Liu and Jia *Boundary Value Problems* 2014, 2014:1
http://www.boundaryvalueproblems.com/content/2014/1/1

Boundary Value Problems
a SpringerOpen Journal

RESEARCH **Open Access**

Superconvergence patch recovery for the gradient of the tensor-product linear triangular prism element

Jinghong Liu[1*] and Yinsuo Jia[2]

*Correspondence: jhliu1129@sina.com
[1]Department of Fundamental Courses, Ningbo Institute of Technology, Zhejiang University, Qianhu South Road, Ningbo, China
Full list of author information is available at the end of the article

Abstract

In this article, we study superconvergence of the finite element approximation to the solution of a general second-order elliptic boundary value problem in three dimensions over a fully uniform mesh of piecewise tensor-product linear triangular prism elements. First, we give the superclose property of the gradient between the finite element solution u_h and the interpolant Πu. Second, we introduce a superconvergence recovery scheme for the gradient of the finite element solution. Finally, superconvergence of the recovered gradient is derived.

Keywords: superconvergence patch recovery; superclose property; triangular prism element

1 Introduction

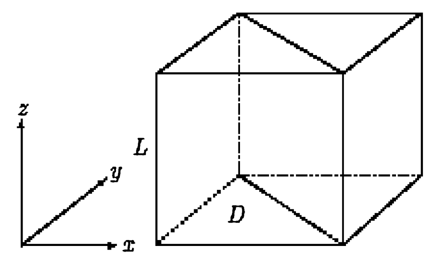
Superconvergence of the gradient for the finite element approximation is a phenomenon whereby the convergent order of the derivatives of the finite element solutions exceeds the optimal global rate. Up to now, superconvergence is still an active research topic (see [1–6]). Recently, we studied the superconvergence patch recovery (SPR) technique introduced by Zienkiewicz and Zhu [7–9] for the linear tetrahedral element and proved pointwise superconvergent property of the recovered gradient by SPR. For the linear tetrahedral element, Chen and Wang [10] also discussed superconvergent properties of the gradients by SPR and obtained superconvergence results of the recovered gradients in the average sense of the L^2-norm. In addition, Chen [11] and Goodsell [12] derived superconvergence estimates of the recovered gradient by the L^2-projection technique and the average technique, respectively. This article will use the SPR technique to obtain a superconvergence estimate for the gradient of the tensor-product linear triangular prism element. In this article, we shall use the letter C to denote a generic constant which may not be the same in each occurrence and also use the standard notations for the Sobolev spaces and their norms.

2 General elliptic boundary value problem and finite element discretization

We consider the model problem

$$\mathcal{L}u \equiv -\sum_{i,j=1}^{3} \partial_j(a_{ij}\partial_i u) + \sum_{i=1}^{3} a_i\partial_i u + a_0 u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{2.1}$$

Figure 1 Triangular prisms partition. This figure gives how to partition the domain Ω. The domain Ω is firstly partitioned into subcubes of side h, and each of these is then subdivided into two triangular prisms.



Here $\Omega = [0,1]^2 \times [0,1] = \Omega_{xy} \times \Omega_z \subset \mathcal{R}^3$ is a rectangular block with boundary $\partial\Omega$ consisting of faces parallel to the x-, y-, and z-axes. We also assume that the given functions $a_{ij}, a_i \in W^{1,\infty}(\Omega)$, $a_0 \in L^\infty(\Omega)$, and $f \in L^2(\Omega)$. In addition, we write $\partial_1 u = \frac{\partial u}{\partial x}$, $\partial_2 u = \frac{\partial u}{\partial y}$, and $\partial_3 u = \frac{\partial u}{\partial z}$, which are usual partial derivatives.

To discretize the problem, one proceeds as follows. The domain Ω is firstly partitioned into subcubes of side h, and each of these is then subdivided into two triangular prisms. We denote by $\{\mathcal{T}^h\}$ these uniform partitions as above. Thus $\bar{\Omega} = \bigcup_{e\in\mathcal{T}^h} \bar{e}$. Obviously, we can write $e = D \times L$ (see Figure 1), where D and L represent a triangle parallel to the xy-plane and a one-dimensional interval parallel to the z-axes, respectively.

We introduce a tensor-product linear polynomial space denoted by $\mathcal{P}$, that is,

$$q(x,y,z) = \sum_{(i,j,k)\in\mathcal{I}} a_{ijk} x^i y^j z^k, \quad a_{ijk} \in \mathcal{R}, q \in \mathcal{P},$$

where $\mathcal{P} = \mathcal{P}_{xy} \otimes \mathcal{P}_z$, $\mathcal{P}_{xy}$ stands for the linear polynomial space with respect to (x,y), and $\mathcal{P}_z$ is the linear polynomial space with respect to z. The indexing set $\mathcal{I}$ satisfies $\mathcal{I} = \{(i,j,k) | i,j,k \geq 0, i+j \leq 1, k \leq 1\}$. Let Π^e_{xy} be the linear interpolation operator with respect to $(x,y) \in D$, and let Π^e_z be the linear interpolation operator with respect to $z \in L$. Thus we may define the tensor-product linear interpolation operator by $\Pi^e : H^1_0(e) \to \mathcal{P}(e)$. Obviously, $\Pi^e = \Pi^e_{xy} \otimes \Pi^e_z = \Pi^e_z \otimes \Pi^e_{xy}$. In addition, the weak form of problem (2.1) is

$$a(u,v) = (f,v) \quad \forall v \in H^1_0(\Omega), \tag{2.2}$$

where

$$a(u,v) \equiv \int_\Omega \left(\sum_{i,j=1}^{3} a_{ij}\partial_i u \, \partial_j v + \sum_{i=1}^{3} a_i \partial_i u v + a_0 u v \right) dx\,dy\,dz,$$

and

$$(f,v) = \int_\Omega f v \, dx\,dy\,dz.$$

Define the tensor-product linear triangular prism finite element space by

$$S^h_0(\Omega) = \left\{ v \in H^1_0(\Omega) : v|_e \in \mathcal{P}(e) \ \forall e \in \mathcal{T}^h \right\}.$$

Thus, the finite element method of problem (2.2) is to find $u_h \in S^h_0(\Omega)$ such that

$$a(u_h, v) = (f,v) \quad \forall v \in S^h_0(\Omega).$$

Moreover, from the definitions of Π^e and $S_0^h(\Omega)$, we may define a global tensor-product linear interpolation operator $\Pi : H_0^1(\Omega) \to S_0^h(\Omega)$. Here $(\Pi u)|_e = \Pi^e u$. As for this interpolation operator, the following Lemma 2.1 holds (see [13]).

Lemma 2.1 *For Πu and u_h, the tensor-product linear interpolant and the tensor-product linear triangular prism finite element approximation to u, respectively, we have the supercloseness estimate*

$$|u_h - \Pi u|_{1,\infty,\Omega} \le Ch^2 |\ln h|^{\frac{4}{3}} \|u\|_{3,\infty,\Omega}. \tag{2.3}$$

3 Gradient recovery and superconvergence

For $v \in S_0^h(\Omega)$, we consider a SPR scheme of ∇v. We denote by R_h the SPR-recovery operator for ∇v and begin by defining the point values of $R_h v$ at the element nodes. After the recovered values at all nodes are obtained, we construct a tensor-product linear interpolation by using these values, namely SPR-recovery gradient $R_h v$. Obviously, $R_h v \in (S_0^h(\Omega))^3$.

Let us first assume that N is an interior node of the partition $\mathcal{T}^h$, and denote by ω the element patch around N containing 12 triangular prisms. Under the local coordinate system centered N, we let $Q_i(x_i, y_i, z_i)$ be the barycenter of a triangular prism $e_i \subset \omega$, $i = 1, 2, \ldots, 12$. Obviously, $\sum_{i=1}^{12}(x_i, y_i, z_i) = (0,0,0)$. SPR uses the discrete least-squares fitting to seek linear vector function $\mathbf{p} \in (P_1(\omega))^3$ such that

$$\sum_{i=1}^{12} \bigl[\mathbf{p}(Q_i) - \nabla v(Q_i)\bigr] q(Q_i) = (0,0,0) \quad \forall q \in P_1(\omega), \tag{3.1}$$

where $v \in S_0^h(\Omega)$. We define $R_h v(N) = \mathbf{p}(0,0,0)$. If N is a boundary node, we calculate $R_h v(N)$ by linear extrapolation from the values of $R_h v$ already obtained at two neighboring interior nodes, N_1 and N_2 (with diagonal directions being used for edge nodes and corner nodes) (see Figure 2). Namely,

$$R_h v(N) = 2R_h v(N_1) - R_h v(N_2).$$

Lemma 3.1 *Let ω be the element patch around an interior node N, and $u \in W^{3,\infty}(\omega)$. For $\Pi u \in S_0^h(\Omega)$ the interpolant to u, we have*

$$\bigl|\nabla u(N) - R_h \Pi u(N)\bigr| \le Ch^2 \|u\|_{3,\infty,\omega}. \tag{3.2}$$

Proof Choose $v = \Pi u$ and set $q = 1$ in (3.1) to obtain $\sum_{i=1}^{12} \mathbf{p}(Q_i) = \sum_{i=1}^{12} \nabla \Pi u(Q_i)$. Therefore,

$$\begin{aligned} R_h \Pi u(N) - \frac{1}{12}\sum_{i=1}^{12} \nabla \Pi u(Q_i) &= \mathbf{p}(0,0,0) - \frac{1}{12}\sum_{i=1}^{12} \mathbf{p}(x_i, y_i, z_i) \\ &= -\frac{1}{12}\sum_{i=1}^{12} \bigl[\partial_1 \mathbf{p}(0,0,0) x_i + \partial_2 \mathbf{p}(0,0,0) y_i + \partial_3 \mathbf{p}(0,0,0) z_i\bigr] \\ &= (0,0,0). \end{aligned}$$

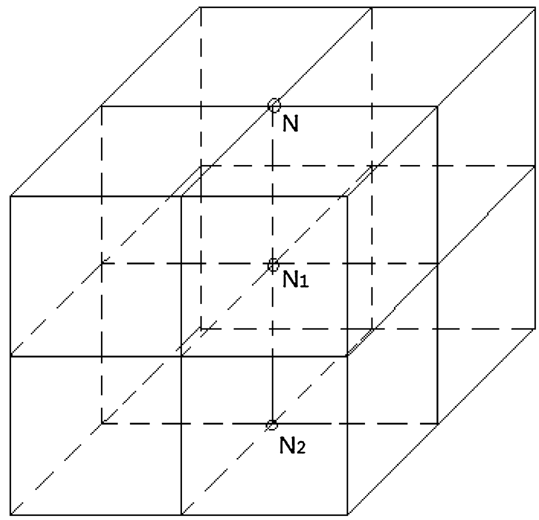


Figure 2 ***N*, a boundary node.** N_1 and N_2 are interior nodes. We can calculate $R_h v(N)$ by linear extrapolation from the values of $R_h v$ already obtained at two neighboring interior nodes, N_1 and N_2.

That is,

$$R_h \Pi u(N) = \frac{1}{12}\sum_{i=1}^{12} \nabla \Pi u(Q_i). \tag{3.3}$$

Further,

$$\begin{aligned}
&\frac{1}{12}\sum_{i=1}^{12} \nabla \Pi u(Q_i) - \nabla u(N) \\
&\quad = \frac{1}{12}\sum_{i=1}^{12} \nabla(\Pi u - u)(Q_i) + \frac{1}{12}\sum_{i=1}^{12}\bigl[\nabla u(Q_i) - \nabla u(N)\bigr] \\
&\quad = \frac{1}{12}\sum_{i=1}^{12} \nabla(\Pi u - u)(Q_i) \\
&\qquad + \frac{1}{12}\sum_{i=1}^{12}\bigl[\partial_1 \nabla u(N) x_i + \partial_2 \nabla u(N) y_i + \partial_3 \nabla u(N) z_i\bigr] + \mathbf{r}(u),
\end{aligned} \tag{3.4}$$

where $\frac{1}{12}\sum_{i=1}^{12}[\partial_1 \nabla u(N) x_i + \partial_2 \nabla u(N) y_i + \partial_3 \nabla u(N) z_i] = (0,0,0)$, and the high-order term $\mathbf{r}(u)$ satisfies $|\mathbf{r}(u)| \le Ch^2 |u|_{3,\infty,\omega}$.

In (3.4), we write $\mathbf{f}(u) = \frac{1}{12}\sum_{i=1}^{12} \nabla(\Pi u - u)(Q_i)$. For every $u \in P_2(\omega)$, it is not difficult to verify $\mathbf{f}(u) = (0,0,0)$. Thus, by the Bramble-Hilbert lemma [14], we have $|\mathbf{f}(u)| \le Ch^2 |u|_{3,\infty,\omega}$. Therefore,

$$\left|\frac{1}{12}\sum_{i=1}^{12} \nabla \Pi u(Q_i) - \nabla u(N)\right| \le Ch^2 |u|_{3,\infty,\omega}. \tag{3.5}$$

Combining (3.3) and (3.5), we obtain the result (3.2). □

Lemma 3.2 *For $\Pi u \in S_0^h(\Omega)$ the tensor-product linear interpolant to u, the solution of* (2.2), *and R_h the SPR recovery operator, we have the superconvergent estimate*

$$|\nabla u - R_h \Pi u|_{0,\infty,\Omega} \le Ch^2 \|u\|_{3,\infty,\Omega}. \tag{3.6}$$

Proof Denote by $F : \hat{e} \to e$ an affine transformation. Obviously, there exists an element $e \in \mathcal{T}^h$, using the triangle inequality and the Sobolev embedding theorem [15], and (3.3), such that

$$\begin{aligned}
|\nabla u - R_h \Pi u|_{0,\infty,\Omega} &= |\nabla u - R_h \Pi u|_{0,\infty,e} \\
&\le Ch^{-1} |\nabla \hat{u} - R_h \widehat{\Pi u}|_{0,\infty,\hat{e}} \\
&\le Ch^{-1} \bigl[|\nabla \hat{u}|_{0,\infty,\hat{e}} + |R_h \widehat{\Pi u}|_{0,\infty,\hat{e}} \bigr] \\
&\le Ch^{-1} \bigl[|\nabla \hat{u}|_{0,\infty,\hat{\chi}} + |\widehat{\Pi u}|_{1,\infty,\hat{\chi}} \bigr] \\
&\le Ch^{-1} \|\hat{u}\|_{3,\infty,\hat{\chi}},
\end{aligned}$$

where $\hat{\chi}$ is a small patch of elements surrounding the triangular prism, $\hat{e}$. Due to the fact that for $\hat{u}$ quadratic over $\hat{\chi}$,

$$\nabla \hat{u} - R_h \widehat{\Pi u} = (0,0,0) \quad \text{in } \hat{\chi},$$

so, from the Bramble-Hilbert lemma [14],

$$|\nabla u - R_h \Pi u|_{0,\infty,\Omega} \le Ch^{-1} |\hat{u}|_{3,\infty,\hat{\chi}} \le Ch^2 |u|_{3,\infty,\Omega},$$

which completes the proof of the result (3.6). Finally, we give the main result in this article. □

Theorem 3.1 *For $u_h \in S_0^h(\Omega)$ the tensor-product linear triangular prism finite element approximation to u, the solution of* (2.2), *and R_h the SPR recovery operator, we have the superconvergence estimate*

$$|\nabla u - R_h u_h|_{0,\infty,\Omega} \le Ch^2 |\ln h|^{\frac{4}{3}} \|u\|_{3,\infty,\Omega}. \tag{3.7}$$

Proof Using the triangle inequality, we have

$$\begin{aligned}
|\nabla u - R_h u_h|_{0,\infty,\Omega} &\le \bigl| R_h (u_h - \Pi u) \bigr|_{0,\infty,\Omega} \\
&\quad + |\nabla u - R_h \Pi u|_{0,\infty,\Omega} \\
&\le |u_h - \Pi u|_{1,\infty,\Omega} \\
&\quad + |\nabla u - R_h \Pi u|_{0,\infty,\Omega},
\end{aligned}$$

which combined with (2.3) and (3.6) completes the proof of the result (3.7). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The first author gave the idea of this article and the methods of proving the main results. He also proved Lemmas 3.1 and 3.2. The second author provided the proof of Theorem 3.1 and the correction of the English language.

Author details
[1]Department of Fundamental Courses, Ningbo Institute of Technology, Zhejiang University, Qianhu South Road, Ningbo, China. [2]School of Mathematics and Computer Science, Shangrao Normal University, Zhimin Road, Shangrao, 334001, China.

Acknowledgements
This work is supported by the National Natural Science Foundation of China Grant 11161039, the Zhejiang Provincial Natural Science Foundation of China Grant LY13A010007, and the Natural Science Foundation of Ningbo City Grant 2013A610104.

Received: 29 July 2013 Accepted: 6 December 2013 Published: 02 Jan 2014

References
1. Babuška, I, Strouboulis, T: The Finite Element Method and Its Reliability, Numerical Mathematics and Scientific Computation. Oxford Science Publications, Oxford (2001)
2. Chen, CM, Huang, YQ: High Accuracy Theory of Finite Element Methods. Hunan Science and Technology Press, Changsha (1995)
3. Chen, CM: Construction Theory of Superconvergence of Finite Elements. Hunan Science and Technology Press, Changsha (2001)
4. Lin, Q, Yan, NN: Construction and Analysis of High Efficient Finite Elements. Hebei University Press, Baoding (1996)
5. Wahlbin, L: Superconvergence in Galerkin Finite Element Methods. Springer, Berlin (1995)
6. Zhu, QD, Lin, Q: Superconvergence Theory of the Finite Element Methods. Hunan Science and Technology Press, Changsha (1989)
7. Zienkiewicz, OC, Zhu, JZ: A simple estimator and adaptive procedure for practical engineering analysis. Int. J. Numer. Methods Eng. **24**, 337-357 (1987)
8. Zienkiewicz, OC, Zhu, JZ: The superconvergence patch recovery and a posteriori error estimates. I. The recovery techniques. Int. J. Numer. Methods Eng. **33**, 1331-1364 (1992)
9. Zienkiewicz, OC, Zhu, JZ: The superconvergence patch recovery and a posteriori error estimates. II. Error estimates and adaptivity. Int. J. Numer. Methods Eng. **33**, 1365-1382 (1992)
10. Chen, J, Wang, DS: Three-dimensional finite element superconvergent gradient recovery on Par6 patterns. Numer. Math., Theory Methods Appl. **3**, 178-194 (2010)
11. Chen, L: Superconvergence of tetrahedral linear finite elements. Int. J. Numer. Anal. Model. **3**, 273-282 (2006)
12. Goodsell, G: Pointwise superconvergence of the gradient for the linear tetrahedral element. Numer. Methods Partial Differ. Equ. **10**, 651-666 (1994)
13. Liu, JH, Deng, YJ, Zhu, QD: High accuracy analysis of tensor-product linear pentahedral finite elements for variable coefficient elliptic equations. J. Syst. Sci. Complex. **25**, 410-416 (2012)
14. Bramble, JH, Hilbert, SR: Estimation of linear functionals on Sobolev spaces with applications to Fourier transforms and spline interpolation. SIAM J. Numer. Anal. **7**, 112-118 (1970)
15. Adams, RA: Sobolev Spaces. Academic Press, New York (1975)

10.1186/1687-2770-2014-1
Cite this article as: Liu and Jia: **Superconvergence patch recovery for the gradient of the tensor-product linear triangular prism element.** *Boundary Value Problems* 2014, 2014:1