f. In Theorem 3, unimprovable bounds of the difference between the maximum and the minimum of a solution are established.

We use the following notation: $\mathbb{R}$ is the space of real numbers, $\mathbf{C}$ is the space of continuous functions $x : [a,b] \to \mathbb{R}$ with the norm $\|x\|_{\mathbf{C}} = \max_{t\in[a,b]} |x(t)|$; $\mathbf{L}$ is the space of integrable functions $z : [a,b] \to \mathbb{R}$ with the norm $\|z\|_{\mathbf{L}} = \int_a^b |z(t)|\,dt$; a linear bounded operator $T : \mathbf{C} \to \mathbf{L}$ is called *positive* if it maps non-negative functions from $\mathbf{C}$ into almost everywhere non-negative functions from $\mathbf{L}$.

Consider the periodic boundary value problem (1)-(2), where $f \in \mathbf{L}$, $T^+, T^- : \mathbf{C} \to \mathbf{L}$ are linear positive operators with norms $\mathcal{T}^+ \equiv \|T^+\|_{\mathbf{C}\to\mathbf{L}} = \int_a^b (T^+\mathbf{1})(t)\,dt$, $\mathcal{T}^- \equiv \|T^-\|_{\mathbf{C}\to\mathbf{L}} = \int_a^b (T^-\mathbf{1})(t)\,dt$, $\mathbf{1}$ is the unit function. An absolutely continuous function $x : [a,b] \to \mathbb{R}$ is called *a solution* of the problem if it satisfies the periodic boundary condition (2) and equation (1) for almost all $t \in [a,b]$. We have to solve problem (1)-(2) if, for example, we search for periodic solutions of the equation with delay

$$\dot{x}(t) = p(t)x\bigl(t - \tau(t)\bigr) + f(t), \quad t \in \mathbb{R}, \tag{5}$$

where $p, f : \mathbb{R} \to \mathbb{R}$ are $(b-a)$-periodic locally integrable functions, $\tau : \mathbb{R} \to \mathbb{R}$ is a measurable $(b-a)$-periodic non-negative delay. Indeed, suppose that linear operators T^+ and T^- are defined by the equalities

$$\bigl(T^+x\bigr)(t) = \frac{p(t) + |p(t)|}{2}x\bigl(\tilde{\tau}(t)\bigr), \qquad \bigl(T^-x\bigr)(t) = \frac{|p(t)| - p(t)}{2}x\bigl(\tilde{\tau}(t)\bigr), \quad t \in [a,b],$$

where $\tilde{\tau}(t) = t - \tau(t) + k(t)(b-a)$ and the integer numbers $k(t)$ are such that $\tilde{\tau}(t) \in [a,b]$ for almost all $t \in \mathbb{R}$. It is easy to show that problem (1)-(2) has a solution if and only if equation (5) has a periodic solution with the period $b-a$.

The conditions (3), (4) for the norms of the operators $T^+, T^- : \mathbf{C} \to \mathbf{L}$ are well known [15]. They guarantee the existence and uniqueness of solutions of problem (1)-(2). Note that these conditions are unimprovable in the following sense: if non-negative numbers $\mathcal{T}^+$, $\mathcal{T}^-$ satisfy neither (3) nor (4), then problem (1)-(2) has no solution for some linear positive operators $T^+, T^- : \mathbf{C} \to \mathbf{L}$ with norms $\|T^+\|_{\mathbf{C}\to\mathbf{L}} = \mathcal{T}^+$, $\|T^-\|_{\mathbf{C}\to\mathbf{L}} = \mathcal{T}^-$ and for some $f \in \mathbf{L}$.

2 The main results

In what follows, we suppose that one of conditions (3), (4) is fulfilled. First, we formulate the results only for the simplest problem (1)-(2) with the null operator T^+:

$$\begin{aligned} &\dot{x}(t) = -\bigl(T^-x\bigr)(t) + f(t), \quad t \in [a,b], \\ &x(a) = x(b), \end{aligned} \tag{6}$$

where $T^- : \mathbf{C} \to \mathbf{L}$ is a linear positive operator with norm $\mathcal{T}^-$, $f \in \mathbf{L}$. The assertions of the following Theorems 1, 2, 3 for problem (6) are as follows.

The solution x of (6) satisfies the estimates

$$\max_{t\in[a,b]}\bigl|x(t)\bigr| \le \begin{cases} \frac{1+\mathcal{T}^-}{\mathcal{T}^-}\int_a^b |f(t)|\,dt & \text{if } 0<\mathcal{T}^-\le 3,\\ \frac{4}{\mathcal{T}^-(4-\mathcal{T}^-)}\int_a^b |f(t)|\,dt & \text{if } 3<\mathcal{T}^-<4,\end{cases} \tag{7}$$

$$\max_{t\in[a,b]} x(t) - \min_{t\in[a,b]} x(t) \le \begin{cases} \int_a^b |f(t)|\,dt & \text{if } 0<\mathcal{T}^-\le 1,\\ \frac{1}{2\sqrt{\mathcal{T}^-}-\mathcal{T}^-}\int_a^b |f(t)|\,dt & \text{if } 1<\mathcal{T}^-<4.\end{cases} \tag{8}$$

If a function f is non-negative, the solution x of (6) satisfies the estimates

$$\begin{aligned} &\frac{1-\mathcal{T}^-}{\mathcal{T}^-}\int_a^b f(t)\,dt \le x(t) \le \frac{1+\mathcal{T}^-}{\mathcal{T}^-}\int_a^b f(t)\,dt && \text{if } 0<\mathcal{T}^-\le 2,\\ &-\frac{1}{4-\mathcal{T}^-}\int_a^b f(t)\,dt \le x(t) \le \frac{1+\mathcal{T}^-}{\mathcal{T}^-}\int_a^b f(t)\,dt && \text{if } 2<\mathcal{T}^-\le 3,\\ &-\frac{1}{4-\mathcal{T}^-}\int_a^b f(t)\,dt \le x(t) \le \frac{4}{\mathcal{T}^-(4-\mathcal{T}^-)}\int_a^b f(t)\,dt && \text{if } 3<\mathcal{T}^-<4. \end{aligned} \tag{9}$$

All estimates (7), (8) and (9), which are proved in Theorems 1, 2, 3 in the general case, are best possible (see Remarks 3, 5, 6).

Remark 1 Consider also the non-linear periodic problem

$$\dot{x}(t) = \bigl(F^+x\bigr)(t) - \bigl(F^-x\bigr)(t) + f(t), \quad t\in[a,b], \tag{10}$$

$$x(a) = x(b), \tag{11}$$

provided there exist non-negative functions $p^+, p^- \in \mathbf{L}$ with norms

$$\bigl\|p^+\bigr\|_{\mathbf{L}} = \mathcal{T}^+, \qquad \bigl\|p^-\bigr\|_{\mathbf{L}} = \mathcal{T}^- \tag{12}$$

such that the operators $F^+, F^- : \mathbf{C}\to\mathbf{L}$ satisfy the inequalities

$$p^+(t)\min_{t\in[a,b]} x(t) \le \bigl(F^+x\bigr)(t) \le p^+(t)\max_{t\in[a,b]} x(t) \quad \text{for a.a. } t\in[a,b], \tag{13}$$

$$p^-(t)\min_{t\in[a,b]} x(t) \le \bigl(F^-x\bigr)(t) \le p^-(t)\max_{t\in[a,b]} x(t) \quad \text{for a.a. } t\in[a,b] \tag{14}$$

for all $x\in\mathbf{C}$.

It follows from Lemma 3 and the proofs of Theorems 1, 2, 3 that all statements of these theorems are also valid for solutions of periodic problem (10)-(11) (if the solutions exist).

Theorem 1 *If the norms $\mathcal{T}^+ < \mathcal{T}^-$ of the linear positive operators $T^+, T^- : \mathbf{C}\to\mathbf{L}$ satisfy the conditions*

$$3 \le \mathcal{T}^- < 2\bigl(1+\sqrt{1-\mathcal{T}^+}\bigr), \qquad \mathcal{T}^+ < 3/4, \tag{15}$$

and x is a solution of (1)-(2), *then the inequality*

$$\max_{t\in[a,b]}\bigl|x(t)\bigr| \le \frac{1}{\mathcal{T}^-(1-\mathcal{T}^-/4)-\mathcal{T}^+}\int_a^b \bigl|f(t)\bigr|\,dt \tag{16}$$

holds.

If the norms $\mathcal{T}^+ < \mathcal{T}^-$ *of the operators* $T^+, T^- : \mathbf{C} \to \mathbf{L}$ *satisfy*

$$\frac{\mathcal{T}^+}{1 - \mathcal{T}^+} < \mathcal{T}^- \leq 3, \qquad \mathcal{T}^+ < 3/4, \tag{17}$$

and x is a solution of problem (1)-(2), *then the inequality*

$$\max_{t \in [a,b]} |x(t)| \leq \frac{1 + \mathcal{T}^-}{\mathcal{T}^-(1 - \mathcal{T}^+) - \mathcal{T}^+} \int_a^b |f(t)| \, dt \tag{18}$$

holds.

Remark 2 ([15]) If $\mathcal{T}^- > \mathcal{T}^+ \geq 0$ and both of the conditions (15), (17) are not fulfilled, then there exist linear positive operators T^-, T^+ with norms $\mathcal{T}^-$, $\mathcal{T}^+$ and a function $f \in \mathbf{L}$ such that problem (1)-(2) has no solution.

Remark 3 From the proof of Theorem 1 it follows that estimates (16), (18) are best possible: if non-negative numbers $\mathcal{T}^-$, $\mathcal{T}^+$ satisfy (15) (or (17)), then equality holds in condition (16) (or (18)) for a unique solution x of problem (1)-(2) for some linear positive operators T^-, T^+ with norms $\mathcal{T}^-$, $\mathcal{T}^+$ and for some function $f \in \mathbf{L}$, $f \not\equiv 0$.

The estimates of solutions (1)-(2) for $\mathcal{T}^- < \mathcal{T}^+$ can be obtained in the same way.

Theorem 1* *If the norms* $\mathcal{T}^+ > \mathcal{T}^-$ *of the linear positive operators* $T^+, T^- : \mathbf{C} \to \mathbf{L}$ *satisfy the conditions*

$$3 \leq \mathcal{T}^+ < 2\bigl(1 + \sqrt{1 - \mathcal{T}^-}\bigr), \qquad \mathcal{T}^- < 3/4, \tag{19}$$

and x is a solution of (1)-(2), *then the inequality*

$$\max_{t \in [a,b]} |x(t)| \leq \frac{1}{\mathcal{T}^+(1 - \mathcal{T}^+/4) - \mathcal{T}^-} \int_a^b |f(t)| \, dt \tag{20}$$

holds.

If the norms $\mathcal{T}^+ > \mathcal{T}^-$ *of the operators* $T^+, T^- : \mathbf{C} \to \mathbf{L}$ *satisfy*

$$\frac{\mathcal{T}^-}{1 - \mathcal{T}^-} < \mathcal{T}^+ \leq 3, \qquad \mathcal{T}^- < 3/4, \tag{21}$$

and x is a solution of problem (1)-(2), *then the inequality*

$$\max_{t \in [a,b]} |x(t)| \leq \frac{1 + \mathcal{T}^+}{\mathcal{T}^+(1 - \mathcal{T}^-) - \mathcal{T}^-} \int_a^b |f(t)| \, dt \tag{22}$$

holds.

Remark 2* ([15]) If $\mathcal{T}^- > \mathcal{T}^+ \geq 0$ and both of conditions (19), (21) are not fulfilled, then there exist linear positive operators T^- and T^+ with norms $\mathcal{T}^-$, $\mathcal{T}^+$ and a function $f \in \mathbf{L}$ such that problem (1)-(2) has no solution.

Remark 3* From the proof of Theorem 1 it follows that estimates (20), (22) are best possible: if non-negative numbers $\mathcal{T}^-$, $\mathcal{T}^+$ satisfy (19) (or (21)), then equality holds in condition (20) (or (22)) for a unique solution x of problem (1)-(2) for some linear positive operators T^-, T^+ with norms $\mathcal{T}^-$, $\mathcal{T}^+$ and for some function $f \in \mathbf{L}$, $f \not\equiv 0$.

In the next statement we get the best possible lower bounds for solutions of problem (1)-(2) for non-negative f.

Theorem 2 *Let x be a solution of problem* (1)-(2) *for some non-negative f.*
If the norms $\mathcal{T}^+$, $\mathcal{T}^-$ of the operators $T^+, T^- : \mathbf{C} \to \mathbf{L}$ satisfy the conditions

$$\max\left\{1+\sqrt{1-\mathcal{T}^+}, \frac{\mathcal{T}^+}{1-\mathcal{T}^+}\right\} < \mathcal{T}^- < 2\bigl(1+\sqrt{1-\mathcal{T}^+}\bigr), \qquad \mathcal{T}^+ < 3/4, \tag{23}$$

then

$$\min_{t\in[a,b]} x(t) \ge -\frac{1}{2(1+\sqrt{1-\mathcal{T}^+}) - \mathcal{T}^-}\int_a^b f(t)\,dt; \tag{24}$$

if the norms $\mathcal{T}^+$, $\mathcal{T}^-$ of the operators $T^+, T^- : \mathbf{C} \to \mathbf{L}$ satisfy the conditions

$$\max\left\{1, \frac{\mathcal{T}^+}{1-\mathcal{T}^+}\right\} < \mathcal{T}^- \le 1+\sqrt{1-\mathcal{T}^+}, \tag{25}$$

then

$$\min_{t\in[a,b]} x(t) \ge -\frac{\mathcal{T}^- - 1}{\mathcal{T}^- - \mathcal{T}^+}\int_a^b f(t)\,dt; \tag{26}$$

if the norms $\mathcal{T}^+$, $\mathcal{T}^-$ of the operators $T^+, T^- : \mathbf{C} \to \mathbf{L}$ satisfy the conditions

$$\frac{\mathcal{T}^+}{1-\mathcal{T}^+} < \mathcal{T}^- \le 1, \qquad \mathcal{T}^+ < 1/2, \tag{27}$$

then

$$\min_{t\in[a,b]} x(t) \ge \frac{1-\mathcal{T}^-}{\mathcal{T}^-(1+\mathcal{T}^+) - \mathcal{T}^+}\int_a^b f(t)\,dt. \tag{28}$$

Remark 4 ([15]) If $\mathcal{T}^- > \mathcal{T}^+$ and all of conditions (23), (25), (27) are not fulfilled, then there exist linear positive operators T^- and T^+ with norms $\mathcal{T}^-$, $\mathcal{T}^+$ and a function $f \in \mathbf{L}$ such that problem (1)-(2) has no solution.

Remark 5 From the proof of Theorem 2 it follows that estimates (24), (26), (28) are best possible: if non-negative numbers $\mathcal{T}^+$, $\mathcal{T}^-$ satisfy (23) ((25) or (27)), then equality holds in condition (24) ((26) or (28)) for a unique solution x of problem (1)-(2) for some linear positive operators T^-, T^+ with norms $\mathcal{T}^-$, $\mathcal{T}^+$ and for some function $f \in \mathbf{L}$, $f \not\equiv 0$.

Now we estimate the difference between the maximum and the minimum of solutions.

Theorem 3 *Let the solvability conditions* (4) *be fulfilled and x be a unique solution of* (1)-(2). *If*

$$\mathcal{T}^- > 1, \qquad \mathcal{T}^+ < \mathcal{T}^- \left(\frac{\mathcal{T}^- - 1}{\mathcal{T}^- + 1}\right)^2,$$

then

$$\max_{t\in[a,b]} x(t) - \min_{t\in[a,b]} x(t) \le \frac{1}{2\sqrt{\mathcal{T}^- - \mathcal{T}^+} - \mathcal{T}^-} \int_a^b \left|f(s)\right| ds; \tag{29}$$

otherwise

$$\max_{t\in[a,b]} x(t) - \min_{t\in[a,b]} x(t) \le \frac{\mathcal{T}^-}{\mathcal{T}^- - \mathcal{T}^+ - \mathcal{T}^-\mathcal{T}^+} \int_a^b \left|f(s)\right| ds. \tag{30}$$

Remark 6 From the proof of Theorem 3 it follows that inequalities (29) and (30) are unimprovable. It means that for every number $\mathcal{T}^+$, $\mathcal{T}^-$ satisfying the conditions of the theorem, equality holds in conditions (29) or (30) for the solution x of problem (1)-(2) for some positive operators $T^+, T^- : \mathbf{C} \to \mathbf{L}$ with norms $\mathcal{T}^-$, $\mathcal{T}^+$, and for some non-negative function $f \in \mathbf{L}, f \not\equiv 0$.

Remark 7 Theorems 2, 3, as Theorem 1, can be easily reformulated for the case $\mathcal{T}^+ > \mathcal{T}^-$ when the solvability condition (3) holds.

3 Proofs

We need three lemmas to prove the main theorems.

Lemma 1 *Let $T^+, T^- : \mathbf{C} \to \mathbf{L}$ be linear positive operators, $p^+ = T^+\mathbf{1}$, $p^- = T^-\mathbf{1}$, $y \in \mathbf{C}$. Then there exist points $t_1, t_2 \in [a,b]$ and a function $p_1 \in \mathbf{L}$ satisfying*

$$-p^-(t) \le p_1(t) \le p^+(t) \quad \textit{for a.a. } t \in [a,b] \tag{31}$$

such that the equality

$$\left(T^+y\right)(t) - \left(T^-y\right)(t) = p_1(t)y(t_1) + \left(p^+(t) - p^-(t) - p_1(t)\right)y(t_2) \quad \textit{for a.a. } t \in [a,b] \tag{32}$$

holds.

Proof Let $y(t_1) = \max_{t\in[a,b]} y(t)$, $y(t_2) = \min_{t\in[a,b]} y(t)$. Since $y \in \mathbf{C}$ and the linear operators $T^+, T^- : \mathbf{C} \to \mathbf{L}$ are positive, we have

$$p^+(t)y(t_2) - p^-(t)y(t_1) \le \left(T^+y\right)(t) - \left(T^-y\right)(t) \le p^+(t)y(t_1) - p^-(t)y(t_2) \quad \text{for a.a. } t \in [a,b].$$

Therefore, for some function $p_1 \in \mathbf{L}$ satisfying (31), equality (32) holds. $\square$

Lemma 2 *If $y \in \mathbf{C}$, functions $p^+, p^- \in \mathbf{L}$ are non-negative, and $p_1 \in \mathbf{L}$ satisfies* (31), *then there exist linear positive operators $T^+, T^- : \mathbf{C} \to \mathbf{L}$ with the norms*

$$\|T^+\|_{\mathbf{C}\to\mathbf{L}} = \|p^+\|_{\mathbf{L}}, \qquad \|T^-\|_{\mathbf{C}\to\mathbf{L}} = \|p^-\|_{\mathbf{L}} \tag{33}$$

such that equality (32) *holds.*

Proof Let $p_1^+(t) = (|p_1(t)| + p_1(t))/2$, $p_1^-(t) = (|p_1(t)| - p_1(t))/2$, $t \in [a,b]$. Then the operators T^+, T^- defined by the equalities

$$\begin{aligned}
\big(T^+x\big)(t) &= p_1^+(t)x(t_1) + \big(p^+(t) - p_1^+(t)\big)x(t_2), \quad t \in [a,b],\\
\big(T^-x\big)(t) &= p_1^-(t)x(t_1) + \big(p^-(t) - p_1^-(t)\big)x(t_2), \quad t \in [a,b],
\end{aligned}$$

satisfy the conditions of the lemma. □

Lemma 3 *Let $F^+, F^- : \mathbf{C} \to \mathbf{L}$ satisfy* (13)-(14), $y \in \mathbf{C}$. *Then there exist a function $p_1 \in \mathbf{L}$ satisfying* (31) *and points $t_1, t_2 \in [a,b]$ such that the equality*

$$\big(F^+y\big)(t) - \big(F^-y\big)(t) = p_1(t)y(t_1) + \big(p^+(t) - p^-(t) - p_1(t)\big)y(t_2) \quad \textit{for a.a. } t \in [a,b] \tag{34}$$

holds.

Proof Let $y(t_1) = \max_{t\in[a,b]} y(t)$, $y(t_2) = \min_{t\in[a,b]} y(t)$. Since $y \in \mathbf{C}$ and using (13), (14), we get

$$p^+(t)y(t_2) - p^-(t)y(t_1) \le \big(F^+y\big)(t) - \big(F^-y\big)(t) \le p^+(t)y(t_1) - p^-(t)y(t_2) \quad \text{for a.a. } t \in [a,b].$$

Therefore, for some function $p_1 \in \mathbf{L}$ satisfying (31), equality (34) holds. □

Remark 8 It is obvious that one can choose the points t_1 and t_2 in Lemmas 1 and 3 in such a way that the solution y takes its maximum and minimum at these points.

Proofs of Theorems 1, 2, 3 If x is a solution of problem (1)-(2) ((10)-(11)), then by Lemma 1 (3) this solution satisfies the boundary value problem

$$\dot{x}(t) = p_1(t)x(t_1) + \big(p^+(t) - p^-(t) - p_1(t)\big)x(t_2) + f(t), \quad t \in [a,b], \tag{35}$$

$$x(a) = x(b), \tag{36}$$

where $p_1 \in \mathbf{L}$ and non-negative $p^+, p^- \in \mathbf{L}$ satisfy (31), (33). If condition (3) or (4) holds, then problem (35)-(36) has a unique solution, which can be easily found explicitly. Since we are only interested in the maximal and minimal values of the solutions, by Remark 8, we have to obtain only representations for values $x(t_1)$ and $x(t_2)$.

Let $a \le t_1 < t_2 \le b$, $E \equiv [t_1, t_2]$, $I \equiv [a, t_1] \cup [t_2, b]$,

$$\begin{aligned}
\triangle \equiv {} & \int_I p_1(s)\,ds \int_E \big(p^+(s) - p^-(s)\big)\,ds - \int_E p_1(s)\,ds \int_I \big(p^+(s) - p^-(s)\big)\,ds\\
& - \int_a^b \big(p^+(s) - p^-(s)\big)\,ds.
\end{aligned}$$

For $x(t_1)$, $x(t_2)$ we have

$$\begin{aligned} x(t_1) &= \frac{1}{\triangle}\biggl(-\int_I f(s)\,ds\int_E \bigl(p^+(s)-p^-(s)-p_1(s)\bigr)\,ds \\ &\quad + \int_E f(s)\,ds\int_I \bigl(p^+(s)-p^-(s)-p_1(s)\bigr)\,ds + \int_a^b f(s)\,ds\biggr), \end{aligned} \tag{37}$$

$$x(t_2) = \frac{1}{\triangle}\biggl(\int_I f(s)\,ds\int_E p_1(s)\,ds - \int_E f(s)\,ds\int_I p_1(s)\,ds + \int_a^b f(s)\,ds\biggr) \tag{38}$$

and

$$\begin{aligned} x(t_1)-x(t_2) &= \frac{1}{\triangle}\biggl(-\int_I f(s)\,ds\int_E \bigl(p^+(s)-p^-(s)\bigr)\,ds \\ &\quad + \int_E f(s)\,ds\int_I \bigl(p^+(s)-p^-(s)\bigr)\,ds\biggr). \end{aligned} \tag{39}$$

Suppose here that $\mathcal{T}^- > \mathcal{T}^+$ and condition (4) is fulfilled.

Define by P the set of all solutions of problem (35)-(36) for all $a \le t_1 < t_2 \le b$, for all functions $p_1 \in \mathbf{L}$ and non-negative $p^+, p^- \in \mathbf{L}$ such that conditions (12), (31) hold, and for all $f \in \mathbf{L}$ with $\|f\|_{\mathbf{L}} = 1$.

Let S be the subset of P corresponding to non-negative functions f.

From Lemmas 1 and 2, it follows that the set P coincides with the set of all solutions of problem (1)-(2) for all linear positive operators $T^-, T^+ : \mathbf{C} \to \mathbf{L}$ with norms $\|T^+\|_{\mathbf{C}\to\mathbf{L}} = \mathcal{T}^+$, $\|T^-\|_{\mathbf{C}\to\mathbf{L}} = \mathcal{T}^-$ and for all $f \in \mathbf{L}$ with $\|f\|_{\mathbf{L}} = 1$. The subset S consists of all solutions of corresponding problems (1)-(2) with non-negative f.

Define the constants

$$M_1 \equiv \max_{x\in P,t\in[a,b]} \bigl|x(t)\bigr|, \qquad M_2 \equiv \max_{x\in P}\Bigl(\max_{t\in[a,b]} x(t) - \min_{t\in[a,b]} x(t)\Bigr),$$

$$N_1 \equiv \max_{x\in S,t\in[a,b]} x(t), \qquad N_2 \equiv \max_{x\in S}\Bigl(\max_{t\in[a,b]} x(t) - \min_{t\in[a,b]} x(t)\Bigr), \qquad N_3 \equiv \min_{x\in S,t\in[a,b]} x(t).$$

From representations (37), (38), (39), it easily follows that all the constants are defined correctly and

$$M_1 = \max\bigl\{|N_1|, |N_3|\bigr\}, \qquad M_2 = N_2.$$

Moreover, for every solution x of (1)-(2), the following inequalities hold:

$$\bigl|x(t)\bigr| \le M_1 \int_a^b \bigl|f(s)\bigr|\,ds, \quad t \in [a,b],$$

$$\max_{t\in[a,b]} x(t) - \min_{t\in[a,b]} x(t) \le N_2 \int_a^b \bigl|f(s)\bigr|\,ds.$$

If $f \in \mathbf{L}$ is non-negative, then

$$N_3 \int_a^b f(s)\,ds \le x(t) \le N_1 \int_a^b f(s)\,ds, \quad t \in [a,b],$$

where the constants N_1, N_2, N_3, M_1 are best possible.

It remains to find N_1, N_2, N_3.

The numerator and denominator of fractions in (37), (38), (39) are linear with respect to variables $\int_E p_1(s)\,ds$ and $\int_I p_1(s)\,ds$. Therefore $x(t_1)$, $x(t_2)$, and $x(t_1)-x(t_2)$ take their minimal and maximal values at the bounds of restriction (31) with respect to variables p_1 on each of the sets E and I. Hence we have to consider only the following four different cases:

(i) $p_1(t) = \begin{cases} p^+(t) & \text{if } t \in E, \\ -p^-(t) & \text{if } t \in I, \end{cases}$

(ii) $p_1(t) = \begin{cases} -p^-(t) & \text{if } t \in E, \\ p^+(t) & \text{if } t \in I, \end{cases}$

(iii) $p_1 = p^+$,

(iv) $p_1 = -p^-$.

In case (i) we have

$$x(t_1) = \frac{1}{\triangle}\left(\int_I f(s)\,ds\left(\int_E p^-(s)\,ds + 1\right) + \int_E f(s)\,ds\left(\int_I p^+\,ds + 1\right)\right),$$

$$x(t_2) = \frac{1}{\triangle}\left(\int_I f(s)\,ds\left(\int_E p^+(s)\,ds + 1\right) + \int_E f(s)\,ds\left(\int_I p^-\,ds + 1\right)\right),$$

$$\begin{aligned} x(t_1) - x(t_2) = \frac{1}{\triangle}\Bigg(&\int_I f(s)\,ds\left(\int_E \big(p^-(s) - p^+(s)\big)\,ds\right) \\ &- \int_E f(s)\,ds\left(\int_I \big(p^-(s) - p^+(s)\big)\,ds\right)\Bigg), \end{aligned}$$

$$\triangle = \int_I p^-(s)\,ds \int_E p^-(s)\,ds - \int_I p^+(s)\,ds \int_E p^+(s)\,ds + \mathcal{T}^- - \mathcal{T}^+.$$

In case (ii) we have

$$x(t_1) = \frac{1}{\triangle}\left(\int_I f(s)\,ds\left(-\int_E p^+(s)\,ds + 1\right) + \int_E f(s)\,ds\left(-\int_I p^-(s)\,ds + 1\right)\right),$$

$$x(t_2) = \frac{1}{\triangle}\left(\int_I f(s)\,ds\left(-\int_E p^-(s)\,ds + 1\right) + \int_E f(s)\,ds\left(-\int_I p^+(s)\,ds + 1\right)\right),$$

$$\begin{aligned} x(t_1) - x(t_2) = \frac{1}{\triangle}\Bigg(&\int_I f(s)\,ds\left(\int_E \big(p^-(s) - p^+(s)\big)\,ds\right) \\ &- \int_E f(s)\,ds\left(\int_I \big(p^-(s) - p^+(s)\big)\,ds\right)\Bigg), \end{aligned}$$

$$\triangle = \int_I p^+(s)\,ds \int_E p^+(s)\,ds - \int_I p^-(s)\,ds \int_E p^-(s)\,ds + \mathcal{T}^- - \mathcal{T}^+.$$

In case (iii) we have

$$x(t_1) = \frac{1}{\triangle}\left(\int_I f(s)\,ds\left(\int_E p^-(s)\,ds + 1\right) + \int_E f(s)\,ds\left(-\int_I p^-(s)\,ds + 1\right)\right),$$

$$x(t_2) = \frac{1}{\triangle}\left(\int_I f(s)\,ds\left(\int_E p^+(s)\,ds + 1\right) + \int_E f(s)\,ds\left(-\int_I p^+(s)\,ds + 1\right)\right),$$

$$\begin{aligned} x(t_1) - x(t_2) = \frac{1}{\triangle}\Bigg(&\int_I f(s)\,ds\left(\int_E \big(p^-(s) - p^+(s)\big)\,ds\right) \\ &- \int_E f(s)\,ds\left(\int_I \big(p^-(s) - p^+(s)\big)\,ds\right)\Bigg), \end{aligned}$$

$$\triangle = -\int_I p^+(s)\,ds \int_E p^-(s)\,ds + \int_I p^-(s)\,ds \int_E p^+(s)\,ds + \mathcal{T}^- - \mathcal{T}^+.$$

In case (iv) we have

$$\begin{aligned}
x(t_1) &= \frac{1}{\triangle}\left(\int_I f(s)\,ds\left(-\int_E p^+(s)\,ds+1\right)+\int_E f(s)\,ds\left(\int_I p^+(s)\,ds+1\right)\right),\\
x(t_2) &= \frac{1}{\triangle}\left(\int_I f(s)\,ds\left(-\int_E p^-(s)\,ds+1\right)+\int_E f(s)\,ds\left(\int_I p^-(s)\,ds+1\right)\right),\\
x(t_1)-x(t_2) &= \frac{1}{\triangle}\left(\int_I f(s)\,ds\left(\int_E \bigl(p^-(s)-p^+(s)\bigr)\,ds\right)-\int_E f(s)\,ds\left(\int_I \bigl(p^-(s)-p^+(s)\bigr)\,ds\right)\right),\\
\triangle &= \int_I p^+(s)\,ds\int_E p^-(s)\,ds-\int_I p^-(s)\,ds\int_E p^+(s)\,ds+\mathcal{T}^- - \mathcal{T}^+.
\end{aligned}$$

Let $S_{(\mathrm{i})}$, $S_{(\mathrm{ii})}$, $S_{(\mathrm{iii})}$, $S_{(\mathrm{iv})}$ be the subsets of S for p_1 corresponding to cases (i), (ii), (iii), (iv). We can easily calculate the minimal and maximal values in every case.

In case (iv) we have

$$\begin{aligned}
&\max_{x\in S_{(\mathrm{iv})}}\bigl\{x(t_1),x(t_2)\bigr\} = \frac{\mathcal{T}^- +1}{\mathcal{T}^- - \mathcal{T}^+ - \mathcal{T}^-\mathcal{T}^+},\\
&\min_{x\in S_{(\mathrm{iv})}}\bigl\{x(t_1),x(t_2)\bigr\} = \begin{cases} \frac{1-\mathcal{T}^-}{\mathcal{T}^- - \mathcal{T}^+} & \text{if } \mathcal{T}^- > 1,\\ \frac{1-\mathcal{T}^-}{\mathcal{T}^- - \mathcal{T}^+ + \mathcal{T}^-\mathcal{T}^+} & \text{if } \mathcal{T}^- \le 1,\end{cases}\\
&\min_{x\in S_{(\mathrm{iv})}}\bigl(x(t_1)-x(t_2)\bigr) = \frac{-\mathcal{T}^-}{\mathcal{T}^- - \mathcal{T}^+ - \mathcal{T}^-\mathcal{T}^+},\\
&\max_{x\in S_{(\mathrm{iv})}}\bigl(x(t_1)-x(t_2)\bigr) = \begin{cases} \frac{\mathcal{T}^-}{\mathcal{T}^- - \mathcal{T}^+ + \mathcal{T}^-\mathcal{T}^+} & \text{if } \mathcal{T}^- < 1,\\ 1 & \text{if } \mathcal{T}^- \ge 1.\end{cases}
\end{aligned}$$

In case (iii) we have

$$\begin{aligned}
&\max_{x\in S_{(\mathrm{iii})}}\bigl\{x(t_1),x(t_2)\bigr\} = \frac{\mathcal{T}^- +1}{\mathcal{T}^- - \mathcal{T}^+ - \mathcal{T}^-\mathcal{T}^+},\\
&\min_{x\in S_{(\mathrm{iii})}}\bigl\{x(t_1),x(t_2)\bigr\} = \begin{cases} \frac{1-\mathcal{T}^-}{\mathcal{T}^- - \mathcal{T}^+} & \text{if } \mathcal{T}^- > 1,\\ \frac{1-\mathcal{T}^-}{\mathcal{T}^- - \mathcal{T}^+ + \mathcal{T}^-\mathcal{T}^+} & \text{if } \mathcal{T}^- \le 1,\end{cases}\\
&\max_{x\in S_{(\mathrm{iii})}}\bigl(x(t_1)-x(t_2)\bigr) = \frac{\mathcal{T}^-}{\mathcal{T}^- - \mathcal{T}^+ - \mathcal{T}^-\mathcal{T}^+},\\
&\min_{x\in S_{(\mathrm{iii})}}\bigl(x(t_1)-x(t_2)\bigr) = \begin{cases} \frac{-\mathcal{T}^-}{\mathcal{T}^- - \mathcal{T}^+ + \mathcal{T}^-\mathcal{T}^+} & \text{if } \mathcal{T}^- < 1,\\ -1 & \text{if } \mathcal{T}^- \ge 1.\end{cases}
\end{aligned}$$

Therefore, in cases (iii) and (iv) we have

$$\begin{aligned}
&\max_{x\in S_{(\mathrm{iii})}\cup S_{(\mathrm{iv})}}\bigl\{x(t_1),x(t_2)\bigr\} = \frac{\mathcal{T}^- +1}{\mathcal{T}^- - \mathcal{T}^+ - \mathcal{T}^-\mathcal{T}^+},\\
&\min_{x\in S_{(\mathrm{iii})}\cup S_{(\mathrm{iv})}}\bigl\{x(t_1),x(t_2)\bigr\} = \begin{cases} \frac{1-\mathcal{T}^-}{\mathcal{T}^- - \mathcal{T}^+} & \text{if } \mathcal{T}^- > 1,\\ \frac{1-\mathcal{T}^-}{\mathcal{T}^- - \mathcal{T}^+ + \mathcal{T}^-\mathcal{T}^+} & \text{if } \mathcal{T}^- \le 1,\end{cases}\\
&\max_{x\in S_{(\mathrm{iii})}\cup S_{(\mathrm{iv})}}\bigl|x(t_1)-x(t_2)\bigr| = \frac{\mathcal{T}^-}{\mathcal{T}^- - \mathcal{T}^+ - \mathcal{T}^-\mathcal{T}^+}.
\end{aligned}$$

In case (i) we have

$$\max_{x\in S_{(\mathrm{i})}}\{x(t_1),x(t_2)\} = \frac{\mathcal{T}^- + 1}{\mathcal{T}^- - \mathcal{T}^+ - (\mathcal{T}^+)^2/4},$$

$$\min_{x\in S_{(\mathrm{i})}}\{x(t_1),x(t_2)\} = \frac{1}{\mathcal{T}^- - \mathcal{T}^+ + (\mathcal{T}^-)^2/4},$$

$$\max_{x\in S_{(\mathrm{i})}}\bigl|x(t_1)-x(t_2)\bigr| = \max_{z\in[0,\mathcal{T}^+]}\left\{\frac{\mathcal{T}^- - z}{\mathcal{T}^- - \mathcal{T}^+ - z(\mathcal{T}^+ - z)}, \frac{z}{\mathcal{T}^- - \mathcal{T}^+ - z(\mathcal{T}^+ - z)}\right\}.$$

In case (ii) we have

$$\max_{x\in S_{(\mathrm{ii})}}\{x(t_1),x(t_2)\} = \frac{1}{\mathcal{T}^- - \mathcal{T}^+ - (\mathcal{T}^-)^2/4},$$

$$\min_{x\in S_{(\mathrm{ii})}}\{x(t_1),x(t_2)\} = \begin{cases} \min\{K,G\} & \text{if } \mathcal{T}^- \le 1, \\ \frac{1-\mathcal{T}^-}{\mathcal{T}^- - \mathcal{T}^+} & \text{if } 1 < \mathcal{T}^- \le 1 + \sqrt{1-\mathcal{T}^+}, \\ -\frac{1}{2(1+\sqrt{1-\mathcal{T}^+})-\mathcal{T}^-} & \text{if } 1 + \sqrt{1-\mathcal{T}^+} < \mathcal{T}^-, \end{cases}$$

where $K = \min_{z\in[0,\mathcal{T}^+]} \frac{1-z}{\mathcal{T}^- - \mathcal{T}^+ + z(\mathcal{T}^+ - z)}$, $G = \min_{z\in[0,\mathcal{T}^-]} \frac{1-z}{\mathcal{T}^- - \mathcal{T}^+ + (\mathcal{T}^+)^2/4 - z(\mathcal{T}^- - z)}$,

$$\max_{x\in S_{(\mathrm{ii})}}\bigl|x(t_1)-x(t_2)\bigr| = \max_{z\in[0,\mathcal{T}^-]}\left\{\frac{z}{\mathcal{T}^- - \mathcal{T}^+ - z(\mathcal{T}^- - z)}, \frac{\mathcal{T}^+ - z}{\mathcal{T}^- - \mathcal{T}^+ - z(\mathcal{T}^- - z)}\right\}.$$

Considering extremal values in all cases (i), (ii), (iii) and (vi), by elementary calculation, we obtain

$$N_1 = \begin{cases} \frac{1}{\mathcal{T}^- - \mathcal{T}^+ - (\mathcal{T}^-)^2/4} & \text{if } \mathcal{T}^- > 3, \\ \frac{\mathcal{T}^- + 1}{\mathcal{T}^- - \mathcal{T}^+ - \mathcal{T}^-\mathcal{T}^+} & \text{if } \mathcal{T}^- \le 3, \end{cases}$$

$$N_3 = \begin{cases} \frac{1-\mathcal{T}^-}{\mathcal{T}^- - \mathcal{T}^+ + \mathcal{T}^-\mathcal{T}^+} & \text{if } \mathcal{T}^- \le 1, \\ \frac{1-\mathcal{T}^-}{\mathcal{T}^- - \mathcal{T}^+} & \text{if } 1 < \mathcal{T}^- \le 1 + \sqrt{1-\mathcal{T}^+}, \\ -\frac{1}{2(1+\sqrt{1-\mathcal{T}^+})-\mathcal{T}^-} & \text{if } 1 + \sqrt{1-\mathcal{T}^+} < \mathcal{T}^-. \end{cases}$$

If $0 \le \mathcal{T}^+ < \mathcal{T}^-(1-\mathcal{T}^-/4)$, $3 \le \mathcal{T}^-$ or $0 \le \mathcal{T}^+ \le \frac{\mathcal{T}^-(\mathcal{T}^- - 1)^2}{(\mathcal{T}^- + 1)^2}$, $1 \le \mathcal{T}^- \le 3$, then

$$N_2 = \frac{1}{2\sqrt{\mathcal{T}^- - \mathcal{T}^+} - \mathcal{T}^-}.$$

If $0 < \mathcal{T}^+ < \frac{\mathcal{T}^-}{1+\mathcal{T}^-}$, $0 < \mathcal{T}^- \le 1$ or $\frac{\mathcal{T}^-(\mathcal{T}^- - 1)^2}{(\mathcal{T}^- + 1)^2} < \mathcal{T}^+ < \frac{\mathcal{T}^-}{1+\mathcal{T}^-}$, $1 < \mathcal{T}^- \le 3$, then

$$N_2 = \frac{\mathcal{T}^-}{\mathcal{T}^- - \mathcal{T}^+ - \mathcal{T}^-\mathcal{T}^+}.$$

This proves all Theorems 1, 2, 3. □

Competing interests

The author declares that he has no competing interests.

Author's contributions
The author read and approved the final manuscript.

Acknowledgements
Research was supported by the Russian Foundation for Basic Research (14-01-0033814). The author would like to thank both reviewers for their careful reading of the manuscript and valuable remarks.

Received: 29 January 2014 Accepted: 6 May 2014 Published: 15 May 2014

References
1. Krawcewicz, W, Ma, S, Wu, J: Multiple slowly oscillating periodic solutions in coupled lossless transmission lines. Nonlinear Anal., Real World Appl. **5**(2), 309-354 (2004)
2. Kang, S, Zhang, G: Existence of nontrivial periodic solutions for first order functional differential equations. Appl. Math. Lett. **18**(1), 101-107 (2005)
3. Wu, J, Wang, Z: Periodic solutions of neutral functional differential systems with two parameters. Nonlinear Anal., Real World Appl. **9**(3), 1012-1023 (2008)
4. Padhi, S, Srivastava, S: Multiple periodic solutions for a nonlinear first order functional differential equations with applications to population dynamics. Appl. Math. Comput. **203**(1), 1-6 (2008)
5. Schwabik, S, Tvrdy, M, Vejvoda, O: Differential and Integral Equations. Boundary Value Problems and Adjoints. Czechoslovak Academy of Sciences, Dordrecht (1979)
6. Hale, JK, Verduyn Lunel, SM: Introduction to Functional Differential Equations. Springer, New York (1993)
7. Kolmanovskii, V, Myshkis, A: Introduction to the Theory and Applications of Functional Differential Equations. Kluwer Academic, Dordrecht (1999)
8. Kiguradze, I, Půža, B: Boundary Value Problems for Systems of Linear Functional Differential Equations. Masaryk University, Brno (2003)
9. Azbelev, NV, Maksimov, VP, Rakhmatullina, LF: Introduction to the Theory of Functional Differential Equations. Methods and Applications. Hindawi Publishing Corporation, New York (2007)
10. Skubachevskii, AL: Nonclassical boundary value problems. I. J. Math. Sci. (N.Y.) **155**(2), 199-334 (2008)
11. Agarwal, RP, Berezansky, L, Braverman, E, Domoshnitsky, AI: Nonoscillation Theory of Functional Differential Equations with Applications. Springer, Berlin (2012)
12. Kiguradze, I, Půža, B: On periodic solutions to nonlinear functional-differential equations. Georgian Math. J. **6**(1), 45-64 (1999)
13. Hakl, R, Lomtatidze, A, Šremr, J: On a periodic-type boundary value problem for first-order nonlinear functional differential equations. Nonlinear Anal., Theory Methods Appl. **51**(3), 425-447 (2002)
14. Hakl, R, Lomtatidze, A, Půža, B: On periodic solutions of first order linear functional differential equations. Nonlinear Anal., Theory Methods Appl. **49**(7), 929-945 (2002)
15. Hakl, R, Lomtatidze, A, Šremr, J: On a boundary-value problem of periodic type for first-order linear functional differential equations. Nonlinear Oscil. **5**(3), 408-425 (2002)
16. Hakl, R, Lomtatidze, A, Šremr, J: Some Boundary Value Problems for First Order Scalar Functional Differential Equations. Masaryk University, Brno (2002)
17. Hakl, R, Lomtatidze, A, Šremr, J: On constant sign solutions of a periodic type boundary value problem for first order functional differential equations. Mem. Differ. Equ. Math. Phys. **26**, 65-90 (2002)
18. Hakl, R, Lomtatidze, A, Šremr, J: Solvability and the unique solvability of a periodic type boundary value problem for first order scalar functional differential equations. Georgian Math. J. **9**(3), 525-547 (2002)
19. Šremr, J, Šremr, P: On a two point boundary problem for first order linear differential equations with a deviating argument. Mem. Differ. Equ. Math. Phys. **29**, 75-124 (2003)
20. Nieto, JJ, Rodríguez-López, R: Remarks on periodic boundary value problems for functional differential equations. J. Comput. Appl. Math. **158**(2), 339-353 (2003)
21. Hakl, R, Lomtatidze, A, Šremr, J: Solvability of a periodic type boundary value problem for first order scalar functional differential equations. Arch. Math. **40**(1), 89-109 (2004)
22. Nieto, JJ, Rodríguez-López, R: Monotone method for first-order functional differential equations. Comput. Math. Appl. **52**(3-4), 471-484 (2006)
23. Bai, D, Xu, Y: Periodic solutions of first order functional differential equations with periodic deviations. Comput. Math. Appl. **53**(9), 1361-1366 (2007)

10.1186/1687-2770-2014-119
Cite this article as: Bravyi: On estimates of solutions of the periodic boundary value problem for first-order functional differential equations. *Boundary Value Problems* 2014, 2014:119