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Multiplicity of positive solutions for Kirchhoff type problem involving critical exponent and sign-changing weight functions

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Abstract

This paper is devoted to the study of a class of Kirchhoff type problems with critical exponent, concave nonlinearity, and sign-changing weight functions. By means of variational methods, the multiplicity of the positive solutions to this problem is obtained.

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1 Introduction and main results

This paper is concerned with the existence and multiplicity of positive solutions for the following problem:

$$\begin{cases} -(a + b\int_\Omega |\nabla u|^2\,dx)\Delta u = \lambda f(x)u^q + g(x)u^5, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a smooth bounded domain in R^3 with $0 < q < 1$ and the parameters $a, b, \lambda > 0$. The weight functions $f(x)$, $g(x)$ satisfy the following conditions:

(f_1) $f(x) \in L^{q_0}(\Omega)$ and $f^+ = \max\{f, 0\} \not\equiv 0$, where $q_0 = \frac{6}{5-q}$;

(f_2) there exist positive constants β_0 and δ_0 such that $B(x_0, 2\delta_0) \subset \Omega$ and $f(x) \geq \beta_0$ in $B(x_0, 2\delta_0)$;

(g_1) $g(x) \in L^\infty(\Omega)$ and $g^+ = \max\{g, 0\} \not\equiv 0$;

(g_2) $g(x_0) = \|g\|_\infty$ and $g(x) > 0$ for all $x \in B(x_0, 2\delta_0)$;

(g_3) there exists $k > 3$ such that $g(x) = g(x_0) + o(|x - x_0|^k)$ as $x \to x_0$.

In (1), if we replace $\lambda f(x)u^q + g(x)u^5$ by $h(x,u)$, it reduces to the following Dirichlet problem of Kirchhoff type:

$$\begin{cases} -(a + b\int_\Omega |\nabla u|^2\,dx)\Delta u = h(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{2}$$

Problem (2) is related to the stationary analogue of the equation

$$u_{tt} - \left(a + b\int_\Omega |\nabla u|^2\,dx\right)\Delta u = h(x,u),$$

proposed by Kirchhoff in [1] as an extension of the classical d'Alembert wave equation for free vibrations of elastic strings. Kirchhoff's model takes into account the changes in the length of the string produced by transverse vibrations. It received great attention only after Lions [2] proposed an abstract framework for the problem. The solvability of the Kirchhoff type problem (2) has been paid much attention to by various authors. The positive solutions of such a problem are obtained by using variational methods [3–5]. Perera and Zhang [6] obtained a nontrivial solution of problem (2) via the Yang index and the critical group. He and Zou [7] obtained infinitely many solutions by using the local minimum methods and the fountain theorems. Recently, when $h(x,u)$ is a continuous superlinear nonlinearity with critical growth, the existence of positive solutions of the Kirchhoff type problem has been studied [8–13]. Moreover, the paper [14] considered problem (2) with concave and convex nonlinearities by using a Nehari manifold and fibering map methods, and one obtained the existence of multiple positive solutions. In addition, the corresponding results of the Kirchhoff type problem can be found in [15–25], and the references therein.

In the present paper, we deal with problem (1) and consider the existence and multiplicity of positive solutions of problem (1). About the critical growth situation, the aforementioned papers only showed the existence of positive solutions of the Kirchhoff type problem. Moreover, involving the concave and convex nonlinearities, [14] only considered the subcritical growth case. Therefore, our purpose is to extend the result of [14] to critical growth. The main results of this paper extend the corresponding results in [11] and [14].

Before stating our results, we give some notations and assumptions. Let $\|w\| = (\int_\Omega |\nabla w|^2\,dx)^{\frac{1}{2}}$, $\|w\|_s = (\int_\Omega |w|^s\,dx)^{\frac{1}{s}}$ $(1 < s < \infty)$, $B(x_0,\delta) = \{x \in R^3 : |x - x_0| < \delta\}$. In addition, we denote positive constants by $C, C_1, C_2, \ldots$. The main results of this paper are as follows.

Theorem 1 *Let $a > 0$, $b > 0$ and $0 < q < 1$. Suppose that (f_1) and (g_1) hold, then there exists $\Lambda > 0$ such that problem (1) for all $\lambda \in (0,\Lambda)$ has at least one positive solution.*

Theorem 2 *Let $a > 0$, $b > 0$ and $0 < q < 1$. Suppose that (f_1), (f_2), (g_1), (g_2) and (g_3) hold, then there exists $\lambda^* > 0$ such that problem (1) for all $\lambda \in (0,\lambda^*)$ has at least two positive solutions.*

Remark 1 Our Theorem 2 extends the results for the critical case of Theorem 1.1 in [11]. Our Theorem 2 shows that we have at least two positive solutions of problem (1), but the authors of the reference only obtain at least one positive solution of problem (1). In addition, the results of Theorem 2.1 in [14] are extended to critical growth.

This paper is organized as follows. In Section 2, we give the local Palais-Smale condition. The proof of Theorems 1 and 2 is provided in Section 3.

2 The local Palais-Smale condition

In this section, we show that the corresponding functional of problem (1) satisfies the $(PS)_c$ condition. Let $u^\pm = \max\{\pm u, 0\}$, the corresponding functional of problem (1) is

$$I(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{\lambda}{q+1}\int_\Omega f\left(u^+\right)^{q+1}dx - \frac{1}{6}\int_\Omega g\left(u^+\right)^6 dx, \quad u \in H_0^1(\Omega).$$

It is well known that the critical points of the functional I in $H_0^1(\Omega)$ are weak solutions of problem (1). By the definition of weak solution u of problem (1), it means that $u \in H_0^1(\Omega)$ satisfies

$$\langle I'(u), v\rangle = \left(a + b\|u\|^2\right) \int_\Omega \nabla u \cdot \nabla v \, dx - \lambda \int_\Omega f\left(u^+\right)^q v \, dx - \int_\Omega g\left(u^+\right)^5 v \, dx$$

for any $v \in H_0^1(\Omega)$.

Define the best Sobolev constant,

$$S = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\|u\|^2}{(\int_\Omega |u|^6 \, dx)^{\frac{1}{3}}}. \tag{3}$$

From [26], we know that S is attained when $\Omega = R^3$ by functions

$$y_\varepsilon = \frac{(3\varepsilon)^{\frac{1}{4}}}{(\varepsilon + |x - x_0|^2)^{\frac{1}{2}}}.$$

Definition A sequence $\{u_n\} \subset H_0^1(\Omega)$ is called a $(PS)_c$ sequence of I if $I(u_n) \to c$ and $I'(u_n) \to 0$ as $n \to \infty$. We say that I satisfies the $(PS)_c$ condition if any $(PS)_c$ sequence $\{u_n\} \subset H_0^1(\Omega)$ of I has a convergent subsequence.

Lemma 1 *Let $a > 0$, $b > 0$ and $0 < q < 1$. Assume that (f_1) and (g_1) hold. If $\{u_n\} \subset H_0^1(\Omega)$ is a $(PS)_c$ sequence of I, then $\{u_n\}$ is bounded in $H_0^1(\Omega)$.*

Proof By the Hölder inequality and the Young inequality, it follows from (3) and (f_1) that

$$\begin{aligned} \left|\frac{\lambda}{q+1} \int_\Omega f\left(u^+\right)^{q+1} dx\right| &\le \frac{\lambda}{q+1} \|f\|_{q_0} \|u\|_6^{q+1} \\ &\le \frac{\lambda}{q+1} S^{-\frac{q+1}{2}} \|f\|_{q_0} \|u\|^{q+1} \\ &\le \eta \|u\|^2 + C(\eta) \lambda^{\frac{2}{1-q}} \end{aligned} \tag{4}$$

for any $u \in H_0^1(\Omega)$, where $C(\eta) = (\frac{\|f\|_{q_0}}{q+1})^{\frac{2}{1-q}} (\eta S)^{\frac{q+1}{q-1}}$. Let $\{u_n\}$ be a $(PS)_c$ sequence of I. It follows from (4) that

$$6I(u_n) - \langle I'(u_n), u_n\rangle \ge \left[2a - (5-q)\eta\right] \|u_n\|^2 - (5-q) C(\eta) \lambda^{\frac{2}{1-q}},$$

which implies

$$\left[2a - (5-q)\eta\right] \|u_n\|^2 \le 6c + (5-q) C(\eta) \lambda^{\frac{2}{1-q}} + o\left(\|u_n\|\right).$$

Set $\eta < \frac{2a}{5-q}$, we see that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. □

Lemma 2 *Let $a > 0$, $b > 0$, and $0 < q < 1$. Assume that (f_1) and (g_1) hold. If $\{u_n\}$ is a $(PS)_c$ sequence of I, then there exists a positive constant A depending on a, q, S, and $\|f\|_{q_0}$ such that*

$$I'(u) = 0 \quad \textit{and} \quad I(u) \ge -A\lambda^{\frac{2}{1-q}}.$$

Proof Let $\{u_n\}$ be a $(PS)_c$ sequence of I. By Lemma 1, we know that $\{u_n\}$ is bounded. Therefore, up to a subsequence, there exists $u \in H_0^1(\Omega)$ such that u_n converges weakly in $H_0^1(\Omega)$, strongly in $L^s(\Omega)$ with $1 \le s < 6$ and a.e. in Ω. By (f_1), (f_2), and the Dominated Convergence Theorem, we have

$$\lambda \int_\Omega f\bigl(u_n^+\bigr)^q (u_n - u)\,dx + \int_\Omega g\bigl(u_n^+\bigr)^5 (u_n - u)\,dx \to 0.$$

Thus, by using also the fact that $\langle I'(u_n), u_n - u\rangle \to 0$, we get

$$\bigl(a + b\|u_n\|^2\bigr) \int_\Omega \nabla u_n \cdot (\nabla u_n - \nabla u)\,dx \to 0,$$

from which it follows that $u_n \to u$ in $H_0^1(\Omega)$. Since I is C^1, we obtain $I'(u) = 0$. In particular, we have $\langle I'(u), u\rangle = 0$, which implies that

$$a\|u\|^2 + b\|u\|^4 = \lambda \int_\Omega f\bigl(u^+\bigr)^{q+1}\,dx + \int_\Omega g\bigl(u^+\bigr)^6\,dx.$$

It follows from (4) that

$$\begin{aligned} I(u) &= I(u) - \frac{1}{6}\bigl\langle I'(u), u\bigr\rangle \\ &= \frac{a}{3}\|u\|^2 + \frac{b}{12}\|u\|^4 - \frac{5-q}{6(q+1)}\lambda \int_\Omega f\bigl(u^+\bigr)^{q+1}\,dx \\ &\ge \frac{a}{3}\|u\|^2 - \frac{5-q}{6}\bigl(\eta\|u\|^2 + C(\eta)\lambda^{\frac{2}{1-q}}\bigr). \end{aligned}$$

Set $\eta = \frac{2a}{5-q}$, and we have $I(u) \ge -A\lambda^{\frac{2}{1-q}}$. □

Lemma 3 *Let $a > 0$, $b > 0$ and $0 < q < 1$. Assume that (f_1) and (g_1) hold, then I satisfies the $(PS)_c$ condition with $c < c^* = \frac{ab}{4}\|g\|_\infty^{-1}S^3 + \frac{b^3}{24}\|g\|_\infty^{-2}S^6 + \frac{aS}{6}\sqrt{b^2\|g\|_\infty^{-2}S^4 + 4a\|g\|_\infty^{-1}S} + \frac{b^2}{24}\|g\|_\infty^{-1}S^4\sqrt{b^2\|g\|_\infty^{-2}S^4 + 4a\|g\|_\infty^{-1}S} - A\lambda^{\frac{2}{1-q}}$, where A is the positive constant given in Lemma* 2.

Proof Let $\{u_n\} \subset H_0^1(\Omega)$ be a $(PS)_c$ sequence of I with $c < c^*$. By Lemma 1, we know that $\{u_n\}$ is bounded. Up to a subsequence, we may assume that

$$\begin{cases} u_n \rightharpoonup u, & \text{in } H_0^1(\Omega), \\ u_n \to u, & \text{a.e. on } \Omega, \\ u_n \to u, & \text{in } L^s(\Omega), 1 \le s < 6. \end{cases}$$

From Lemma 2, we have $I'(u) = 0$. By (f_1) and the Dominated Convergence Theorem, we obtain

$$\int_\Omega f\bigl(u_n^+\bigr)^{q+1}\,dx = \int_\Omega f\bigl(u^+\bigr)^{q+1}\,dx + o(1).$$

Let $w_n = u_n - u$; by the Brezis-Lieb lemma [27], one has

$$\|u_n\|^2 = \|w_n\|^2 + \|u\|^2 + o(1), \qquad \|u_n\|^4 = \|w_n\|^4 + 2\|w_n\|^2\|u\|^2 + \|u\|^4 + o(1)$$

and

$$\int_\Omega g\left(u_n^+\right)^6\,dx = \int_\Omega g\left(w_n^+\right)^6\,dx + \int_\Omega g\left(u^+\right)^6\,dx + o(1).$$

Since $I(u_n) = c + o(1)$, we obtain

$$\frac{a}{2}\|w_n\|^2 + \frac{b}{4}\|w_n\|^4 + \frac{b}{2}\|w_n\|^2\|u\|^2 - \frac{1}{6}\int_\Omega g\left(w_n^+\right)^6\,dx = c - I(u) + o(1). \tag{5}$$

According to $I'(u_n) = o(1)$ and $\langle I'(u), u\rangle = 0$, we get

$$a\|w_n\|^2 + b\|w_n\|^4 + b\|w_n\|^2\|u\|^2 - \int_\Omega g\left(w_n^+\right)^6\,dx = o(1). \tag{6}$$

Assume that $\|w_n\| \to l$, it follows from (6) that

$$\int_\Omega g\left(w_n^+\right)^6\,dx \to al^2 + bl^4 + bl^2\|u\|^2.$$

From (3), we have

$$\|w_n\|^6 \geq S^3\int_\Omega |w_n|^6\,dx \geq S^3\|g\|_\infty^{-1}\int_\Omega g\left(w_n^+\right)^6\,dx.$$

As $n \to \infty$, we deduce that

$$l^2 \geq \frac{b}{2}\|g\|_\infty^{-1}S^3 + \frac{S}{2}\sqrt{b^2\|g\|_\infty^{-2}S^4 + 4\left(a + b\|u\|^2\right)\|g\|_\infty^{-1}S}.$$

It follows from (5), (6), and Lemma 2 that

$$\begin{aligned}
c &= \frac{a}{3}l^2 + \frac{b}{12}l^4 + \frac{b}{3}l^2\|u\|^2 + I(u) \\
&\geq \frac{a}{3}l^2 + \frac{b}{12}l^4 - A\lambda^{\frac{2}{1-q}} \\
&\geq \frac{ab}{4}\|g\|_\infty^{-1}S^3 + \frac{b^3}{24}\|g\|_\infty^{-2}S^6 + \frac{aS}{6}\sqrt{b^2\|g\|_\infty^{-2}S^4 + 4\left(a + b\|u\|^2\right)\|g\|_\infty^{-1}S} \\
&\quad + \frac{b^2}{24}\|g\|_\infty^{-1}S^4\sqrt{b^2\|g\|_\infty^{-2}S^4 + 4\left(a + b\|u\|^2\right)\|g\|_\infty^{-1}S} - A\lambda^{\frac{2}{1-q}} \\
&\geq c^*,
\end{aligned}$$

which contradicts the fact that $c < c^*$. Therefore, we have $l = 0$, which implies that $u_n \to u$ in $H_0^1(\Omega)$. Hence I satisfies the $(PS)_c$ condition with $c < c^*$. □

3 The proof of the main results

In this section, we show the proofs of our Theorems 1 and 2. Before we come to the proof of Theorem 1, we first recall the following lemma in [28].

Lemma 4 *Let $r, s > 1$, $\psi \in L^s(\Omega)$ and $\psi^+ = \max\{\psi, 0\} \not\equiv 0$. Then there exists $w_0 \in C_0^\infty(\Omega)$ such that $\int_\Omega \psi(w_0^+)^r\,dx > 0$.*

Proof of Theorem 1 Using the hypotheses (f_1) and (g_1), it follows from (3) and (4) that

$$I(u) \geq \left(\frac{a}{2} - \eta\right)\|u\|^2 - C(\eta)\lambda^{\frac{2}{1-q}} - \frac{1}{6}\|g\|_\infty S^{-3}\|u\|^6.$$

Let $\eta = \frac{a}{4}$, we can find $\rho > 0$ and $\Lambda_1 > 0$ such that for all $\lambda \in (0, \Lambda_1)$

$$I(u) > 0 \quad \text{if } \|u\| = \rho \quad \text{and} \quad I(u) > -C_1 \quad \text{if } \|u\| \leq \rho, \tag{7}$$

where $C_1 = (\frac{\|f\|_{q_0}\Lambda_1}{q+1})^{\frac{2}{1-q}}(\frac{a}{4}S)^{\frac{q+1}{q-1}}$.

From Lemma 4, we obtain the result that there exists $\varphi_0 \in C_0^\infty(\Omega) \subset H_0^1(\Omega)$ such that

$$\int_\Omega f\left(\varphi_0^+\right)^{q+1} dx > 0. \tag{8}$$

Therefore, one has

$$I(k\varphi_0) \leq \frac{a}{2}k^2\|\varphi_0\|^2 + \frac{b}{4}k^4\|\varphi_0\|^4 - \frac{\lambda}{q+1}k^{q+1}\int_\Omega f\left(\varphi_0^+\right)^{q+1} dx + \frac{1}{6}k^6\|g\|_\infty\|\varphi_0\|^6.$$

Fix $\lambda \in (0, \Lambda_1)$; noticing that $0 < q < 1$, it implies from (8) that there exists $k_0 = k(\lambda) > 0$ small enough such that $I(k_0\varphi_0) < 0$. Thus we deduce that

$$c_\lambda = \inf_{u \in B_\rho(0)} I(u) < 0 < \inf_{u \in \partial B_\rho(0)} I(u).$$

By applying the Ekeland's variational principle in $\overline{B_\rho(0)}$ [29], we obtain the result that there exists a $(PS)_{c_\lambda}$ sequence $\{u_n\} \subset \overline{B_\rho(0)}$ of I.

By the expression of c^*, we can choose $0 < \Lambda < \Lambda_1$ such that $c^* > 0$ for all $\lambda \in (0, \Lambda)$. It follows from $c_\lambda < 0$ and Lemma 3 that I satisfies the $(PS)_{c_\lambda}$ condition. Therefore, one has a subsequence still denoted by $\{u_n\}$ and $u_\lambda \in H_0^1(\Omega)$ such that $u_n \to u_\lambda$ in $H_0^1(\Omega)$ and

$$I(u_\lambda) = c_\lambda, \qquad I'(u_\lambda) = 0,$$

which implies that u_λ is a solution of problem (1). After a direct calculation, we derive $\|u_\lambda^-\| = \langle I'(u_\lambda), -u_\lambda^-\rangle = 0$, which implies $u_\lambda \geq 0$. Since $I(u_\lambda) = c_\lambda < 0 = I(0)$, we have $u_\lambda \not\equiv 0$. Applying the Harnack inequality [30], we see that u_λ is a positive solution of problem (1). The proof of Theorem 1 is completed. □

Lemma 5 *Let* $a > 0$, $b > 0$ *and* $0 < q < 1$. *Assume that* (f_1), (f_2), (g_1), (g_2), *and* (g_3) *hold, then there exists* $\Lambda^* > 0$, *such that for any* $\lambda \in (0, \Lambda^*)$, *we can find* $\bar{u}_\lambda \in H_0^1(\Omega)$ *such that* $\sup_{t\geq 0} I(t\bar{u}_\lambda) < c^*$.

Proof For convenience, we consider the functional $J : H_0^1(\Omega) \to R$ defined by

$$J(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{1}{6}\int_\Omega g\left(u^+\right)^6 dx$$

for all $u \in H_0^1(\Omega)$. According to (g_2) and (g_3), we can choose such a cut-off function $\phi(x) \in C_0^\infty(\Omega)$ that $\phi(x) = 1$ for $x \in B(x_0, \delta_0)$, $\phi(x) = 0$ for $x \in R^3 \setminus B(x_0, 2\delta_0)$, $0 \leq \phi(x) \leq 1$ and

$|\nabla\phi| \le C_2$, where $C_2 > 0$ is a positive constant. Define

$$u_\varepsilon(x) = \frac{\phi(x)}{(\varepsilon + |x - x_0|^2)^{\frac{1}{2}}}.$$

According to (g_2) and (g_3), similar to the calculation of [31], we have the following estimate (as $\varepsilon \to 0$)

$$\begin{aligned}
&\left(\int_\Omega g|u_\varepsilon|^6\,dx\right)^{\frac{1}{3}} = \varepsilon^{-\frac{1}{2}}\|g\|_\infty^{\frac{1}{3}}|U|^2_{L^6(R^3)} + O(\varepsilon),\\
&\|u_\varepsilon\|^2 = \varepsilon^{-\frac{1}{2}}|\nabla U|^2_{L^2(R^3)} + O(1),\\
&\frac{\|u_\varepsilon\|^2}{(\int_\Omega g|u_\varepsilon|^6\,dx)^{\frac{1}{3}}} = \|g\|_\infty^{-\frac{1}{3}}S + O\big(\varepsilon^{\frac{1}{2}}\big),
\end{aligned} \tag{9}$$

where $U(x) = (1 + |x - x_0|^2)^{-\frac{1}{2}}$. Define

$$h(t) = J(tu_\varepsilon) = \frac{a}{2}t^2\|u_\varepsilon\|^2 + \frac{b}{4}t^4\|u_\varepsilon\|^4 - \frac{1}{6}t^6\int_\Omega g|u_\varepsilon|^6\,dx$$

for all $t \ge 0$. From (9), we have $\lim_{t\to+\infty} h(t) = -\infty$. Note that $h(0) = 0$ and $h(t) > 0$ for $t \to 0^+$, so $\sup_{t\ge0} h(t)$ is attained for some $t_\varepsilon > 0$. By

$$0 = h'(t_\varepsilon) = t_\varepsilon\left(a\|u_\varepsilon\|^2 + bt_\varepsilon^2\|u_\varepsilon\|^4 - t_\varepsilon^4\int_\Omega g|u_\varepsilon|^6\,dx\right),$$

one has

$$t_\varepsilon^2 = \frac{b\|u_\varepsilon\|^4 + \sqrt{b^2\|u_\varepsilon\|^8 + 4a\|u_\varepsilon\|^2\int_\Omega g|u_\varepsilon|^6\,dx}}{2\int_\Omega g|u_\varepsilon|^6\,dx}.$$

Therefore, we deduce from (9) that

$$\begin{aligned}
\sup_{t\ge0} J(tu_\varepsilon) &= \frac{ab\|u_\varepsilon\|^6}{4\int_\Omega g|u_\varepsilon|^6\,dx} + \frac{a\sqrt{b^2\|u_\varepsilon\|^{12} + 4a\|u_\varepsilon\|^6\int_\Omega g|u_\varepsilon|^6\,dx}}{6\int_\Omega g|u_\varepsilon|^6\,dx}\\
&\quad + \frac{b^3\|u_\varepsilon\|^{12}}{24(\int_\Omega g|u_\varepsilon|^6\,dx)^2} + \frac{b^2\sqrt{b^2\|u_\varepsilon\|^{24} + 4a\|u_\varepsilon\|^{18}\int_\Omega g|u_\varepsilon|^6\,dx}}{24(\int_\Omega g|u_\varepsilon|^6\,dx)^2}\\
&= \frac{ab}{4}\|g\|_\infty^{-1}S^3 + O\big(\varepsilon^{\frac{1}{2}}\big) + \frac{b^3}{24}\|g\|_\infty^{-2}S^6 + O\big(\varepsilon^{\frac{1}{2}}\big)\\
&\quad + \frac{aS}{6}\sqrt{b^2\|g\|_\infty^{-2}S^4 + 4a\|g\|_\infty^{-1}S} + O\big(\varepsilon^{\frac{1}{4}}\big)\\
&\quad + \frac{b^2}{24}\|g\|_\infty^{-1}S^4\sqrt{b^2\|g\|_\infty^{-2}S^4 + 4a\|g\|_\infty^{-1}S} + O\big(\varepsilon^{\frac{1}{4}}\big)\\
&= c^* + A\lambda^{\frac{2}{1-q}} + O\big(\varepsilon^{\frac{1}{4}}\big).
\end{aligned} \tag{10}$$

By the expression of c^*, we can choose $\Lambda_2 > 0$ such that $c^* > 0$ for all $\lambda \in (0, \Lambda_2)$. Using the definitions of I and u_ε, from (f_2) and (g_2), we have

$$I(tu_\varepsilon) \le \frac{a}{2}t^2\|u_\varepsilon\|^2 + \frac{b}{4}t^4\|u_\varepsilon\|^4$$

for all $t \geq 0$ and $\lambda > 0$. It follows that there exist $T \in (0,1)$ and $\varepsilon_1 > 0$ such that

$$\sup_{0 \leq t \leq T} I(tu_\varepsilon) \leq c^* \tag{11}$$

for all $0 < \lambda < \Lambda_2$ and $0 < \varepsilon < \varepsilon_1$. Moreover, using the definitions of I and u_ε, it follows from (f_2) and (10) that

$$\begin{aligned}\sup_{t \geq T} I(tu_\varepsilon) &= \sup_{t \geq T}\left(J(tu_\varepsilon) - \frac{\lambda}{q+1} t^{q+1} \int_\Omega f|u_\varepsilon|^{q+1}\,dx\right) \\ &\leq c^* + A\lambda^{\frac{2}{1-q}} + O\big(\varepsilon^{\frac{1}{4}}\big) - \frac{\lambda \beta_0}{q+1} T^{q+1} \int_{B(x_0,\delta_0)} |u_\varepsilon|^{q+1}\,dx.\end{aligned}$$

Let $\varepsilon = \lambda^{\frac{8}{1-q}} \in (0, \delta_0^2)$, it follows that

$$\begin{aligned}\int_{B(x_0,\delta_0)} |u_\varepsilon|^{q+1}\,dx &= \int_{B(x_0,\delta_0)} \frac{1}{(\varepsilon + |x - x_0|^2)^{\frac{q+1}{2}}}\,dx \\ &\geq \int_{B(x_0,\delta_0)} \frac{1}{(2\delta_0^2)^{\frac{q+1}{2}}}\,dx \\ &= C_3.\end{aligned}$$

By the above two inequalities, for any $0 < \lambda < \delta_0^{\frac{1-q}{4}}$, we have

$$\sup_{t \geq T} I(tu_\varepsilon) \leq c^* + O\big(\lambda^{\frac{2}{1-q}}\big) - \frac{C_3 \beta_0 \lambda}{q+1} T^{q+1}. \tag{12}$$

Hence, we can choose $\Lambda_3 > 0$ such that for all $0 < \lambda < \Lambda_3$

$$O\big(\lambda^{\frac{2}{1-q}}\big) - \frac{C_3 \beta_0 \lambda}{q+1} T^{q+1} < 0.$$

Therefore, for all $0 < \lambda < \Lambda_3$ and $\varepsilon = \lambda^{\frac{8}{1-q}}$, we have

$$\sup_{t \geq T} I(tu_\varepsilon) \leq c^*. \tag{13}$$

Set $\Lambda^* = \min\{\Lambda_2, \Lambda_3, \varepsilon_1^{\frac{1-q}{8}}\}$. Let $\lambda \in (0, \Lambda^*)$, $\varepsilon = \lambda^{\frac{8}{1-q}}$ and $\bar{u}_\lambda = u_\varepsilon$, we deduce from (11) and (13) that

$$\sup_{t \geq 0} I(t\bar{u}_\lambda) < c^*. \qquad \square$$

Proof of Theorem 2 Choose $\lambda^* = \min\{\Lambda, \Lambda^*\}$, from the proof of Theorem 1, we have already seen that problem (1) for any $\lambda \in (0, \lambda^*)$ has a positive solution u_λ with $I(u_\lambda) < 0$. Now we only need to find the second positive solution of problem (1). According to (f_1), we can see that (4) and (7) hold. It follows from (g_1) and Lemma 4 that there exists $\phi_0 \in C_0^\infty(\Omega)$ such that

$$\int_\Omega g\big(\phi_0^+\big)^6\,dx > 0.$$

According to (4), we have

$$I(t\phi_0) \le \left(\frac{a}{2} + \eta\right)t^2\|\phi_0\|^2 + \frac{1}{4}t^4\|\phi_0\|^4 - \frac{1}{6}t^6\int_\Omega g\left(\phi_0^+\right)^6\,dx + C(\eta)\lambda^{\frac{2}{1-q}},$$

which implies that

$$I(t\phi_0) \to -\infty \quad \text{as } t \to +\infty.$$

Hence, there exists a positive number t_0 such that $\|t_0\phi_0\| > \rho$ and $I(t_0\phi_0) < 0$ for any $\lambda \in (0, \lambda^*)$. It implies from (7) that the functional I has the mountain pass geometry. Define

$$\Gamma = \left\{\gamma \in C\left([0,1], H_0^1(\Omega)\right) \mid \gamma(0) = 0, \gamma(1) = t_0\phi_0\right\}, \qquad \tilde{c}_\lambda = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I\left(\gamma(t)\right).$$

From Lemma 5, we have $\tilde{c}_\lambda < c^*$. Applying Lemma 3, we know that I satisfies the $(PS)_{\tilde{c}_\lambda}$ condition. By the Mountain Pass Theorem [32], we obtain the result that problem (1) has the second solution $\tilde{u}_\lambda$ with $I(\tilde{u}_\lambda) > 0$. After a direct calculation, we derive

$$\left\|\tilde{u}_\lambda^-\right\|^2 = \left\langle I'(\tilde{u}_\lambda), -\tilde{u}_\lambda^-\right\rangle = 0,$$

which implies that $\tilde{u}_\lambda^- = 0$. Hence we have $\tilde{u}_\lambda \ge 0$. Since $I(\tilde{u}_\lambda) > 0 = I(0)$, we have $\tilde{u}_\lambda \not\equiv 0$. By the Harnack inequality, we obtain the result that $\tilde{u}_\lambda$ is the second positive solution of problem (1). The proof of Theorem 2 is completed. □

Competing interests
The author declares that they have no competing interests.

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