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Triple solutions of complementary Lidstone boundary value problems via fixed point theorems

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Abstract

We consider the following complementary Lidstone boundary value problem:

$$\begin{aligned}(-1)^m y^{(2m+1)}(t) &= F(t, y(t), y'(t)), \quad t \in [0, 1], \\ y(0) &= 0, \quad y^{(2k-1)}(0) = y^{(2k-1)}(1) = 0, \quad 1 \leq k \leq m.\end{aligned}$$

By using fixed point theorems of Leggett-Williams and Avery, we offer several criteria for the existence of three positive solutions of the boundary value problem. Examples are also included to illustrate the results obtained. We note that the nonlinear term F depends on y' and this derivative dependence is seldom investigated in the literature and a *new* technique is required to tackle the problem.

MSC: 34B15; 34B18

Keywords: positive solutions; complementary Lidstone boundary value problems; derivative-dependent nonlinearity; fixed point theorems

1 Introduction

In this paper we shall consider the *complementary Lidstone* boundary value problem

$$\begin{aligned}(-1)^m y^{(2m+1)}(t) &= F(t, y(t), y'(t)), \quad t \in [0, 1], \\ y(0) &= 0, \quad y^{(2k-1)}(0) = y^{(2k-1)}(1) = 0, \quad 1 \leq k \leq m,\end{aligned}\tag{1.1}$$

where $m \geq 1$ and F is continuous at least in the interior of the domain of interest. It is noted that the nonlinear term F involves y' , a derivative of the dependent variable. Most research papers on boundary value problems consider nonlinear terms that involve y only, and derivative-dependent nonlinearities are seldom tackled as special techniques are required.

The *complementary Lidstone* interpolation and boundary value problems have been very recently introduced in [1], and drawn on by Agarwal *et al.* in [2, 3] where they consider an $(2m + 1)$ th order differential equation together with boundary data at the odd order derivatives

$$y(0) = a_0, \quad y^{(2k-1)}(0) = a_k, \quad y^{(2k-1)}(1) = b_k, \quad 1 \leq k \leq m.\tag{1.2}$$

The boundary conditions (1.2) are known as *complementary Lidstone* boundary conditions, they naturally complement the *Lidstone* boundary conditions [4–7] which involve even order derivatives. To be precise, the *Lidstone* boundary value problem comprises an $2m$ th order differential equation and the *Lidstone* boundary conditions

$$y^{(2k)}(0) = a_k, \quad y^{(2k)}(1) = b_k, \quad 0 \leq k \leq m - 1. \quad (1.3)$$

There is a vast literature on Lidstone interpolation and boundary value problems. In fact, the Lidstone interpolation was first introduced by Lidstone [8] in 1929 and further characterized in the work of [9–16]. More recent research on Lidstone interpolation as well as Lidstone spline can be found in [1, 17–23]. Meanwhile, the Lidstone boundary value problems and several of its particular cases have been the subject matter of numerous investigations, see [4, 18, 24–37] and the references cited therein. In most of these works the nonlinear terms considered do *not* involve derivatives of the dependent variable, only a handful of papers [30, 31, 34, 35] tackle nonlinear terms that involve even order derivatives. In the present work, our study of the *complementary Lidstone* boundary value problem (1.1) where F depends on a *derivative* certainly extends and complements the rich literature on boundary value problems and notably on Lidstone boundary value problems. The literature on *complementary Lidstone* boundary value problems pales in comparison with that on *Lidstone* boundary value problems - after the first work [2] on complementary Lidstone boundary value problems, the recent paper [38] discusses the eigenvalue problem, while in [39] the existence of at least *one* or *two* positive solutions of the complementary Lidstone boundary value problem is derived by Leray-Schauder alternative and Krasnosel'skii's fixed point theorem in a cone.

In the present work, we shall establish the existence of at least *three* positive solutions using fixed point theorems of Leggett and Williams [40] as well as of Avery [41]. Estimates on the norms of these solutions will also be provided. Besides achieving *new* results, we also compare the results in terms of generality and illustrate the importance of the results through some examples. As remarked earlier, the presence of the derivative y' in the nonlinear term F requires a *special* technique to tackle the problem.

The paper is organized as follows. Section 2 contains the necessary definitions and fixed point theorems. The existence criteria are developed and discussed in Section 3. Finally, examples are presented in Section 4 to illustrate the importance of the results obtained.

2 Preliminaries

In this section we shall state some necessary definitions, the relevant fixed point theorems and properties of certain Green's function. Let B be a Banach space equipped with the norm $\|\cdot\|$.

Definition 2.1 Let $C (\subset B)$ be a nonempty closed convex set. We say that C is a *cone* provided the following conditions are satisfied:

- (a) If $x \in C$ and $\alpha \geq 0$, then $\alpha x \in C$;
- (b) If $x \in C$ and $-x \in C$, then $x = 0$.

Definition 2.2 Let $C (\subset B)$ be a cone. A map ψ is a *nonnegative continuous concave functional* on C if the following conditions are satisfied:

- (a) $\psi : C \rightarrow [0, \infty)$ is continuous;
- (b) $\psi(ty + (1 - t)z) \geq t\psi(y) + (1 - t)\psi(z)$ for all $y, z \in C$ and $0 \leq t \leq 1$.

Definition 2.3 Let $C (\subset B)$ be a cone. A map β is a *nonnegative continuous convex functional* on C if the following conditions are satisfied:

- (a) $\beta : C \rightarrow [0, \infty)$ is continuous;
- (b) $\beta(ty + (1 - t)z) \leq t\beta(y) + (1 - t)\beta(z)$ for all $y, z \in C$ and $0 \leq t \leq 1$.

Let γ, β, Θ be nonnegative continuous convex functionals on C and α, ψ be nonnegative continuous concave functionals on C . For nonnegative numbers $w_i, 1 \leq i \leq 3$, we shall introduce the following notations:

$$\begin{aligned}
 C(w_1) &= \{x \in C \mid \|x\| < w_1\}, \\
 C(\psi, w_1, w_2) &= \{x \in C \mid \psi(x) \geq w_1 \text{ and } \|x\| \leq w_2\}, \\
 P(\gamma, w_1) &= \{x \in C \mid \gamma(x) < w_1\}, \\
 P(\gamma, \alpha, w_1, w_2) &= \{x \in C \mid \alpha(x) \geq w_1 \text{ and } \gamma(x) \leq w_2\}, \\
 Q(\gamma, \beta, w_1, w_2) &= \{x \in C \mid \beta(x) \leq w_1 \text{ and } \gamma(x) \leq w_2\}, \\
 P(\gamma, \Theta, \alpha, w_1, w_2, w_3) &= \{x \in C \mid \alpha(x) \geq w_1, \Theta(x) \leq w_2 \text{ and } \gamma(x) \leq w_3\}, \\
 Q(\gamma, \beta, \psi, w_1, w_2, w_3) &= \{x \in C \mid \psi(x) \geq w_1, \beta(x) \leq w_2 \text{ and } \gamma(x) \leq w_3\}.
 \end{aligned}$$

The following fixed point theorems are our main tools, the first is usually called *Leggett-Williams' fixed point theorem*, and the second is known as the *five-functional fixed point theorem*.

Theorem 2.1 [40] *Let $C (\subset B)$ be a cone, and $w_4 > 0$ be given. Assume that ψ is a nonnegative continuous concave functional on C such that $\psi(x) \leq \|x\|$ for all $x \in \overline{C}(w_4)$, and let $S : \overline{C}(w_4) \rightarrow \overline{C}(w_4)$ be a continuous and completely continuous operator. Suppose that there exist numbers w_1, w_2, w_3 , where $0 < w_1 < w_2 < w_3 \leq w_4$, such that*

- (a) $\{x \in C(\psi, w_2, w_3) \mid \psi(x) > w_2\} \neq \emptyset$, and $\psi(Sx) > w_2$ for all $x \in C(\psi, w_2, w_3)$;
- (b) $\|Sx\| < w_1$ for all $x \in \overline{C}(w_1)$;
- (c) $\psi(Sx) > w_2$ for all $x \in C(\psi, w_2, w_4)$ with $\|Sx\| > w_3$.

Then S has (at least) three fixed points x_1, x_2 and x_3 in $\overline{C}(w_4)$. Furthermore, we have

$$\begin{aligned}
 x_1 \in C(w_1), \quad x_2 \in \{x \in C(\psi, w_2, w_4) \mid \psi(x) > w_2\} \quad \text{and} \\
 x_3 \in \overline{C}(w_4) \setminus (C(\psi, w_2, w_4) \cup \overline{C}(w_1)).
 \end{aligned} \tag{2.1}$$

Theorem 2.2 [41] *Let $C (\subset B)$ be a cone. Assume that there exist positive numbers w_5, M , nonnegative continuous convex functionals γ, β, Θ on C , and nonnegative continuous concave functionals α, ψ on C , with*

$$\alpha(x) \leq \beta(x) \quad \text{and} \quad \|x\| \leq M\gamma(x)$$

for all $x \in \overline{P}(\gamma, w_5)$. Let $S : \overline{P}(\gamma, w_5) \rightarrow \overline{P}(\gamma, w_5)$ be a continuous and completely continuous operator. Suppose that there exist nonnegative numbers $w_i, 1 \leq i \leq 4$, with $0 < w_2 < w_3$ such that

- (a) $\{x \in P(\gamma, \Theta, \alpha, w_3, w_4, w_5) \mid \alpha(x) > w_3\} \neq \emptyset$, and $\alpha(Sx) > w_3$ for all $x \in P(\gamma, \Theta, \alpha, w_3, w_4, w_5)$;
- (b) $\{x \in Q(\gamma, \beta, \psi, w_1, w_2, w_5) \mid \beta(x) < w_2\} \neq \emptyset$, and $\beta(Sx) < w_2$ for all $x \in Q(\gamma, \beta, \psi, w_1, w_2, w_5)$;
- (c) $\alpha(Sx) > w_3$ for all $x \in P(\gamma, \alpha, w_3, w_5)$ with $\Theta(Sx) > w_4$;
- (d) $\beta(Sx) < w_2$ for all $x \in Q(\gamma, \beta, w_2, w_5)$ with $\psi(Sx) < w_1$.

Then S has (at least) three fixed points x_1, x_2 and x_3 in $\bar{P}(\gamma, w_5)$. Furthermore, we have

$$\beta(x_1) < w_2, \quad \alpha(x_2) > w_3 \quad \text{and} \quad \beta(x_3) > w_2 \quad \text{with} \quad \alpha(x_3) < w_3. \tag{2.2}$$

We also require the definition of an L^1 -Carathéodory function.

Definition 2.4 [42] A function $P : [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is an L^1 -Carathéodory function if the following conditions hold:

- (a) The map $t \rightarrow P(t, u)$ is measurable for all $u \in \mathbb{R}^2$.
- (b) The map $u \rightarrow P(t, u)$ is continuous for almost all $t \in [0, 1]$.
- (c) For any $r > 0$, there exists $\mu_r \in L^1[0, 1]$ such that $|u| \leq r$ implies that $|P(t, u)| \leq \mu_r(t)$ for almost all $t \in [0, 1]$.

To tackle the *complementary Lidstone* boundary value problem (1.1), let us review certain attributes of the *Lidstone* boundary value problem. Let $g_m(t, s)$ be the Green's function of the *Lidstone* boundary value problem

$$\begin{aligned} x^{(2m)}(t) &= 0, \quad t \in [0, 1], \\ x^{(2k)}(0) &= x^{(2k)}(1) = 0, \quad 0 \leq k \leq m - 1. \end{aligned} \tag{2.3}$$

The Green's function $g_m(t, s)$ can be expressed as [4, 5]

$$g_m(t, s) = \int_0^1 g_1(t, u) g_{m-1}(u, s) du, \quad m \geq 2, \tag{2.4}$$

where

$$g_1(t, s) = \begin{cases} t(s-1), & 0 \leq t \leq s \leq 1, \\ s(t-1), & 0 \leq s \leq t \leq 1. \end{cases}$$

Further, it is known that

$$|g_m(t, s)| = (-1)^m g_m(t, s) \quad \text{and} \quad g_m(t, s) = g_m(s, t), \quad (t, s) \in (0, 1) \times (0, 1). \tag{2.5}$$

The following two lemmas give the upper and lower bounds of $|g_m(t, s)|$, they play an important role in subsequent development. We remark that the bounds in the two lemmas are *sharper* than those given in the literature [4, 5, 35, 37].

Lemma 2.1 [38] For $(t, s) \in [0, 1] \times [0, 1]$, we have

$$|g_m(t, s)| \leq \frac{1}{\pi^{2m-1}} \sin \pi s.$$

Lemma 2.2 [38] *Let $\delta \in (0, \frac{1}{2})$ be given. For $(t, s) \in [\delta, 1 - \delta] \times [0, 1]$, we have*

$$|g_m(t, s)| \geq \frac{2\delta}{\pi^{2m}} \sin \pi s.$$

3 Triple positive solutions

In this section, we shall use the fixed point theorems stated in Section 2 to obtain the existence of at least three positive solutions of the complementary Lidstone boundary value problem (1.1). By a *positive solution* y of (1.1), we mean a nontrivial $y \in C[0, 1]$ satisfying (1.1) and $y(t) \geq 0$ for $t \in [0, 1]$.

To tackle (1.1), we first consider the initial value problem

$$\begin{aligned} y'(t) &= x(t), \quad t \in [0, 1], \\ y(0) &= 0 \end{aligned} \tag{3.1}$$

whose solution is simply

$$y(t) = \int_0^t x(s) ds. \tag{3.2}$$

Taking into account (3.1) and (3.2), the *complementary Lidstone* boundary value problem (1.1) reduces to the *Lidstone* boundary value problem

$$\begin{aligned} (-1)^m x^{(2m)}(t) &= F\left(t, \int_0^t x(s) ds, x(t)\right), \quad t \in [0, 1], \\ x^{(2k-2)}(0) &= x^{(2k-2)}(1) = 0, \quad 1 \leq k \leq m. \end{aligned} \tag{3.3}$$

If (3.3) has a solution x^* , then by virtue of (3.2), the boundary value problem (1.1) has a solution given by

$$y^*(t) = \int_0^t x^*(s) ds. \tag{3.4}$$

So the existence of a solution of the *complementary Lidstone* boundary value problem (1.1) follows from the existence of a solution of the *Lidstone* boundary value problem (3.3). It is clear from (3.4) that $\|y^*\| \leq \|x^*\|$; moreover if x^* is positive, so is y^* . With the tools in Section 2 and a technique to handle the nonlinear term F , we shall study the boundary value problem (1.1) via (3.3).

Let the Banach space $B = C[0, 1]$ be equipped with the norm $\|x\| = \sup_{t \in [0, 1]} |x(t)|$ for $x \in B$. Define the operator $S : C[0, 1] \rightarrow C[0, 1]$ by

$$\begin{aligned} Sx(t) &= \int_0^1 (-1)^m g_m(t, s) F\left(s, \int_0^s x(\tau) d\tau, x(s)\right) ds \\ &= \int_0^1 |g_m(t, s)| F\left(s, \int_0^s x(\tau) d\tau, x(s)\right) ds, \quad t \in [0, 1], \end{aligned} \tag{3.5}$$

where $g_m(t, s)$ is the Green's function given in (2.4). A fixed point x^* of the operator S is clearly a solution of the boundary value problem (3.3), and as seen earlier $y^*(t) = \int_0^t x^*(s) ds$ is a solution of (1.1).

For easy reference, we shall list the conditions that are needed later. In these conditions the sets K and \tilde{K} are defined by

$$\tilde{K} = \{x \in B \mid x(t) \geq 0, t \in [0, 1]\}, \tag{3.6}$$

$$K = \{x \in \tilde{K} \mid x(t) > 0 \text{ on some subset of } [0, 1] \text{ of positive measure}\}.$$

(C1) $F : [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is an L^1 -Carathéodory function.

(C2) We have

$$F(t, u, v) \geq 0, \quad u, v \in \tilde{K}, \text{ a.e. } t \in (0, 1) \quad \text{and}$$

$$F(t, u, v) > 0, \quad u, v \in K, \text{ a.e. } t \in (0, 1).$$

(C3) There exist continuous functions f, v, μ with $f : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$ and $v, \mu : (0, 1) \rightarrow [0, \infty)$ such that

$$\mu(t)f(u, v) \leq F(t, u, v) \leq v(t)f(u, v), \quad u, v \in \tilde{K}, \text{ a.e. } t \in (0, 1).$$

(C4) There exists a number $0 < \theta \leq 1$ such that

$$\mu(t) \geq \theta v(t), \quad \text{a.e. } t \in (0, 1).$$

If (C2) and (C3) hold, then it follows from (3.5) that for $x \in \tilde{K}$ and $t \in [0, 1]$,

$$\begin{aligned} 0 &\leq \int_0^1 |g_m(t, s)| \mu(s) f\left(\int_0^s x(\tau) d\tau, x(s)\right) ds \leq Sx(t) \\ &\leq \int_0^1 |g_m(t, s)| v(s) f\left(\int_0^s x(\tau) d\tau, x(s)\right) ds. \end{aligned} \tag{3.7}$$

Let $\delta \in (0, \frac{1}{2})$ be fixed. We define a cone C in B as

$$C = \left\{ x \in B \mid x(t) \geq 0 \text{ for } t \in [0, 1], \text{ and } \min_{t \in [\delta, 1-\delta]} x(t) \geq \frac{2\delta\theta}{\pi} \|x\| \right\}, \tag{3.8}$$

where θ is given in (C4). Clearly, we have $C \subseteq \tilde{K}$.

Lemma 3.1 *Let (C1)-(C4) hold. Then the operator S defined in (3.5) is continuous and completely continuous, and S maps C into C .*

Proof From (2.4) we have $g_m(t, s) \in C[0, 1] \subseteq L^\infty[0, 1]$, $t \in [0, 1]$ and the map $t \rightarrow g_m(t, s)$ is continuous from $[0, 1]$ to $C[0, 1]$. This together with $F : [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is an L^1 -Carathéodory function ensures (as in [42, Theorem 4.2.2]) that S is continuous and completely continuous.

Let $x \in C$. From (3.7) we have $Sx(t) \geq 0$ for $t \in [0, 1]$. Next, using (3.7) and Lemma 2.1 gives for $t \in [0, 1]$,

$$\begin{aligned} Sx(t) &\leq \int_0^1 |g_m(t, s)| v(s) f\left(\int_0^s x(\tau) d\tau, x(s)\right) ds \\ &\leq \frac{1}{\pi^{2m-1}} \int_0^1 v(s) f\left(\int_0^s x(\tau) d\tau, x(s)\right) \sin \pi s ds. \end{aligned} \tag{3.9}$$

Hence, we have

$$\|Sx\| \leq \frac{1}{\pi^{2m-1}} \int_0^1 v(s) f\left(\int_0^s x(\tau) d\tau, x(s)\right) \sin \pi s ds. \tag{3.10}$$

Now, employing (3.7), Lemma 2.2, (C4) and (3.10), we find for $t \in [\delta, 1 - \delta]$,

$$\begin{aligned} Sx(t) &\geq \int_0^1 |g_m(t, s)| \mu(s) f\left(\int_0^s x(\tau) d\tau, x(s)\right) ds \\ &\geq \frac{2\delta}{\pi^{2m}} \int_0^1 \mu(s) f\left(\int_0^s x(\tau) d\tau, x(s)\right) \sin \pi s ds \\ &\geq \frac{2\delta}{\pi^{2m}} \int_0^1 \theta v(s) f\left(\int_0^s x(\tau) d\tau, x(s)\right) \sin \pi s ds \\ &\geq \frac{2\delta\theta}{\pi} \|Sx\|. \end{aligned}$$

This leads to

$$\min_{t \in [\delta, 1-\delta]} Sx(t) \geq \frac{2\delta\theta}{\pi} \|Sx\|.$$

We have shown that $Sx \in C$. □

For subsequent results, we define the following constants for fixed $\delta \in (0, \frac{1}{2})$ and $\tau_1, \tau_2, \tau_3, \tau_4 \in [0, 1]$:

$$\begin{cases} q = \frac{1}{\pi^{2m-1}} \int_0^1 v(s) \sin \pi s ds, \\ r = \min_{t \in [\delta, 1-\delta]} \int_{\frac{1}{2}}^{1-\delta} |g_m(t, s)| \mu(s) ds, \\ p_1 = \min_{t \in [\tau_2, \tau_3]} \int_{\frac{1}{2}}^{\tau_3} |g_m(t, s)| \mu(s) ds, \\ p_2 = \frac{1}{\pi^{2m-1}} \int_{\tau_1}^{\tau_4} v(s) \sin \pi s ds, \\ p_3 = \frac{1}{\pi^{2m-1}} \int_0^{\tau_1} v(s) \sin \pi s ds + \frac{1}{\pi^{2m-1}} \int_{\tau_4}^1 v(s) \sin \pi s ds. \end{cases} \tag{3.11}$$

Lemma 3.2 *Let (C1)-(C4) hold, and assume*

(C5) *the function $v(s) \sin \pi s > 0$ on a subset of $[0, 1]$ of positive measure.*

Suppose that there exists a number $d > 0$ such that for $u, v \in [0, d]$,

$$f(u, v) < \frac{d}{q}. \tag{3.12}$$

Then

$$S(\overline{C}(d)) \subseteq C(d) \subset \overline{C}(d). \tag{3.13}$$

Proof Let $x \in \overline{C}(d)$. So $\|x\| \leq d$, which implies immediately that

$$\int_0^s x(\tau) d\tau \leq d \quad \text{and} \quad x(s) \leq d, \quad s \in [0, 1].$$

Then, using (3.9), (C5) and (3.12), we find for $t \in [0, 1]$,

$$\begin{aligned} |Sx(t)| &\leq \frac{1}{\pi^{2m-1}} \int_0^1 v(s) f\left(\int_0^s x(\tau) d\tau, x(s)\right) \sin \pi s ds \\ &< \frac{1}{\pi^{2m-1}} \int_0^1 v(s) \sin \pi s ds \cdot \frac{d}{q} \\ &= q \cdot \frac{d}{q} = d. \end{aligned}$$

This implies $\|Sx\| < d$. Together with the fact that $Sx \in C$ (Lemma 3.1), we have shown that $Sx \in C(d)$. Conclusion (3.13) is now immediate. \square

Using a similar argument as Lemma 3.2, we have the following lemma.

Lemma 3.3 *Let (C1)-(C4) hold. Suppose that there exists a number $d > 0$ such that for $u, v \in [0, d]$,*

$$f(u, v) \leq \frac{d}{q}.$$

Then

$$S(\overline{C}(d)) \subseteq \overline{C}(d).$$

We are now ready to establish the existence of three positive solutions for the complementary Lidstone boundary value problem (1.1). The first result below uses Leggett-Williams' fixed point theorem (Theorem 2.1).

Theorem 3.1 *Let $\delta \in (0, \frac{1}{2})$ be fixed. Let (C1)-(C5) hold, and assume*

(C6) for each $t \in [\delta, 1 - \delta]$, the function $|g_m(t, s)|\mu(s) > 0$ on a subset of $[\frac{1}{2}, 1 - \delta]$ of positive measure.

Suppose that there exist numbers w_1, w_2, w_3 with

$$0 < w_1 < w_2 \left(\frac{1}{2} - \delta\right) < w_2 < \frac{\pi w_2}{2\delta\theta} \leq w_3$$

such that the following hold:

(P) $f(u, v) < \frac{w_1}{q}$ for $u, v \in [0, w_1]$;

(Q) *one of the following holds:*

(Q1) $\limsup_{u \rightarrow \infty, v \rightarrow \infty} \frac{f(u, v)}{u} < \frac{1}{q}$ or $\limsup_{u \rightarrow \infty, v \rightarrow \infty} \frac{f(u, v)}{v} < \frac{1}{q}$;

(Q2) *there exists a number $d (\geq w_3)$ such that $f(u, v) \leq \frac{d}{q}$ for $u, v \in [0, d]$;*

(R) $f(u, v) > \frac{w_2}{r}$ for $u \in [w_2(\frac{1}{2} - \delta), w_3(\frac{1}{2} - \delta)]$ and $v \in [w_2, w_3]$.

Then we have the following conclusions:

(a) *The Lidstone boundary value problem (3.3) has (at least) three positive solutions $x_1, x_2, x_3 \in C$ (where C is defined in (3.8)) such that*

$$\begin{cases} \|x_1\| < w_1; \\ x_2(t) > w_2, & t \in [\delta, 1 - \delta]; \\ \|x_3\| > w_1 \quad \text{and} \quad \min_{t \in [\delta, 1 - \delta]} x_3(t) < w_2. \end{cases} \tag{3.14}$$

(b) *The complementary Lidstone boundary value problem (1.1) has (at least) three positive solutions y_1, y_2, y_3 such that for $i = 1, 2, 3$,*

$$\begin{cases} y_i(t) = \int_0^t x_i(s) ds, & t \in [0, 1]; \\ \|y_i\| \leq \|x_i\|; & y_i(t) \geq \frac{2\delta\theta}{\pi} \|x_i\| (t - \delta), \quad t \in [\delta, 1 - \delta] \end{cases} \quad (3.15)$$

(where x_i 's are those in conclusion (a)). We further have

$$\begin{cases} \|y_1\| < w_1; \\ y_2(t) > w_2(t - \delta), & t \in [\delta, 1 - \delta]; \\ y_3(t) > \frac{2\delta\theta}{\pi} w_1(t - \delta), & t \in [\delta, 1 - \delta]. \end{cases} \quad (3.16)$$

Proof We shall employ Theorem 2.1 with the cone C defined in (3.8). First, we shall prove that condition (Q) implies the existence of a number w_4 , where $w_4 \geq w_3$, such that

$$S(\overline{C}(w_4)) \subseteq \overline{C}(w_4). \quad (3.17)$$

Suppose that (Q2) holds. Then by Lemma 3.3 we immediately have (3.17) where we pick $w_4 = d$. Suppose now that $\limsup_{u \rightarrow \infty, v \rightarrow \infty} \frac{f(u,v)}{u} < \frac{1}{q}$ of (Q1) is satisfied. Then there exist $N > 0$ and $\epsilon < \frac{1}{q}$ such that

$$\frac{f(u,v)}{u} < \epsilon, \quad u, v > N. \quad (3.18)$$

Let

$$L = \max_{u,v \in [0,N]} f(u,v).$$

Noting (3.18), it is then clear that for $u, v \geq 0$,

$$f(u,v) \leq L + \epsilon u. \quad (3.19)$$

Now, pick the number w_4 so that

$$w_4 > \max \left\{ w_3, L \left(\frac{1}{q} - \epsilon \right)^{-1} \right\}. \quad (3.20)$$

Let $x \in \overline{C}(w_4)$. Using (3.9), (3.19) and (3.20) yields for $t \in [0, 1]$,

$$\begin{aligned} |Sx(t)| &\leq \frac{1}{\pi^{2m-1}} \int_0^1 v(s) f \left(\int_0^s x(\tau) d\tau, x(s) \right) \sin \pi s ds \\ &\leq \frac{1}{\pi^{2m-1}} \int_0^1 v(s) \left(L + \epsilon \int_0^s x(\tau) d\tau \right) \sin \pi s ds \\ &\leq \frac{1}{\pi^{2m-1}} \int_0^1 v(s) (L + \epsilon w_4) \sin \pi s ds \\ &= q(L + \epsilon w_4) \\ &< q \left[w_4 \left(\frac{1}{q} - \epsilon \right) + \epsilon w_4 \right] = w_4. \end{aligned}$$

Hence, $\|Sx\| < w_4$ and so $Sx \in C(w_4) \subset \overline{C}(w_4)$. Thus, (3.17) follows immediately. Note that the argument is similar if we assume that $\limsup_{u \rightarrow \infty, v \rightarrow \infty} \frac{f(u,v)}{v} < \frac{1}{q}$ of (Q1) is satisfied.

Let $\psi : C \rightarrow [0, \infty)$ be defined by

$$\psi(x) = \min_{t \in [\delta, 1-\delta]} x(t).$$

Clearly, ψ is a nonnegative continuous concave functional on C and $\psi(x) \leq \|x\|$ for all $x \in C$.

We shall verify that condition (a) of Theorem 2.1 is satisfied. It is obvious that

$$x(t) = \frac{w_2 + w_3}{2} \in \{x \in C(\psi, w_2, w_3) \mid \psi(x) > w_2\}$$

and so $\{x \in C(\psi, w_2, w_3) \mid \psi(x) > w_2\} \neq \emptyset$. Next, let $x \in C(\psi, w_2, w_3)$. Then $\psi(x) \geq w_2$ and $\|x\| \leq w_3$ which imply

$$x(s) \in [w_2, w_3], \quad s \in [\delta, 1-\delta] \quad \text{and} \quad \int_{\delta}^{\frac{1}{2}} x(\tau) d\tau \in \left[w_2 \left(\frac{1}{2} - \delta \right), w_3 \left(\frac{1}{2} - \delta \right) \right]. \quad (3.21)$$

Using (3.7), (3.21), (C6) and (R), it follows that

$$\begin{aligned} \psi(Sx) &= \min_{t \in [\delta, 1-\delta]} Sx(t) \\ &\geq \min_{t \in [\delta, 1-\delta]} \int_0^1 |g_m(t, s)| \mu(s) f\left(\int_0^s x(\tau) d\tau, x(s)\right) ds \\ &\geq \min_{t \in [\delta, 1-\delta]} \int_{\frac{1}{2}}^{1-\delta} |g_m(t, s)| \mu(s) f\left(\int_{\delta}^{\frac{1}{2}} x(\tau) d\tau, x(s)\right) ds \\ &> \min_{t \in [\delta, 1-\delta]} \int_{\frac{1}{2}}^{1-\delta} |g_m(t, s)| \mu(s) ds \cdot \frac{w_2}{r} \\ &= r \cdot \frac{w_2}{r} = w_2. \end{aligned}$$

Therefore, we have shown that $\psi(Sx) > w_2$ for all $x \in C(\psi, w_2, w_3)$.

Next, by condition (P) and Lemma 3.2 (with $d = w_1$), we have $S(\overline{C}(w_1)) \subseteq C(w_1)$. Hence, condition (b) of Theorem 2.1 is fulfilled.

Finally, we shall show that condition (c) of Theorem 2.1 holds. Let $x \in C(\psi, w_2, w_4)$ with $\|Sx\| > w_3$. Using (3.7), Lemma 2.2, (C4), (3.10) and the inequality $\frac{\pi w_2}{2\delta\theta} \leq w_3$, we find

$$\begin{aligned} \psi(Sx) &\geq \min_{t \in [\delta, 1-\delta]} \int_0^1 |g_m(t, s)| \mu(s) f\left(\int_0^s x(\tau) d\tau, x(s)\right) ds \\ &\geq \frac{2\delta}{\pi^{2m}} \int_0^1 \mu(s) f\left(\int_0^s x(\tau) d\tau, x(s)\right) \sin \pi s ds \\ &\geq \frac{2\delta}{\pi^{2m}} \int_0^1 \theta v(s) f\left(\int_0^s x(\tau) d\tau, x(s)\right) \sin \pi s ds \\ &\geq \frac{2\delta\theta}{\pi} \|Sx\| \\ &> \frac{2\delta\theta}{\pi} w_3 \geq w_2. \end{aligned}$$

Hence, we have proved that $\psi(Sx) > w_2$ for all $x \in C(\psi, w_2, w_4)$ with $\|Sx\| > w_3$.

It now follows from Theorem 2.1 that the Lidstone boundary value problem (3.3) has (at least) three positive solutions $x_1, x_2, x_3 \in \overline{C}(w_4)$ satisfying (2.1). It is easy to see that here (2.1) reduces to (3.14). This completes the proof of conclusion (a).

Finally, it is observed from (3.4) that the complementary Lidstone boundary value problem (1.1) has (at least) three positive solutions y_1, y_2, y_3 such that for $i = 1, 2, 3$,

$$y_i(t) = \int_0^t x_i(s) ds, \quad t \in [0, 1] \quad \text{and} \quad \|y_i\| \leq \|x_i\|. \tag{3.22}$$

Moreover, since $x_i \in C$, we get for $t \in [\delta, 1 - \delta]$,

$$y_i(t) = \int_0^t x_i(s) ds \geq \int_\delta^t x_i(s) ds \geq \int_\delta^t \frac{2\delta\theta}{\pi} \|x_i\| ds = \frac{2\delta\theta}{\pi} \|x_i\| (t - \delta). \tag{3.23}$$

Combining (3.22) and (3.23) gives (3.15) immediately.

Further, since $x_2(t) > w_2$ for $t \in [\delta, 1 - \delta]$, we have for $t \in [\delta, 1 - \delta]$,

$$y_2(t) = \int_0^t x_2(s) ds > \int_\delta^t x_2(s) ds > \int_\delta^t w_2 ds = w_2(t - \delta). \tag{3.24}$$

Hence, noting (3.14), (3.15) and (3.24), we get (3.16). This completes the proof of conclusion (b). □

We shall now employ the five-functional fixed point theorem (Theorem 2.2) to give other existence criteria. In applying Theorem 2.2 it is possible to choose the functionals and constants in different ways, indeed we shall do so and derive two results. Our first result below turns out to be a generalization of Theorem 3.1.

Theorem 3.2 *Let $\delta \in (0, \frac{1}{2})$ be fixed. Let (C1)-(C4) hold. Assume that there exist numbers $\tau_j, 1 \leq j \leq 4$, with*

$$0 \leq \tau_1 \leq \delta \leq \tau_2 < \frac{1}{2} < \tau_3 \leq 1 - \delta \leq \tau_4 \leq 1$$

such that

(C7) *for each $t \in [\tau_2, \tau_3]$, the function $|g_m(t, s)|\mu(s) > 0$ on a subset of $[\frac{1}{2}, \tau_3]$ of positive measure;*

(C8) *the function $v(s) \sin \pi s > 0$ on a subset of $[\tau_1, \tau_4]$ of positive measure.*

Suppose that there exist numbers $w_i, 2 \leq i \leq 5$, with

$$\begin{cases} 0 < w_2 < w_3 < \frac{\pi w_3}{2\delta\theta} \leq w_4 \leq w_5 < \frac{qw_2}{p_3}, \\ \tau_1 w_5 + (\tau_4 - \tau_1)w_2 < w_3(\frac{1}{2} - \tau_2) \end{cases}$$

such that the following hold:

(P) $f(u, v) < \frac{1}{p_2}(w_2 - \frac{w_5 p_3}{q})$ for $u \in [0, \tau_1 w_5 + (\tau_4 - \tau_1)w_2]$ and $v \in [0, w_2]$;

(Q) $f(u, v) \leq \frac{w_5}{q}$ for $u, v \in [0, w_5]$;

(R) $f(u, v) > \frac{w_3}{p_1}$ for $u \in [w_3(\frac{1}{2} - \tau_2), w_4(\frac{1}{2} - \tau_2)]$ and $v \in [w_3, w_4]$.

Then we have the following conclusions:

(a) *The Lidstone boundary value problem (3.3) has (at least) three positive solutions $x_1, x_2, x_3 \in \overline{C}(w_5)$ (where C is defined in (3.8)) such that*

$$\begin{cases} x_1(t) < w_2, & t \in [\tau_1, \tau_4]; \\ x_2(t) > w_3, & t \in [\tau_2, \tau_3]; \\ \max_{t \in [\tau_1, \tau_4]} x_3(t) > w_2 \quad \text{and} \quad \min_{t \in [\tau_2, \tau_3]} x_3(t) < w_3. \end{cases} \tag{3.25}$$

(b) *The complementary Lidstone boundary value problem (1.1) has (at least) three positive solutions y_1, y_2, y_3 such that (3.15) holds for $i = 1, 2, 3$. We further have*

$$\begin{cases} y_1(t) < \tau_1 \max_{s \in [0, \tau_1]} x_1(s) + (\tau_4 - \tau_1)w_2, & t \in [\tau_1, \tau_4]; \\ y_2(t) > w_3(t - \tau_2), & t \in [\tau_2, \tau_3]; \\ y_3(t) > \frac{2\delta\theta}{\pi} w_2(t - \delta), & t \in [\delta, 1 - \delta]. \end{cases} \tag{3.26}$$

Proof We shall apply Theorem 2.2 with the cone C defined in (3.8). We define the following five functionals on the cone C :

$$\begin{cases} \gamma(x) = \|x\|, \\ \psi(x) = \min_{t \in [\delta, 1 - \delta]} x(t), \\ \beta(x) = \Theta(x) = \max_{t \in [\tau_1, \tau_4]} x(t), \\ \alpha(x) = \min_{t \in [\tau_2, \tau_3]} x(t). \end{cases} \tag{3.27}$$

First, we shall show that the operator S maps $\overline{P}(\gamma, w_5)$ into $\overline{P}(\gamma, w_5)$. Note that $\overline{P}(\gamma, w_5) = \overline{C}(w_5)$. By (Q) and Lemma 3.3 (with $d = w_5$), we immediately have $S(\overline{C}(w_5)) \subseteq \overline{C}(w_5)$.

Next, to see that condition (a) of Theorem 2.2 is fulfilled, we note that

$$\{x \in P(\gamma, \Theta, \alpha, w_3, w_4, w_5) \mid \alpha(x) > w_3\} \neq \emptyset$$

since it has an element $x(t) = \frac{1}{2}(w_3 + w_4)$. Let $x \in P(\gamma, \Theta, \alpha, w_3, w_4, w_5)$. Then by definition we have $\alpha(x) \geq w_3$ and $\Theta(x) \leq w_4$, which imply

$$\begin{aligned} x(s) &\in [w_3, w_4], \quad s \in [\tau_2, \tau_3] \quad \text{and} \\ \int_{\tau_2}^{\frac{1}{2}} x(\tau) d\tau &\in \left[w_3 \left(\frac{1}{2} - \tau_2 \right), w_4 \left(\frac{1}{2} - \tau_2 \right) \right]. \end{aligned} \tag{3.28}$$

Noting (3.7), (3.28), (C7) and (R), we find

$$\begin{aligned} \alpha(Sx) &= \min_{t \in [\tau_2, \tau_3]} Sx(t) \\ &\geq \min_{t \in [\tau_2, \tau_3]} \int_0^1 |g_m(t, s)| \mu(s) f \left(\int_0^s x(\tau) d\tau, x(s) \right) ds \\ &\geq \min_{t \in [\tau_2, \tau_3]} \int_{\frac{1}{2}}^{\tau_3} |g_m(t, s)| \mu(s) f \left(\int_{\tau_2}^{\frac{1}{2}} x(\tau) d\tau, x(s) \right) ds \\ &> \min_{t \in [\tau_2, \tau_3]} \int_{\frac{1}{2}}^{\tau_3} |g_m(t, s)| \mu(s) ds \cdot \frac{w_3}{p_1} \\ &= p_1 \cdot \frac{w_3}{p_1} = w_3. \end{aligned}$$

Hence, $\alpha(Sx) > w_3$ for all $x \in P(\gamma, \Theta, \alpha, w_3, w_4, w_5)$.

We shall now verify that condition (b) of Theorem 2.2 is satisfied. Let w_1 be such that $0 < w_1 < w_2$. Note that

$$\{x \in Q(\gamma, \beta, \psi, w_1, w_2, w_5) \mid \beta(x) < w_2\} \neq \emptyset$$

because it has an element $x(t) = \frac{1}{2}(w_1 + w_2)$. Let $x \in Q(\gamma, \beta, \psi, w_1, w_2, w_5)$. Then we have $\beta(x) \leq w_2$ and $\gamma(x) \leq w_5$, *i.e.*,

$$x(s) \in [0, w_2], \quad s \in [\tau_1, \tau_4] \quad \text{and} \quad x(s) \in [0, w_5], \quad s \in [0, 1], \tag{3.29}$$

which lead to the following:

$$\begin{aligned} \int_0^s x(\tau) d\tau &\leq w_5, \quad s \in [0, \tau_1] \cup [\tau_4, 1]; \\ \int_0^s x(\tau) d\tau &\leq \int_0^{\tau_4} x(\tau) d\tau = \int_0^{\tau_1} x(\tau) d\tau + \int_{\tau_1}^{\tau_4} x(\tau) d\tau \\ &\leq \tau_1 w_5 + (\tau_4 - \tau_1) w_2, \quad s \in [\tau_1, \tau_4]. \end{aligned} \tag{3.30}$$

Using (3.9), (3.29), (3.30), (C8), (P) and (Q) successively, we find

$$\begin{aligned} \beta(Sx) &= \max_{t \in [\tau_1, \tau_4]} Sx(t) \\ &\leq \frac{1}{\pi^{2m-1}} \int_0^1 v(s) f\left(\int_0^s x(\tau) d\tau, x(s)\right) \sin \pi s ds \\ &= \frac{1}{\pi^{2m-1}} \left(\int_0^{\tau_1} + \int_{\tau_1}^{\tau_4} + \int_{\tau_4}^1\right) v(s) f\left(\int_0^s x(\tau) d\tau, x(s)\right) \sin \pi s ds \\ &< \left[\frac{1}{\pi^{2m-1}} \int_0^{\tau_1} v(s) \sin \pi s ds + \frac{1}{\pi^{2m-1}} \int_{\tau_4}^1 v(s) \sin \pi s ds\right] \frac{w_5}{q} \\ &\quad + \left[\frac{1}{\pi^{2m-1}} \int_{\tau_1}^{\tau_4} v(s) \sin \pi s ds\right] \frac{1}{p_2} \left(w_2 - \frac{w_5 p_3}{q}\right) \\ &= p_3 \cdot \frac{w_5}{q} + p_2 \cdot \frac{1}{p_2} \left(w_2 - \frac{w_5 p_3}{q}\right) = w_2. \end{aligned}$$

Therefore, $\beta(Sx) < w_2$ for all $x \in Q(\gamma, \beta, \psi, w_1, w_2, w_5)$.

Next, we shall show that condition (c) of Theorem 2.2 is met. Let $x \in C$. Clearly, we have

$$\Theta(Sx) = \max_{t \in [\tau_1, \tau_4]} Sx(t) \leq \|Sx\|. \tag{3.31}$$

Moreover, using the fact that S maps C into C , we find

$$\alpha(Sx) = \min_{t \in [\tau_2, \tau_3]} Sx(t) \geq \min_{t \in [\delta, 1-\delta]} Sx(t) \geq \frac{2\delta\theta}{\pi} \|Sx\|. \tag{3.32}$$

Combining (3.31) and (3.32) yields

$$\alpha(Sx) \geq \frac{2\delta\theta}{\pi} \Theta(Sx), \quad x \in C. \tag{3.33}$$

Now, let $x \in P(\gamma, \alpha, w_3, w_5)$ with $\Theta(Sx) > w_4$. Then it follows from (3.33) and the inequality $\frac{\pi w_3}{2\delta\theta} \leq w_4$ that

$$\alpha(Sx) \geq \frac{2\delta\theta}{\pi} \Theta(Sx) > \frac{2\delta\theta}{\pi} w_4 \geq \frac{2\delta\theta}{\pi} \frac{\pi w_3}{2\delta\theta} = w_3. \tag{3.34}$$

Thus, $\alpha(Sx) > w_3$ for all $x \in P(\gamma, \alpha, w_3, w_5)$ with $\Theta(Sx) > w_4$.

Finally, we shall prove that condition (d) of Theorem 2.2 is fulfilled. Let $x \in Q(\gamma, \beta, w_2, w_5)$ with $\psi(Sx) < w_1$. Then we have $\beta(x) \leq w_2$ and $\gamma(x) \leq w_5$ which give (3.29) and (3.30). As in proving condition (b), we get $\beta(Sx) < w_2$. Hence, condition (d) of Theorem 2.2 is satisfied.

It now follows from Theorem 2.2 that the Lidstone boundary value problem (3.3) has (at least) three positive solutions $x_1, x_2, x_3 \in \bar{P}(\gamma, w_5) = \bar{C}(w_5)$ satisfying (2.2). Furthermore, (2.2) reduces to (3.25) immediately. This completes the proof of conclusion (a).

Finally, as in the proof of Theorem 3.1, we see that (3.15) holds for the positive solutions $y_i, i = 1, 2, 3$, of the complementary Lidstone boundary value problem (1.1). Moreover, noting that $x_1(t) < w_2$ for $t \in [\tau_1, \tau_4]$, we find for $t \in [\tau_1, \tau_4]$,

$$\begin{aligned} y_1(t) &= \int_0^t x_1(s) ds \leq \int_0^{\tau_4} x_1(s) ds = \int_0^{\tau_1} x_1(s) ds + \int_{\tau_1}^{\tau_4} x_1(s) ds \\ &< \tau_1 \max_{s \in [0, \tau_1]} x_1(s) + (\tau_4 - \tau_1)w_2. \end{aligned}$$

Next, noting $x_2(t) > w_3$ for $t \in [\tau_2, \tau_3]$, we get for $t \in [\tau_2, \tau_3]$,

$$y_2(t) = \int_0^t x_2(s) ds > \int_{\tau_2}^t x_2(s) ds > w_3(t - \tau_2).$$

Lastly, using (3.15) and $\|x_3\| \geq \max_{t \in [\tau_1, \tau_4]} x_3(t) > w_2$, we find for $t \in [\delta, 1 - \delta]$,

$$y_3(t) \geq \frac{2\delta\theta}{\pi} \|x_3\| (t - \delta) > \frac{2\delta\theta}{\pi} w_2 (t - \delta).$$

The proof of conclusion (b) is complete. □

We shall now consider the special case of Theorem 3.2 when

$$\tau_1 = 0, \quad \tau_2 = \delta, \quad \tau_3 = 1 - \delta \quad \text{and} \quad \tau_4 = 1.$$

Then, from definitions (3.11), we see that

$$p_1 = r, \quad p_2 = q \quad \text{and} \quad p_3 = 0.$$

In this case Theorem 3.2 yields the following corollary.

Corollary 3.1 *Let $\delta \in (0, \frac{1}{2})$ be fixed. Let (C1)-(C4) hold, and assume*

(C7)' *for each $t \in [\delta, 1 - \delta]$, the function $|g_m(t, s)|\mu(s) > 0$ on a subset of $[\frac{1}{2}, 1 - \delta]$ of positive measure;*

(C8)' *the function $v(s) \sin \pi s > 0$ on a subset of $[0, 1]$ of positive measure.*

Suppose that there exist numbers w_i , $2 \leq i \leq 5$, with

$$0 < w_2 < w_3 \left(\frac{1}{2} - \delta \right) < w_3 < \frac{\pi w_3}{2\delta\theta} \leq w_4 \leq w_5$$

such that the following hold:

- (P) $f(u, v) < \frac{w_2}{q}$ for $u, v \in [0, w_2]$;
- (Q) $f(u, v) \leq \frac{w_5}{q}$ for $u, v \in [0, w_5]$;
- (R) $f(u, v) > \frac{w_3}{r}$ for $u \in [w_3(\frac{1}{2} - \delta), w_4(\frac{1}{2} - \delta)]$ and $v \in [w_3, w_4]$.

Then we have the following conclusions:

- (a) The Lidstone boundary value problem (3.3) has (at least) three positive solutions $x_1, x_2, x_3 \in \overline{C}(w_5)$ (where C is defined in (3.8)) such that

$$\begin{cases} \|x_1\| < w_2; \\ x_2(t) > w_3, \quad t \in [\delta, 1 - \delta]; \\ \|x_3\| > w_2 \quad \text{and} \quad \min_{t \in [\delta, 1 - \delta]} x_3(t) < w_3. \end{cases} \tag{3.35}$$

- (b) The complementary Lidstone boundary value problem (1.1) has (at least) three positive solutions y_1, y_2, y_3 such that (3.15) holds for $i = 1, 2, 3$. We further have

$$\begin{cases} \|y_1\| < w_2; \\ y_2(t) > w_3(t - \delta), \quad t \in [\delta, 1 - \delta]; \\ y_3(t) > \frac{2\delta\theta}{\pi} w_2(t - \delta), \quad t \in [\delta, 1 - \delta]. \end{cases} \tag{3.36}$$

Remark 3.1 Corollary 3.1 is actually Theorem 3.1. Since Corollary 3.1 is a special case of Theorem 3.2, this shows that Theorem 3.2 is *more general* than Theorem 3.1.

The next theorem illustrates another application of Theorem 2.2. Compared to the conditions in Theorem 3.2, here the numbers w_1, τ_1 and τ_4 have different ranges and condition (P) is also different. Note that in the proof of Theorem 3.3 the functionals ψ and Θ are chosen differently from those in Theorem 3.2.

Theorem 3.3 Let $\delta \in (0, \frac{1}{2})$ be fixed. Let (C1)-(C4) hold. Assume that there exist numbers $\tau_j, 1 \leq j \leq 4$, with

$$\delta \leq \tau_1 \leq \tau_2 < \frac{1}{2} < \tau_3 \leq \tau_4 \leq 1 - \delta$$

such that (C7) and (C8) hold. Suppose that there exist numbers $w_i, 1 \leq i \leq 5$, with

$$\begin{cases} 0 < w_1 \leq \frac{2\delta\theta w_2}{\pi} < w_2 < w_3 < \frac{\pi w_3}{2\delta\theta} \leq w_4 \leq w_5 < \frac{q w_2}{p_3}, \\ \tau_1 w_5 + (\tau_4 - \tau_1) w_2 < w_3 \left(\frac{1}{2} - \tau_2 \right) \end{cases}$$

such that the following hold:

- (P) $f(u, v) < \frac{1}{p_2} \left(w_2 - \frac{w_5 p_3}{q} \right)$ for $u \in [0, \tau_1 w_5 + (\tau_4 - \tau_1) w_2]$ and $v \in [w_1, w_2]$;
- (Q) $f(u, v) \leq \frac{w_5}{q}$ for $u, v \in [0, w_5]$;
- (R) $f(u, v) > \frac{w_3}{p_1}$ for $u \in [w_3(\frac{1}{2} - \tau_2), w_4(\frac{1}{2} - \tau_2)]$ and $v \in [w_3, w_4]$.

Then we have conclusions (a) and (b) of Theorem 3.2.

Proof To apply Theorem 2.2, we shall define the following functionals on the cone C (see (3.8)):

$$\begin{cases} \gamma(x) = \|x\|, \\ \psi(x) = \min_{t \in [\tau_1, \tau_4]} x(t), \\ \beta(x) = \max_{t \in [\tau_1, \tau_4]} x(t), \\ \alpha(u) = \min_{t \in [\tau_2, \tau_3]} x(t), \\ \Theta(u) = \max_{t \in [\tau_2, \tau_3]} x(t). \end{cases} \quad (3.37)$$

As in the proof of Theorem 3.2, using (Q) and Lemma 3.3 we can show that $S : \overline{P}(\gamma, w_5) \rightarrow \overline{P}(\gamma, w_5)$.

Next, to see that condition (a) of Theorem 2.2 is fulfilled, we use (R) and a similar argument as in the proof of Theorem 3.2.

We shall now prove that condition (b) of Theorem 2.2 is satisfied. Note that

$$x(t) = \frac{1}{2}(w_1 + w_2) \in \{x \in Q(\gamma, \beta, \psi, w_1, w_2, w_5) \mid \beta(x) < w_2\} \neq \emptyset.$$

Let $x \in Q(\gamma, \beta, \psi, w_1, w_2, w_5)$. Then we have $\psi(x) \geq w_1$, $\beta(x) \leq w_2$ and $\gamma(x) \leq w_5$ which imply

$$x(s) \in [w_1, w_2], \quad s \in [\tau_1, \tau_4] \quad \text{and} \quad x(s) \in [0, w_5], \quad s \in [0, 1] \quad (3.38)$$

and also (3.30). In view of (3.9), (3.38), (3.30), (C8), (P) and (Q), we obtain, as in the proof of Theorem 3.2, that $\beta(Sx) < w_2$. Therefore, condition (b) of Theorem 2.2 is fulfilled.

Next, using a similar argument as in the proof of Theorem 3.2, we see that condition (c) of Theorem 2.2 is met.

Finally, we shall verify that condition (d) of Theorem 2.2 is fulfilled. Let $x \in C$. It is clear that

$$\beta(Sx) = \max_{t \in [\tau_1, \tau_4]} Sx(t) \leq \|Sx\|. \quad (3.39)$$

Noting that S maps C into C , we find

$$\psi(Sx) = \min_{t \in [\tau_1, \tau_4]} Sx(t) \geq \min_{t \in [\delta, 1-\delta]} Sx(t) \geq \frac{2\delta\theta}{\pi} \|Sx\|. \quad (3.40)$$

A combination of (3.39) and (3.40) gives

$$\psi(Sx) \geq \frac{2\delta\theta}{\pi} \beta(Sx), \quad x \in C. \quad (3.41)$$

Let $x \in Q(\gamma, \beta, w_2, w_5)$ with $\psi(Sx) < w_1$. Then (3.41) and the inequality $w_1 \leq \frac{2\delta\theta w_2}{\pi}$ lead to

$$\beta(Sx) \leq \frac{\pi}{2\delta\theta} \psi(Sx) < \frac{\pi}{2\delta\theta} w_1 \leq \frac{\pi}{2\delta\theta} \frac{2\delta\theta}{\pi} w_2 = w_2.$$

Thus, $\beta(Sx) < w_2$ for all $x \in Q(\gamma, \beta, w_2, w_5)$ with $\psi(Sx) < w_1$.

Conclusion (a) now follows from Theorem 2.2 immediately, while conclusion (b) is similarly obtained as in Theorem 3.2. \square

4 Examples

In this section, we shall present examples to illustrate the usefulness as well as to compare the generality of the results obtained in Section 3.

Example 4.1 Consider the complementary Lidstone boundary value problem (1.1) with $m = 3$ and the nonlinear term F given by

$$\begin{aligned}
 F(t, u, v) &= f(u, v) \\
 &= \begin{cases} \frac{w_1}{2q}, & (u, v) \in [0, w_1] \times [0, w_1] \equiv A_1, \\ \frac{1}{2}\left(\frac{d}{q} + \frac{w_2}{r}\right), & (u, v) \in [w_2(\frac{1}{2} - \delta), \infty) \times [w_2, \infty) \equiv A_2, \\ \phi(u, v), & (u, v) \in \mathbb{R}^2 \setminus (A_1 \cup A_2), \end{cases} \quad (4.1)
 \end{aligned}$$

where $\phi(u, v)$ is continuous in each argument and satisfies

$$\begin{cases} \phi(0, v) = \phi(w_1, v) = \phi(u, 0) = \phi(u, w_1) = \frac{w_1}{2q}, & u, v \in [0, w_1]; \\ \phi(w_2(\frac{1}{2} - \delta), v) = \phi(u, w_2) = \frac{1}{2}\left(\frac{d}{q} + \frac{w_2}{r}\right), & u \in [w_2(\frac{1}{2} - \delta), \infty), v \in [w_2, \infty); \\ 0 \leq \phi(u, v) \leq \frac{1}{2}\left(\frac{d}{q} + \frac{w_2}{r}\right), & (u, v) \in \mathbb{R}^2 \setminus (A_1 \cup A_2). \end{cases} \quad (4.2)$$

Here, $\delta = \frac{1}{4}$ is fixed and the w_i 's and d are in the context of Theorem 3.1 satisfying

$$0 < w_1 < w_2\left(\frac{1}{2} - \delta\right) < w_2 < \frac{\pi w_2}{2\delta\theta} \leq w_3 \leq d \quad \text{and} \quad d > \frac{qw_2}{r}. \quad (4.3)$$

Let the functions $\mu = v \equiv 1$ (which implies $\theta = 1$). Then it is clear that (C1)-(C6) are fulfilled. Moreover, by direct computation we get $q = 2.080 \times 10^{-3}$, and on using Lemma 2.2 we find

$$r = \min_{t \in [\delta, 1-\delta]} \int_{\frac{1}{2}}^{1-\delta} |g_m(t, s)| \mu(s) ds \geq \frac{2\delta}{\pi^{2m}} \int_{\frac{1}{2}}^{1-\delta} \sin \pi s ds = 1.171 \times 10^{-4}.$$

For convenience, we take $r = 1.171 \times 10^{-4}$ although this will lead to more stringent conditions.

Hence, (4.3) reduces to

$$0 < w_1 < 0.25w_2 < w_2 < 6.283w_2 \leq w_3 \leq d \quad \text{and} \quad d > 17.77w_2 \quad (4.4)$$

and clearly we can easily find numbers w_i 's and d that satisfy (4.4).

We shall check the conditions of Theorem 3.1. First, condition (P) is obviously satisfied. Next, from (4.3) we have $\frac{w_2}{r} < \frac{d}{q}$, therefore for $(u, v) \in [0, d] \times [0, d]$ it follows that

$$f(u, v) \leq \frac{1}{2}\left(\frac{d}{q} + \frac{w_2}{r}\right) < \frac{1}{2}\left(\frac{d}{q} + \frac{d}{q}\right) = \frac{d}{q}.$$

Hence, condition (Q2) is met. Finally, (R) is satisfied since for $(u, v) \in [w_2(\frac{1}{2} - \delta), w_3(\frac{1}{2} - \delta)] \times [w_2, w_3]$, we have

$$f(u, v) = \frac{1}{2}\left(\frac{d}{q} + \frac{w_2}{r}\right) > \frac{1}{2}\left(\frac{w_2}{r} + \frac{w_2}{r}\right) = \frac{w_2}{r}.$$

By Theorem 3.1 (conclusion (b)), the boundary value problem (1.1) with $m = 3$, $\delta = \frac{1}{4}$, (4.1) and (4.2) has (at least) three positive solutions y_1, y_2, y_3 such that (from (3.16))

$$\begin{cases} \|y_1\| < w_1; \\ y_2(t) > w_2(t - \frac{1}{4}), & t \in [\frac{1}{4}, \frac{3}{4}]; \\ y_3(t) > \frac{1}{2\pi} w_1(t - \frac{1}{4}), & t \in [\frac{1}{4}, \frac{3}{4}], \end{cases} \quad (4.5)$$

where w_i 's satisfy (4.4).

Example 4.2 Consider the complementary Lidstone boundary value problem (1.1) with $m = 3$ and the nonlinear term F given by

$$F(t, u, v) = f(u, v) = \begin{cases} \frac{1}{2p_2}(w_2 - \frac{w_5 p_3}{q}), & (u, v) \in [0, \tau_1 w_5 + (\tau_4 - \tau_1)w_2] \times [0, w_2] \equiv A_3, \\ \frac{1}{2}(\frac{w_3}{r} + \frac{w_3}{p_1}), & (u, v) \in [w_3(\frac{1}{2} - \tau_2), \infty) \times [w_3, \infty) \equiv A_4, \\ \kappa(u, v), & (u, v) \in \mathbb{R}^2 \setminus (A_3 \cup A_4), \end{cases} \quad (4.6)$$

where $\kappa(u, v)$ is continuous in each argument and satisfies

$$\begin{cases} \kappa(0, v) = \kappa(\tau_1 w_5 + (\tau_4 - \tau_1)w_2, v) = \kappa(u, 0) = \kappa(u, w_2) = \frac{1}{2p_2}(w_2 - \frac{w_5 p_3}{q}), \\ \hspace{15em} u \in [0, \tau_1 w_5 + (\tau_4 - \tau_1)w_2], v \in [0, w_2]; \\ \kappa(w_3(\frac{1}{2} - \tau_2), v) = \kappa(u, w_3) = \frac{1}{2}(\frac{w_3}{r} + \frac{w_3}{p_1}), & u \in [w_3(\frac{1}{2} - \tau_2), \infty), v \in [w_3, \infty); \\ 0 \leq \kappa(u, v) \leq \frac{1}{2}(\frac{w_3}{r} + \frac{w_3}{p_1}), & (u, v) \in \mathbb{R}^2 \setminus (A_3 \cup A_4). \end{cases} \quad (4.7)$$

Here, we fix

$$\delta = 0.05, \quad \tau_1 = 0, \quad \tau_2 = 0.1, \quad \tau_3 = \tau_4 = 1 - \delta = 0.95 \quad (4.8)$$

and the w_i 's are in the context of Theorem 3.2 satisfying

$$\begin{cases} 0 < w_2 < w_3 < \frac{\pi w_3}{2\delta\theta} \leq w_4 \leq w_5 < \frac{qw_2}{p_3}, \\ \tau_1 w_5 + (\tau_4 - \tau_1)w_2 < w_3(\frac{1}{2} - \tau_2) \quad \text{and} \quad w_5 > \frac{qw_3}{r}. \end{cases} \quad (4.9)$$

Let the functions $\mu = v \equiv 1$ (which implies $\theta = 1$). Then it is clear that (C1)-(C4), (C7) and (C8) are fulfilled. Moreover, by direct computation we have

$$q = 2.080 \times 10^{-3}, \quad p_2 = 2.068 \times 10^{-3}, \quad p_3 = 1.281 \times 10^{-5}.$$

By using Lemma 2.2, we get

$$\begin{aligned} p_1 &= \min_{t \in [0.1, 0.95]} \int_{\frac{1}{2}}^{0.95} |g_m(t, s)| \mu(s) ds \\ &> r = \min_{t \in [0.05, 0.95]} \int_{\frac{1}{2}}^{0.95} |g_m(t, s)| \mu(s) ds \\ &\geq \frac{2\delta}{\pi^{2m}} \int_{\frac{1}{2}}^{0.95} \sin \pi s ds \\ &= 3.270 \times 10^{-5}. \end{aligned}$$

For convenience, we take $r = 3.270 \times 10^{-5}$ although this will lead to more stringent conditions. Hence, (4.9) reduces to

$$\begin{cases} 0 < w_2 < w_3 < 31.42w_3 \leq w_4 \leq w_5 < 162.4w_2, \\ 0.95w_2 < 0.4w_3 \quad \text{and} \quad w_5 > 63.61w_3 \end{cases}$$

or equivalently (combining the first two inequalities)

$$\begin{cases} 0 < w_2 < w_3 < 31.42w_3 \leq w_4 \leq w_5 < 162.4w_2 < 68.38w_3 \quad \text{and} \\ w_5 > 63.61w_3. \end{cases} \tag{4.10}$$

It is clear that we can easily find numbers w_i 's that fulfill (4.10).

We shall check the conditions of Theorem 3.2. First, condition (P) is obviously satisfied. Next, since

$$r < p_1 < p_2 \quad \text{and} \quad \frac{w_3}{r} < \frac{w_5}{q} \quad (\text{i.e., } w_5 > 63.61w_3), \tag{4.11}$$

we find for $(u, v) \in [0, w_5] \times [0, w_5]$,

$$f(u, v) \leq \frac{1}{2} \left(\frac{w_3}{r} + \frac{w_3}{p_1} \right) < \frac{1}{2} \left(\frac{w_3}{r} + \frac{w_3}{r} \right) = \frac{w_3}{r} < \frac{w_5}{q}.$$

Hence, condition (Q) is met. Finally, (R) is satisfied since for $(u, v) \in [w_3(\frac{1}{2} - \tau_2), w_4(\frac{1}{2} - \tau_2)] \times [w_3, w_4]$, using (4.11) we get

$$f(u, v) = \frac{1}{2} \left(\frac{w_3}{r} + \frac{w_3}{p_1} \right) > \frac{1}{2} \left(\frac{w_3}{p_1} + \frac{w_3}{p_1} \right) = \frac{w_3}{p_1}.$$

By Theorem 3.2 (conclusion (b)), the boundary value problem (1.1) with $m = 3$, (4.6), (4.7) and (4.8) has (at least) three positive solutions y_1, y_2, y_3 such that (from (3.26))

$$\begin{cases} y_1(t) < 0.95w_2, & t \in [0, 0.95]; \\ y_2(t) > w_3(t - 0.1), & t \in [0.1, 0.95]; \\ y_3(t) > \frac{1}{10\pi}w_2(t - 0.05), & t \in [0.05, 0.95], \end{cases} \tag{4.12}$$

where w_i 's satisfy (4.10).

Remark 4.1 In Example 4.2, we see that for $(u, v) \in [w_3(\frac{1}{2} - \tau_2), w_4(\frac{1}{2} - \tau_2)] \times [w_3, w_4]$,

$$f(u, v) = \frac{1}{2} \left(\frac{w_3}{r} + \frac{w_3}{p_1} \right) < \frac{1}{2} \left(\frac{w_3}{r} + \frac{w_3}{r} \right) = \frac{w_3}{r}.$$

Thus, condition (R) of Corollary 3.1 is *not* satisfied and so Corollary 3.1 *cannot* be used to establish the existence of triple positive solutions in Example 4.2. Recalling that Corollary 3.1 is actually Theorem 3.1, this illustrates the case when Theorem 3.2 is applicable but *not* Theorem 3.1. Hence, this example shows that Theorem 3.2 is indeed more general than Theorem 3.1.

Competing interests

The author declares that she has no competing interests.

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