Wang et al. *Boundary Value Problems* 2014, 2014:20
http://www.boundaryvalueproblems.com/content/2014/1/20

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Analysis of Abel-type nonlinear integral equations with weakly singular kernels

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Dedicated to Professor Ivan Kiguradze

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Abstract
In this paper, we investigate Abel-type nonlinear integral equations with weakly singular kernels. Existence and uniqueness of nontrivial solution are presented in an order interval of a cone by using fixed point methods. As a byproduct of our method, we improve a gap in the proof of Theorem 5 in Buckwar (Nonlinear Anal. TMA 63:88-96, 2005). As an extension, solutions in closed form of some Erdélyi-Kober-type fractional integral equations are given. Finally theoretical results with three illustrative examples are presented.

MSC: Primary 26A33; secondary 45E10; 45G05

Keywords: Abel-type nonlinear integral equations; weakly singular kernels; existence; numerical solutions

1 Introduction

Abel-type integral equations are associated with a wide range of physical problems such as heat transfer [1], nonlinear diffusion [2], propagation of nonlinear waves [3], and they can also be applied in the theory of neutron transport and in traffic theory. In the past 70 years, many researchers investigated the existence and uniqueness of nontrivial solutions for a large number of Abel-type integral equations by using various analysis methods (see [4–16] and references therein).

Fractional calculus provides a powerful tool for the description of hereditary properties of various materials and memory processes. In particular, integral equations involving fractional integral operators (which can be regarded as an extension of Abel integral equations) appear naturally in the fields of biophysics, viscoelasticity, electrical circuits, and etc. There are some remarkable monographs that provide the main theoretical tools for the qualitative analysis of fractional order differential equations, and at the same time, show the interconnection as well as the contrast between integer order differential models and fractional order differential models [17–24].

It is remarkable that many researchers pay attention to the study of the existence and attractiveness of solutions for fractional integral equations by using functional analysis methods such as the contraction principle, the Schauder fixed point theorem and a Darboux-type fixed point theorem involving a measure of noncompactness (see [25–33] and references therein).

A completely different approach is given in Buckwar [13] to discussing the existence and uniqueness of nontrivial solutions for Abel-type nonlinear integral equation with power-law nonlinearity on an order interval as follows:

$$x^p(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \left[\frac{K(t,s)}{(t-s)^{1-\alpha}}\right] x(s)\,ds, \quad t \in [0,T]. \tag{1}$$

Many analysis techniques are used to construct the suitable order interval (see Lemma 2, [13]) and the spaces with suitable weighted norms.

Motivated by [6, 11, 13, 33], we extend to study the following Abel-type nonlinear integral equation with weakly singular kernels:

$$h\bigl(x(t)\bigr) = \frac{\beta^{1-\alpha}}{\Gamma(\alpha)}\int_0^t \left[\frac{K(t,s)s^{\gamma-1}}{(t^\beta - s^\beta)^{1-\alpha}}\right] g\bigl(x(s)\bigr)\,ds, \quad t \in [0,T], \tag{2}$$

where $h, g \in C([0,M),[0,+\infty))$ are given functions for some $M \in (0,+\infty]$, h is increasing, g is nondecreasing such that

$$a_- x^{p_-} \le h(x) \le a_+ x^{p_+}, \qquad b_- x^{q_-} \le g(x) \le b_+ x^{q_+}, \quad 0 \le x < M, \tag{3}$$

for some positive constants $a_\pm$, $b_\pm$, $p_\pm$, $q_\pm$, $0 < \alpha < 1$, $\gamma \ge \beta > 0$, and $0 < q_+ \le q_- < p_+ \le p_-$, the function $K(t,s)$ is non-negative and it has either the form $K(t,s) = k_1(t^\beta - s^\beta)$ or $K(t,s) = k_2(t,s)$ for some function k_1, k_2 specified later. $\Gamma(\cdot)$ is the Gamma function. Of course, we suppose

$$a_- M^{p_- - p_+} \le a_+, \qquad b_- M^{q_- - q_+} \le b_+. \tag{4}$$

It is obvious that equation (1) or

$$x^p(t) = \frac{\beta^{1-\alpha}}{\Gamma(\alpha)}\int_0^t \left[\frac{K(t,s)s^{\gamma-1}}{(t^\beta - s^\beta)^{1-\alpha}}\right] x^q(s)\,ds, \quad t \in [0,T], \tag{5}$$

are special cases of equation (2), which of course all have trivial solutions.

Thus, the main purpose of this paper is to prove the existence and uniqueness of nontrivial solutions for equation (2). The key difficult comes from the weakly singular kernels $(t^\beta - s^\beta)^{1-\alpha}$ and nonlinear terms in equation (2). Although we are motivated by [13], we have to introduce novel techniques and results to overcome the difficult from the weakly singular kernels and nonlinear terms h and g. For example, the first important step is how to construct a suitable order interval to help us to apply the fixed point theorem in such an order interval. More details of the novel techniques and results will be found in the proof. As a byproduct of our method, we improve a gap in the proof of [13, Theorem 5]. So even for equation (1) (or (5)) we get a new result.

As an extension, we find general solutions in closed form of some Erdélyi-Kober-type fractional integral equations (the special case of equation (5) if $b = 0$):

$$\varphi^m(x) = a x^{\frac{\beta(m-N)}{N}} \bigl({}_{EK}I^\alpha_{0+;\sigma,\eta}\varphi^N\bigr)(x) + b x^{\frac{\beta m}{N}}, \quad x > 0, \tag{6}$$

where $\alpha, b, \sigma \ge 0$, $N \neq 0$, and $\eta \in \mathbb{R}$ and the symbol ${}_{EK}I^\alpha_{0+;\sigma,\eta}\varphi^N$ denotes the Erdélyi-Kober-type fractional integrals [19] of the function φ^N, which is given by

$$\bigl({}_{EK}I^\alpha_{0+;\sigma,\eta}\varphi^N\bigr)(x) := \frac{\sigma x^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)}\int_0^x \frac{t^{\sigma\eta+\sigma-1}\varphi^N(t)\,dt}{(x^\sigma - t^\sigma)^{1-\alpha}}, \quad x > 0.$$

The plan of this paper is as follows. In Section 2, some notation and preparation results are given. Existence and uniqueness results of a nontrivial solution of equation (2) in an order interval are given in Section 3. In Section 4, we find general solutions in closed form of some Erdélyi-Kober-type fractional integral equations, and finally theoretical results with three illustrate examples are presented in Section 5.

2 Preliminary

Let $\mathcal{M}$ be the set $\mathcal{M} := \{f \in C[0,T] : f(0) = 0\}$ with the supremum-norm $\|f\|_{\mathcal{M}} := \sup_{0<t\leq T}\{|f(t)|\}$. Clearly, the set $(\mathcal{M}, \|\cdot\|_{\mathcal{M}})$ is a closed subspace of Banach space $(C[0,T], \|\cdot\|_C)$. Thus, $(\mathcal{M}, \|\cdot\|_{\mathcal{M}})$ is a Banach space.

Let q be a continuous function on $[0,T]$ with $q(t) > 0$ for all $t > 0$ and let $\mathcal{M}_q$ be the set

$$\mathcal{M}_q := \left\{f \in \mathcal{M} : \sup_{0<t\leq T} \frac{|f(t)|}{q(t)} < \infty\right\}$$

with the weighted norm

$$\|f\|_q := \sup_{0<t\leq T}\left\{\frac{|f(t)|}{q(t)}\right\}. \tag{7}$$

Remark 2.1 If $q(0) > 0$, then the set $\mathcal{M}_q$ is the same as the set $\mathcal{M}$, but with an equivalent norm, and the constants can be determined in the following inequality:

$$c_2\|u - v\|_g \leq \|u - v\|_{\mathcal{M}} \leq c_1\|u - v\|_g,$$

where $c_1 = \max_{t\in[0,T]}\{g(t)\}$ and $c_2 = \min_{t\in[0,T]}\{g(t)\}$. Note that the similar inequality (5) of [13] is incorrect.

Consider the cone $P_{\mathcal{M}} := \{u \in \mathcal{M} : u(t) \geq 0, t \in [0,T]\}$ in $\mathcal{M}$. The so-called partial ordering induced by the cone $P_{\mathcal{M}}$ is given by $u \leq v \Longleftrightarrow u(t) \leq v(t)$ for all $u, v \in \mathcal{M}$ and all $t \in [0,T]$. In general [34, 35], a set $[f,g] = \{h \in E : f \leq h \leq g\}$ is called an order interval where E is an ordered Banach space. We know that every order interval $[f,g]$ is closed. Moreover, if $\|f\|_E \leq \|g\|_E$ for all $f, g \in E$ with $0 \leq f \leq g$, then every order interval $[f,g]$ is bounded.

We introduce some conditions on the functions K, k_i, $i = 1,2$ as follows:

(i) $k_1 \in C^{(n)}[0,T]$ where $n \in \{0,1,2,\ldots\}$, and $0 \neq k_2 \in C[0,T]^2$.

(ii) $k_1(t) > 0$ for all $t \in (0,T]$ and $k_2(t,s) \geq 0$ for all $0 \leq s \leq t \leq T$.

(iii) $k_1(0) = k_1^{(1)}(0) = \cdots = k_1^{(n-1)}(0) = 0$, and $k_1^{(n)}(t) \geq k_1^{(n)}(0) > 0$, for all $t \in (0,T]$.

- For $K(t,s) = k_1(t^\beta - s^\beta)$ and $n \geq 1$, we set

$$\mathbb{K}_{\text{low}} = k_1^{(n)}(0), \qquad \mathbb{K}_{\text{up}} = \max_{t\in[0,T]}\left\{k_1^{(n)}(t)\right\}. \tag{8}$$

- For $K(t,s) = k_2(t,s)$, we set $n = 0$ and

$$\mathbb{K}_{\text{low}} = K_{\min} := \min_{0\leq s\leq t\leq T}\left\{k_2(t,s)\right\}, \qquad \mathbb{K}_{\text{up}} = K_{\max} := \max_{0\leq s\leq t\leq T}\left\{k_2(t,s)\right\}. \tag{9}$$

Similarly for $K(t,s) = k_1(t^\beta - s^\beta)$ and $n = 0$.

We note [34] an important estimate on the function K, which will be used in the sequel.

Lemma 2.2 *The function* $K(\cdot,\cdot)$ *has the following estimate*:

$$\frac{1}{n!}\left(t^\beta - s^\beta\right)^n \mathbb{K}_{\mathrm{low}} \le K(t,s) \le \frac{1}{n!}\left(t^\beta - s^\beta\right)^n \mathbb{K}_{\mathrm{up}}, \quad 0 \le s \le t \le T. \tag{10}$$

Proof We only check the case of $K(t,s) = k_1(t^\beta - s^\beta)$ with $n \ge 1$, since the other cases are trivial.

Integrating n times step-by-step all sides of the inequality

$$k_1^{(n)}(0) \le k_1^{(n)}(t) \le \mathbb{K}_{\mathrm{up}}$$

from 0 to t and using $k_1(0) = k_1^{(1)}(0) = \cdots = k_1^{(n-1)}(0) = 0$ we immediately derive

$$\frac{t^n}{n!} k_1^{(n)}(0) \le k_1(t) \le \frac{t^n}{n!}\mathbb{K}_{\mathrm{up}}.$$

Replacing t by $(t^\beta - s^\beta)$, we obtain the desired result. □

To end this section, we collect the following basic facts, which will be used several times in the next section.

Lemma 2.3 *Let* λ, γ, μ, *and* ν *be constants such that* $\lambda > 0$, $\mathrm{Re}(\gamma) > 0$, $\mathrm{Re}(\mu) > 0$, *and* $\mathrm{Re}(\nu) > 0$. *Then*

$$\int_0^t \left(t^\lambda - s^\lambda\right)^{\nu-1} s^{\mu-1}\,ds = \frac{t^{\lambda(\nu-1)+\mu}}{\lambda}\mathbb{B}\left(\frac{\mu}{\lambda},\nu\right), \quad t \in [0,+\infty),$$

and

$$\int_a^t \left(t^\lambda - s^\lambda\right)^{\nu-1} s^{\gamma-1}\left(s^\lambda - a^\lambda\right)^{\mu}\,ds$$
$$\ge \frac{(t^\lambda - a^\lambda)^{\nu+\frac{\gamma}{\lambda}+\mu-1}}{\lambda}\mathbb{B}\left(\frac{\gamma}{\lambda}+\mu,\nu\right), \quad t \in [a,+\infty), a \ge 0,$$

where

$$\mathbb{B}(\xi,\eta) = \int_0^1 s^{\xi-1}(1-s)^{\eta-1}\,ds \quad \left(\mathrm{Re}(\xi) > 0, \mathrm{Re}(\eta) > 0\right)$$

is the well-known Beta function.

Proof The first result have been reported in [36] or [37, Formula 3.251]. We only verify the second inequality. In fact, for any $t \in [a,+\infty)$, $a \ge 0$, we derive

$$\int_a^t \left(t^\lambda - s^\lambda\right)^{\nu-1} s^{\gamma-1}\left(s^\lambda - a^\lambda\right)^{\mu}\,ds$$
$$= \frac{1}{\lambda}\int_{a^\lambda}^{t^\lambda} \left(t^\lambda - u\right)^{\nu-1} u^{\frac{\gamma-1}{\lambda}}\left(u - a^\lambda\right)^{\mu} u^{\frac{1}{\lambda}-1}\,du \quad \left(\text{set } u = s^\lambda\right)$$

$$
\begin{aligned}
&= \frac{1}{\lambda}\int_{a^\lambda}^{t^\lambda} \left(t^\lambda - u\right)^{\nu-1} u^{\frac{\gamma}{\lambda}-1}\left(u - a^\lambda\right)^{\mu}\, du \\
&= \frac{1}{\lambda}\int_{0}^{t^\lambda - a^\lambda} \left(t^\lambda - a^\lambda - z\right)^{\nu-1} \left(a^\lambda + z\right)^{\frac{\gamma}{\lambda}-1} z^{\mu}\, du \quad \left(\text{set } u = a^\lambda + z\right) \\
&\geq \frac{1}{\lambda}\int_{0}^{t^\lambda - a^\lambda} \left(t^\lambda - a^\lambda - z\right)^{\nu-1} z^{\frac{\gamma}{\lambda}+\mu-1}\, du \\
&= \frac{(t^\lambda - a^\lambda)^{\nu+\frac{\gamma}{\lambda}+\mu-1}}{\lambda}\mathbb{B}\left(\frac{\gamma}{\lambda}+\mu, \nu\right).
\end{aligned}
$$

The proof is completed. □

3 Existence and uniqueness of nontrivial solution in an order interval

In this section, we will use the fixed point method to prove the existence and uniqueness of nontrivial solution for equation (2) in an order interval.

For all $t \in [0, T]$, we introduce the following functions:

$$
\begin{aligned}
&F(t) = At^{\tau_-}, \qquad G(t) = Bt^{\tau_+}, \\
&A = \left(\frac{b_-\beta^{-\alpha}\mathbb{K}_{\text{low}}}{a_+\Gamma(\alpha)n!}\mathbb{B}\left(\frac{\beta(n+\alpha-1)q_- + \gamma p_+}{\beta(p_+ - q_-)}, n+\alpha\right)\right)^{\frac{1}{p_+-q_-}}, \\
&B = \left(\frac{b_+\beta^{-\alpha}\mathbb{K}_{\text{up}}}{a_-\Gamma(\alpha)n!}\mathbb{B}\left(\frac{\beta(n+\alpha-1)q_+ + \gamma p_-}{\beta(p_- - q_+)}, n+\alpha\right)\right)^{\frac{1}{p_--q_+}}, \\
&\tau_- = \frac{\beta(n+\alpha-1)+\gamma}{p_+ - q_-}, \qquad \tau_+ = \frac{\beta(n+\alpha-1)+\gamma}{p_- - q_+},
\end{aligned}
$$

where $\mathbb{K}_{\text{low}}$ and $\mathbb{K}_{\text{up}}$ are defined in equation (8) or (9).

Remark 3.1 Note that $\beta(n+\alpha-1)q_\mp + \gamma p_\pm \geq \beta(\alpha-1)q_\mp + \beta p_\pm = \beta(p_\pm - q_\mp) + \alpha\beta q_\mp > 0$ and $\beta(n+\alpha-1)+\gamma \geq \beta(\alpha-1)+\beta = \alpha\beta > 0$. Next, $\tau_- \geq \tau_+$.

The following result is clear.

Lemma 3.2 *If*

$$
AT^{\tau_- - \tau_+} \leq B < MT^{-\tau_+} \tag{11}
$$

then $F(t) \leq G(t) < M$ for all $t \in [0, T]$. Consequently, the order interval $[F, G] \subset P_{\mathcal{M}}$ is well defined.

Remark 3.3 If $p_+ = p_-$, $q_+ = q_-$, and $M = +\infty$ then equation (11) reads

$$
\frac{a_+b_+}{a_-b_-} \geq \frac{\mathbb{K}_{\text{low}}}{\mathbb{K}_{\text{up}}}
$$

which is satisfied, since equation (3) implies $a_+ \geq a_- > 0$, $b_+ \geq b_- > 0$ (see equation (4)) and clearly $\frac{\mathbb{K}_{\text{low}}}{\mathbb{K}_{\text{up}}} \leq 1$. This case occurs for instance when $h(x) = x^p\tilde{h}(x)$ and $g(x) = x^q\tilde{g}(x)$ with $p > q$ and $0 < \inf_{\mathbb{R}} h(x) \leq \sup_{\mathbb{R}} h(x) < \infty$, $0 < \inf_{\mathbb{R}} g(x) \leq \sup_{\mathbb{R}} g(x) < \infty$.

From now on, we suppose that all above assumptions hold: equations (3), (4), (i)-(iii), and (11).

Lemma 3.4 *Any solution $x \in P_{\mathcal{M}}$ of equation* (2), *with $M > x(t) > 0$ for all $t \in (0, T]$, satisfies $x \in [F, G]$.*

Proof Step 1: We prove that $x \leq G$ for a solution x of equation (2).

Set $x_+(t) = \max_{s\in[0,t]} x(s) = x(s_t)$. Then we obtain

$$\begin{aligned} a_- x^{p_-}(t) &\leq h\bigl(x(t)\bigr) \\ &= \frac{\beta^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \frac{K(t,s)s^{\gamma-1}}{(t^\beta - s^\beta)^{1-\alpha}} g\bigl(x(s)\bigr)\,ds \\ &\leq b_+ x_+^{q_+}(t) \frac{\beta^{1-\alpha}}{\Gamma(\alpha)} \int_0^{s_t} \frac{K(s_t,s)s^{\gamma-1}}{(s_t^\beta - s^\beta)^{1-\alpha}}\,ds \\ &\leq b_+ x_+^{q_+}(t) \frac{\beta^{1-\alpha}\mathbb{K}_{\mathrm{up}}}{\Gamma(\alpha)n!} \int_0^{s_t} \bigl(s_t^\beta - s^\beta\bigr)^{n+\alpha-1} s^{\gamma-1}\,ds \\ &= b_+ x_+^{q_+}(t) \frac{\beta^{-\alpha}\mathbb{K}_{\mathrm{up}}}{\Gamma(\alpha)n!} \mathbb{B}\biggl(\frac{\gamma}{\beta}, n+\alpha\biggr) s_t^{\beta(n+\alpha-1)+\gamma} \\ &\leq b_+ x_+^{q_+}(t) \frac{\beta^{-\alpha}\mathbb{K}_{\mathrm{up}}}{\Gamma(\alpha)n!} \mathbb{B}\biggl(\frac{\gamma}{\beta}, n+\alpha\biggr) t^{\beta(n+\alpha-1)+\gamma}, \end{aligned}$$

which implies that

$$x(t) \leq x_+(t) \leq \biggl(\frac{b_+\beta^{-\alpha}\mathbb{K}_{\mathrm{up}}}{a_-\Gamma(\alpha)n!} \mathbb{B}\biggl(\frac{\gamma}{\beta}, n+\alpha\biggr)\biggr)^{\frac{1}{p_--q_+}} t^{\frac{\beta(n+\alpha-1)+\gamma}{p_--q_+}}. \tag{12}$$

Next we set

$$\Xi := \sup_{t\in(0,T]} \frac{x(t)}{t^{\tau_+}} \leq \biggl(\frac{b_+\beta^{-\alpha}\mathbb{K}_{\mathrm{up}}}{a_-\Gamma(\alpha)n!} \mathbb{B}\biggl(\frac{\gamma}{\beta}, n+\alpha\biggr)\biggr)^{\frac{1}{p_--q_+}}.$$

Then we have

$$\begin{aligned} a_- x^{p_-}(t) &\leq h\bigl(x(t)\bigr) \\ &= \frac{\beta^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \frac{K(t,s)s^{\gamma-1}}{(t^\beta - s^\beta)^{1-\alpha}} g\bigl(x(s)\bigr)\,ds \\ &\leq b_+ \Xi^{q_+} \frac{\beta^{1-\alpha}\mathbb{K}_{\mathrm{up}}}{\Gamma(\alpha)n!} \int_0^t \bigl(t^\beta - s^\beta\bigr)^{n+\alpha-1} s^{\gamma-1} s^{q_+\tau_+}\,ds \\ &= b_+ \Xi^{q_+} \frac{\beta^{-\alpha}\mathbb{K}_{\mathrm{up}}}{\Gamma(\alpha)n!} \mathbb{B}\biggl(\frac{\gamma+q_+\tau_+}{\beta}, n+\alpha\biggr) t^{\beta(n+\alpha-1)+\gamma+q_+\tau_+} \\ &\leq b_+ \Xi^{q_+} \frac{\beta^{-\alpha}\mathbb{K}_{\mathrm{up}}}{\Gamma(\alpha)n!} \mathbb{B}\biggl(\frac{\beta(n+\alpha-1)q_+ + \gamma p_-}{\beta(p_- - q_+)}, n+\alpha\biggr) t^{p_-\tau_+}, \end{aligned}$$

and so

$$\frac{x(t)}{t^{\tau_+}} \leq \Xi^{\frac{q_+}{p_-}} \biggl(\frac{b_+\beta^{-\alpha}\mathbb{K}_{\mathrm{up}}}{a_-\Gamma(\alpha)n!} \mathbb{B}\biggl(\frac{\beta(n+\alpha-1)q_+ + \gamma p_-}{\beta(p_- - q_+)}, n+\alpha\biggr)\biggr)^{\frac{1}{p_-}},$$

hence

$$\Xi \leq \Xi^{\frac{q_+}{p_-}} \left(\frac{b_+ \beta^{-\alpha} \mathbb{K}_{\mathrm{up}}}{a_- \Gamma(\alpha) n!} \mathbb{B}\left(\frac{\beta(n+\alpha-1)q_+ + \gamma p_-}{\beta(p_- - q_+)}, n+\alpha \right) \right)^{\frac{1}{p_-}},$$

thus

$$\Xi \leq \left(\frac{b_+ \beta^{-\alpha} \mathbb{K}_{\mathrm{up}}}{a_- \Gamma(\alpha) n!} \mathbb{B}\left(\frac{\beta(n+\alpha-1)q_+ + \gamma p_-}{\beta(p_- - q_+)}, n+\alpha \right) \right)^{\frac{1}{p_- - q_+}},$$

consequently

$$x(t) \leq \left(\frac{b_+ \beta^{-\alpha} \mathbb{K}_{\mathrm{up}}}{a_- \Gamma(\alpha) n!} \mathbb{B}\left(\frac{\beta(n+\alpha-1)q_+ + \gamma p_-}{\beta(p_- - q_+)}, n+\alpha \right) \right)^{\frac{1}{p_- - q_+}} t^{\tau_+} = G(t). \tag{13}$$

Since $\frac{\beta(n+\alpha-1)q_+ + \gamma p_-}{\beta(p_- - q_+)} > \frac{\gamma}{\beta}$ implies $\mathbb{B}(\frac{\beta(n+\alpha-1)q_+ + \gamma p_-}{\beta(p_- - q_+)}, n+\alpha) < \mathbb{B}(\frac{\gamma}{\beta}, n+\alpha)$, estimate (13) is an improvement of equation (12).

Step 2: We prove that $x \geq F$. Fix $a \in (0,T)$ and set

$$\Upsilon_a := \inf_{t \in (a,T]} \frac{x(t)}{(t^\beta - a^\beta)^\Theta} > 0$$

for $\Theta := \frac{\beta(n+\alpha-1)+\gamma}{\beta(p_+ - q_-)} > 0$. Then like above, for $t \in (a,T]$, we get

$$\begin{aligned}
a_+ x^{p_+}(t) &\geq h\bigl(x(t)\bigr) \\
&= \frac{\beta^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \frac{K(t,s) s^{\gamma-1}}{(t^\beta - s^\beta)^{1-\alpha}} g\bigl(x(s)\bigr)\, ds \\
&\geq b_- \Upsilon_a^{q_-} \frac{\beta^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \frac{K(t,s) s^{\gamma-1}}{(t^\beta - s^\beta)^{1-\alpha}} \bigl(s^\beta - a^\beta\bigr)^{q_- \Theta}\, ds \\
&\geq b_- \Upsilon_a^{q_-} \frac{\beta^{1-\alpha} \mathbb{K}_{\mathrm{low}}}{\Gamma(\alpha) n!} \int_a^t \bigl(t^\beta - s^\beta\bigr)^{n+\alpha-1} s^{\gamma-1} \bigl(s^\beta - a^\beta\bigr)^{q_- \Theta}\, ds \\
&\geq b_- \Upsilon_a^{q_-} \frac{\beta^{-\alpha} \mathbb{K}_{\mathrm{low}}}{\Gamma(\alpha) n!} \mathbb{B}\left(\frac{\gamma}{\beta} + q_- \Theta, n+\alpha \right) \bigl(t^\beta - a^\beta\bigr)^{n+\alpha-1+\frac{\gamma}{\beta}+q_- \Theta} \\
&= b_- \Upsilon_a^{q_-} \frac{\beta^{-\alpha} \mathbb{K}_{\mathrm{low}}}{\Gamma(\alpha) n!} \mathbb{B}\left(\frac{\beta(n+\alpha-1)q_- + \gamma p_+}{\beta(p_+ - q_-)}, n+\alpha \right) \bigl(t^\beta - a^\beta\bigr)^{p_+ \Theta},
\end{aligned}$$

which implies

$$\left(\frac{x(t)}{(t^\beta - a^\beta)^\Theta} \right)^{p_+} \geq \Upsilon_a^{q_-} \frac{b_- \beta^{-\alpha} \mathbb{K}_{\mathrm{low}}}{a_+ \Gamma(\alpha) n!} \mathbb{B}\left(\frac{\beta(n+\alpha-1)q_- + \gamma p_+}{\beta(p_+ - q_-)}, n+\alpha \right).$$

Hence

$$\Upsilon_a^{p_+} \geq \Upsilon_a^{q_-} \frac{b_- \beta^{-\alpha} \mathbb{K}_{\mathrm{low}}}{a_+ \Gamma(\alpha) n!} \mathbb{B}\left(\frac{\beta(n+\alpha-1)q_- + \gamma p_+}{\beta(p_+ - q_-)}, n+\alpha \right),$$

and so

$$\Upsilon_a \geq \left(\frac{b_- \beta^{-\alpha} \mathbb{K}_{\mathrm{low}}}{a_+ \Gamma(\alpha) n!} \mathbb{B}\left(\frac{\beta(n+\alpha-1)q_- + \gamma p_+}{\beta(p_+ - q_-)}, n+\alpha \right) \right)^{\frac{1}{p_+ - q_-}}.$$

Consequently, we arrive at

$$\begin{aligned} x(t) &\geq \Upsilon_a\big(t^\beta - a^\beta\big)^\Theta \\ &\geq \left(\frac{b_- \beta^{-\alpha}\mathbb{K}_{\text{low}}}{a_+\Gamma(\alpha)n!}\mathbb{B}\left(\frac{\beta(n+\alpha-1)q_- + \gamma p_+}{\beta(p_+ - q_-)}, n+\alpha\right)\right)^{\frac{1}{p_+ - q_-}} \big(t^\beta - a^\beta\big)^\Theta. \end{aligned}$$

Since $a \in (0,T)$ is arbitrarily, we have

$$x(t) \geq \left(\frac{b_- \beta^{-\alpha}\mathbb{K}_{\text{low}}}{a_+\Gamma(\alpha)n!}\mathbb{B}\left(\frac{\beta(n+\alpha-1)q_- + \gamma p_+}{\beta(p_+ - q_-)}, n+\alpha\right)\right)^{\frac{1}{p_+ - q_-}} t^{\beta\Theta} = F(t).$$

Hence we can complete the proof. □

To solve equation (2), we introduce an operator $S_{h,g} : [F,G] \subset P_{\mathcal{M}} \to C[0,T]$ by

$$S_{h,g}(x)(t) = h^{-1}\left(\frac{\beta^{1-\alpha}}{\Gamma(\alpha)}\int_0^t \left[\frac{K(t,s)s^{\gamma-1}}{(t^\beta - s^\beta)^{1-\alpha}}\right] g\big(x(s)\big)\,ds\right), \quad t \in [0,T]. \tag{14}$$

Lemma 3.5 *The operator $S_{h,g}$ maps the order interval $[F,G]$ into itself.*

Proof To achieve our aim, we only need to verify that $SF \geq F$ and $SG \leq G$:

$$h\big(F(t)\big) \leq \frac{\beta^{1-\alpha}}{\Gamma(\alpha)}\int_0^t \left[\frac{K(t,s)s^{\gamma-1}}{(t^\beta - s^\beta)^{1-\alpha}}\right] g\big(F(s)\big)\,ds, \quad t \in [0,T], \tag{15}$$

$$h\big(G(t)\big) \geq \frac{\beta^{1-\alpha}}{\Gamma(\alpha)}\int_0^t \left[\frac{K(t,s)s^{\gamma-1}}{(t^\beta - s^\beta)^{1-\alpha}}\right] g\big(G(s)\big)\,ds, \quad t \in [0,T]. \tag{16}$$

First we show equation (15):

$$\begin{aligned} &\frac{\beta^{1-\alpha}}{\Gamma(\alpha)}\int_0^t \left[\frac{K(t,s)s^{\gamma-1}}{(t^\beta - s^\beta)^{1-\alpha}}\right] g\big(F(s)\big)\,ds \\ &\quad\geq \frac{b_-\beta^{1-\alpha}A^{q_-}\mathbb{K}_{\text{low}}}{\Gamma(\alpha)n!}\int_0^t \big(t^\beta - s^\beta\big)^{n+\alpha-1} s^{q_-\tau_- + \gamma - 1}\,ds \\ &\quad= \frac{b_-\beta^{-\alpha}A^{q_-}\mathbb{K}_{\text{low}}}{\Gamma(\alpha)n!}\mathbb{B}\left(\frac{q_-\tau_- + \gamma}{\beta}, n+\alpha\right) t^{\beta(n+\alpha-1)+q_-\tau_-+\gamma} \\ &\quad= a_+A^{p_+}t^{p_+\tau_-} \geq h\big(F(t)\big). \end{aligned}$$

Secondly we derive equation (16):

$$\begin{aligned} &\frac{\beta^{1-\alpha}}{\Gamma(\alpha)}\int_0^t \left[\frac{K(t,s)s^{\gamma-1}}{(t^\beta - s^\beta)^{1-\alpha}}\right] g\big(G(s)\big)\,ds \\ &\quad\leq \frac{b_+\beta^{1-\alpha}B^{q_+}\mathbb{K}_{\text{up}}}{\Gamma(\alpha)n!}\int_0^t \big(t^\beta - s^\beta\big)^{n+\alpha-1} s^{q_+\tau_+ + \gamma - 1}\,ds, \\ &\quad= \frac{b_+\beta^{-\alpha}B^{q_+}\mathbb{K}_{\text{up}}}{\Gamma(\alpha)n!}\mathbb{B}\left(\frac{q_+b + \gamma}{\beta}, n+\alpha\right) t^{\beta(n+\alpha-1)+q_+\tau_++\gamma} \\ &\quad= a_-B^{p_-}t^{p_-\tau_+} \leq h\big(G(t)\big). \end{aligned}$$

Since obviously, the operator S is strictly increasing in $[F,G]$ and if $x \in [F,G]$ then $F(t) \leq x(t) \leq G(t) < M$, $t \in [0,T]$. Hence

$$\begin{aligned} h\bigl(F(t)\bigr) &\leq \frac{\beta^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \biggl[\frac{K(t,s)s^{\gamma-1}}{(t^\beta - s^\beta)^{1-\alpha}}\biggr] g\bigl(F(s)\bigr)\,ds \\ &\leq \frac{\beta^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \biggl[\frac{K(t,s)s^{\gamma-1}}{(t^\beta - s^\beta)^{1-\alpha}}\biggr] g\bigl(x(s)\bigr)\,ds \\ &\leq \frac{\beta^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \biggl[\frac{K(t,s)s^{\gamma-1}}{(t^\beta - s^\beta)^{1-\alpha}}\biggr] g\bigl(G(s)\bigr)\,ds \\ &\leq h\bigl(G(t)\bigr), \end{aligned}$$

so

$$\frac{\beta^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \biggl[\frac{K(t,s)s^{\gamma-1}}{(t^\beta - s^\beta)^{1-\alpha}}\biggr] g\bigl(x(s)\bigr)\,ds \in \bigl[0, h\bigl(G(t)\bigr)\bigr] = h\bigl(\bigl[0, G(t)\bigr]\bigr).$$

Consequently, $S_{h,g}$ is well defined and $S_{h,g}([F,G]) \subset [F,G]$. The proof is completed. □

From the Arzela-Ascoli theorem and since $S_{h,g} : [F,G] \to [F,G]$ is nondecreasing, it follows that $S_{h,g}$ is compact, so the Schauder fixed point theorem implies the following existence result [35, 38, 39].

Theorem 3.6 *Equation* (2) *has a solution in* $[F,G]$. *Moreover,*

$$\lim_{n\to\infty} S_{h,g}^n(F) = x_- \quad \textit{and} \quad \lim_{n\to\infty} S_{h,g}^n(G) = x_+$$

are fixed points of $S_{h,g}$ *with*

$$F \leq x_- \leq x_+ \leq G.$$

Now we are ready to state the following uniqueness result. But first we note that the above considerations can be repeated for any $0 < T_1 \leq T$, so we get $\mathbb{K}_{\mathrm{low}}(T_1)$, $\mathbb{K}_{\mathrm{up}}(T_1)$, $A(T_1)$, $B(T_1)$, F_{T_1}, and G_{T_1} as continuous functions of T_1. Note $\mathbb{K}_{\mathrm{low}}(T_1)$ is nonincreasing, $\mathbb{K}_{\mathrm{up}}(T_1)$ is nondecreasing, and $\mathbb{K}_{\mathrm{low}}(T_1)$, $\mathbb{K}_{\mathrm{up}}(T_1)$ can be continuously extended to $T_1 = 0$. Then $\mathbb{K}_{\mathrm{low}}(0) = \mathbb{K}_{\mathrm{up}}(0)$. We still keep the notation $\mathbb{K}_{\mathrm{low}} = \mathbb{K}_{\mathrm{low}}(T)$, $\mathbb{K}_{\mathrm{up}} = \mathbb{K}_{\mathrm{up}}(T)$, $F = F_T$, and $G = G_T$.

Theorem 3.7 *If there are constants* ψ, χ *and continuous functions* $a_g(t) > 0$ *and* $a_h(t) > 0$ *on* $[0,T]$ *such that*

$$\begin{aligned} &a_h(T_1)t^\psi \bigl|x(t) - y(t)\bigr| \leq \bigl|h\bigl(x(t)\bigr) - h\bigl(y(t)\bigr)\bigr|, \\ &\bigl|g\bigl(x(t)\bigr) - g\bigl(y(t)\bigr)\bigr| \leq a_g(T_1)t^\chi \bigl|x(t) - y(t)\bigr|, \end{aligned} \tag{17}$$

for all $T_1 \in (0,T]$, $t \in (0,T_1]$, $x,y \in [F_{T_1}, G_{T_1}]$ *then equation* (2) *has a unique solution in* $[F,G]$ *provided we have*

$$\beta(n + \alpha - 1) + \chi + \gamma \geq \psi, \tag{18}$$

and

$$\Lambda := \frac{a_g(0)\beta^{-\alpha}\mathbb{K}_{\mathrm{up}}(0)}{a_h(0)\Gamma(\alpha)n!}\mathbb{B}\left(\frac{\chi+\gamma+\tau_+}{\beta}, n+\alpha\right)0^{\beta(n+\alpha-1)+\chi+\gamma-\psi} < 1, \tag{19}$$

where we set $0^0 = 1$.

Proof For any $x, y \in [F, G]$ we set $x_1 = S_{h,g}(x)$ and $y_1 = S_{h,g}(y)$. Clearly, we have

$$\|x-y\|_q := \sup_{t\in(0,T]} \frac{|x(t)-y(t)|}{t^{\tau_+}(1+\iota t^{\varpi})} \le 2B$$

for $q(t) = t^{\tau_+}(1+\iota t^{\varpi})$ with $\varpi > 0$ and $\iota > 0$ specified below, so $[F, G] \subset \mathcal{M}_q$. Then for any $t \in (0, T]$, we derive

$$\begin{aligned}
a_h(t)t^{\psi}\left|x_1(t)-y_1(t)\right| &\le \left|h(x_1(t)) - h(y_1(t))\right| \\
&\le \frac{\beta^{1-\alpha}}{\Gamma(\alpha)}\int_0^t \left[\frac{K(t,s)s^{\gamma-1}}{(t^\beta - s^\beta)^{1-\alpha}}\right]\left|g(x(s)) - g(y(s))\right| ds \\
&\le \frac{a_g(t)\beta^{1-\alpha}\mathbb{K}_{\mathrm{up}}(t)}{\Gamma(\alpha)n!}\int_0^t \left(t^\beta - s^\beta\right)^{n+\alpha-1}s^{\chi+\gamma-1}\left|x(s)-y(s)\right| ds \\
&\le \frac{a_g(t)\beta^{1-\alpha}\mathbb{K}_{\mathrm{up}}(t)}{\Gamma(\alpha)n!}\left[\int_0^t \left(t^\beta - s^\beta\right)^{n+\alpha-1}s^{\chi+\gamma+\tau_+-1} ds\right. \\
&\quad \left. + \iota\int_0^t \left(t^\beta - s^\beta\right)^{n+\alpha-1}s^{\chi+\gamma+\tau_++\varpi-1} ds\right]\|x-y\|_q \\
&= \frac{a_g(t)\beta^{1-\alpha}\mathbb{K}_{\mathrm{up}}(t)}{\Gamma(\alpha)n!}\left[\mathbb{B}\left(\frac{\chi+\gamma+\tau_+}{\beta}, n+\alpha\right)t^{\beta(n+\alpha-1)+\chi+\gamma+\tau_+}\right. \\
&\quad \left. + \iota\mathbb{B}\left(\frac{\chi+\gamma+\tau_++\varpi}{\beta}, n+\alpha\right)t^{\beta(n+\alpha-1)+\chi+\gamma+\tau_++\varpi}\right]\|x-y\|_q,
\end{aligned}$$

which implies

$$\begin{aligned}
\frac{|x_1(t)-y_1(t)|}{t^{\tau_+}(1+\iota t^{\varpi})} &\le \frac{a_g(t)\beta^{1-\alpha}\mathbb{K}_{\mathrm{up}}(t)}{a_h(t)\Gamma(\alpha)n!(1+\iota t^{\varpi})}\left[\mathbb{B}\left(\frac{\chi+\gamma+\tau_+}{\beta}, n+\alpha\right)t^{\beta(n+\alpha-1)+\chi+\gamma-\psi}\right. \\
&\quad \left. + \iota\mathbb{B}\left(\frac{\chi+\gamma+\tau_++\varpi}{\beta}, n+\alpha\right)t^{\beta(n+\alpha-1)+\chi+\gamma+\varpi-\psi}\right]\|x-y\|_q,
\end{aligned}$$

consequently, we obtain

$$\left\|S_{h,g}(x) - S_{h,g}(y)\right\|_q \le L\|x-y\|_q \quad \forall x, y \in [F, G] \tag{20}$$

with

$$\begin{aligned}
L &:= \sup_{t\in(0,T]} L(t), \quad L(t) = L_1(t) + L_2(t), \\
L_1(t) &:= \frac{a_g(t)\beta^{1-\alpha}\mathbb{K}_{\mathrm{up}}(t)}{a_h(t)\Gamma(\alpha)n!(1+\iota t^{\varpi})}\mathbb{B}\left(\frac{\chi+\gamma+\tau_+}{\beta}, n+\alpha\right)t^{\beta(n+\alpha-1)+\chi+\gamma-\psi}, \\
L_2(t) &:= \frac{\iota a_g(t)\beta^{1-\alpha}\mathbb{K}_{\mathrm{up}}(t)}{a_h(t)\Gamma(\alpha)n!(1+\iota t^{\varpi})}\mathbb{B}\left(\frac{\chi+\gamma+\tau_++\varpi}{\beta}, n+\alpha\right)t^{\beta(n+\alpha-1)+\chi+\gamma+\varpi-\psi}.
\end{aligned}$$

Since (note equation (18))

$$
\begin{aligned}
L_2(t) &\le \frac{\iota t^{\varpi} a_g(t)\beta^{1-\alpha}\mathbb{K}_{\mathrm{up}}}{a_h(t)\Gamma(\alpha)n!(1+\iota t^{\varpi})}\mathbb{B}\left(\frac{\chi+\gamma+\tau_++\varpi}{\beta}, n+\alpha\right)T^{\beta(n+\alpha-1)+\chi+\gamma-\psi} \\
&\le \frac{a_g(t)\beta^{1-\alpha}\mathbb{K}_{\mathrm{up}}}{a_h(t)\Gamma(\alpha)n!}\mathbb{B}\left(\frac{\chi+\gamma+\tau_++\varpi}{\beta}, n+\alpha\right)T^{\beta(n+\alpha-1)+\chi+\gamma-\psi}
\end{aligned}
$$

and $\mathbb{B}(\frac{\chi+\gamma+\tau_++\varpi}{\beta}, n+\alpha) \to 0$ as $\varpi \to +\infty$, we see that

$$
\sup_{t\in(0,T]} L_2(t) < \frac{1-\Lambda}{4}
$$

for any $\varpi > 0$ sufficiently large uniformly for any $\iota > 0$. So we take and fix such a ϖ. Next, by equation (19) there is a $t_0 \in (0,T]$ so that

$$
L_1(t) \le \frac{a_g(t)\beta^{1-\alpha}\mathbb{K}_{\mathrm{up}}(t)}{a_h(t)\Gamma(\alpha)n!}\mathbb{B}\left(\frac{\chi+\gamma+\tau_+}{\beta}, n+\alpha\right)t^{\beta(n+\alpha-1)+\chi+\gamma-\psi} < \frac{1+\Lambda}{2} < 1
$$

for any $t \in (0,t_0]$. Furthermore, for $t \in [t_0,T]$, we have (note equation (18))

$$
\begin{aligned}
L_1(t) &\le \frac{\max_{t\in[t_0,T]} a_g(t)\beta^{1-\alpha}\mathbb{K}_{\mathrm{up}}}{a_h(t)\Gamma(\alpha)n!}\mathbb{B}\left(\frac{\chi+\gamma+\tau_+}{\beta}, n+\alpha\right)\frac{T^{\beta(n+\alpha-1)+\chi+\gamma-\psi}}{1+\iota t_0^{\varpi}} \\
&\le \frac{1+\Lambda}{2} < 1
\end{aligned}
$$

for any $\iota > 0$ sufficiently large, so we fix such $\iota > 0$. Consequently we get

$$
\sup_{t\in(0,T]} L_1(t) \le \frac{1+\Lambda}{2}.
$$

Summarizing we see that there is $\varpi > 0$ and $\iota > 0$ so that

$$
L \le \frac{1+\Lambda}{2} + \frac{1-\Lambda}{4} = \frac{3+\Lambda}{4} < 1.
$$

This shows that $S_{h,g} : [F,G] \to [F,G]$ is a contraction with respect to the norm $\|\cdot\|_q$ with a constant L. By the contraction mapping principle, one can obtain the result immediately. □

Remark 3.8 Consider equation (5). Of course, we can suppose $p > 1 = q$. Then $p_\pm = p$, $q_\pm = 1$, $a_\pm = b_\pm = 1$, and $\tau_\pm = \tau := \frac{\beta(n+\alpha-1)+\gamma}{p-1}$. Moreover, Remark 3.3 can be applied to get an existence result. If in addition $\mathbb{K}_{\mathrm{low}} > 0$ then $B \ge A > 0$, and it is not difficult to see that $\psi = (p-1)\tau$, $\chi = 0$, $a_g(T_1) = 1$, and

$$
a_h(T_1) = pA^{p-1}(T_1) = p\frac{\beta^{-\alpha}\mathbb{K}_{\mathrm{low}}(T_1)}{\Gamma(\alpha)n!}\mathbb{B}\left(\frac{\beta(n+\alpha-1)+\gamma p}{\beta(p-1)}, n+\alpha\right).
$$

Then $\beta(n+\alpha-1)+\chi+\gamma = \psi$, so equation (18) holds. Next, we derive

$$
\Lambda = \frac{\mathbb{K}_{\mathrm{up}}(0)}{p\mathbb{K}_{\mathrm{low}}(0)}\frac{\mathbb{B}(\frac{\gamma+\tau}{\beta}, n+\alpha)}{\mathbb{B}(\frac{\beta(n+\alpha-1)+\gamma p}{\beta(p-1)}, n+\alpha)} = \frac{\mathbb{K}_{\mathrm{up}}(0)}{p\mathbb{K}_{\mathrm{low}}(0)} = \frac{1}{p} < 1.
$$

Hence condition (19) is satisfied and then we get a uniqueness result by Theorem 3.7. Note there is gap in the proof of [13, Theorem 5]. So here we give its correct proof.

4 General solutions of Erdélyi-Kober-type integral equations

This section is devoted to a derivation of explicit solutions of some Erdélyi-Kober-type integral equations. In order to establish this, we introduce the following useful result.

Lemma 4.1 *Let* $\sigma\eta + \beta > -\sigma$ *and* $\alpha, \sigma > 0$. *Then*

$$\left({}_{EK}I^{\alpha}_{0+;\sigma,\eta} t^{\beta}\right)(x) = \frac{\Gamma(\eta + 1 + \frac{\beta}{\sigma})}{\Gamma(\eta + 1 + \alpha + \frac{\beta}{\sigma})} x^{\beta}.$$

Proof Set $t = xy$. By using Lemma 2.3, we have

$$\begin{aligned}\left({}_{EK}I^{\alpha}_{0+;\sigma,\eta} t^{\beta}\right)(x) &= \frac{\sigma x^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \int_0^x \frac{t^{\sigma\eta+\sigma-1} t^{\beta}\, dt}{(x^{\sigma} - t^{\sigma})^{1-\alpha}} \\ &= \frac{\sigma x^{\beta}}{\Gamma(\alpha)} \frac{1}{\sigma} \mathbb{B}\left(\frac{\sigma\eta + \sigma + \beta}{\sigma}, \alpha\right) \\ &= \frac{\Gamma(\eta + 1 + \frac{\beta}{\sigma})}{\Gamma(\eta + 1 + \alpha + \frac{\beta}{\sigma})} x^{\beta}.\end{aligned}$$

This completes the proof. □

Now we are ready to present our main result of this section.

Theorem 4.2 *Let* $\alpha > 0$, $\sigma > 0$, $\frac{\beta}{\sigma} + \eta + 1 > 0$, $m, b, \beta \in \mathbb{R}$, *and* $a, N, m \neq 0$. *Then equation* (6) *is solvable and its solution* $\varphi(x)$ *can be written as*

$$\varphi(x) = C^{\frac{1}{N}} x^{\frac{\beta}{N}}, \tag{21}$$

where the constant C *satisfies the following equation*:

$$C^{\frac{m}{N}} = aC \frac{\Gamma(\eta + 1 + \frac{\beta}{\sigma})}{\Gamma(\eta + 1 + \alpha + \frac{\beta}{\sigma})} + b. \tag{22}$$

Proof With the help of Lemma 4.1, substituting equation (21) into (6), we find that C satisfies equation (22) which completes the proof. □

5 Illustrative examples

In this section, we pay our attention to show three numerical performance results.

Example 5.1 We consider the problem

$$x^2(t) = \frac{(\frac{1}{4})^{\frac{1}{2}}}{\Gamma(\frac{1}{2})} \int_0^t \left[\frac{t^2 s^{-\frac{1}{2}}}{(t^{\frac{1}{4}} - s^{\frac{1}{4}})^{\frac{1}{2}}}\right] x(s)\, ds, \quad t \in [0, 1]. \tag{23}$$

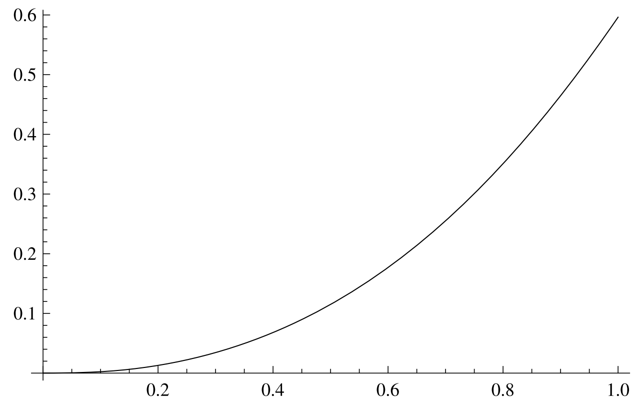

Figure 1 Solution of equation (23) and the boundaries *F* and *G* for Example 5.1 coincide with the unique solution.

First, Theorem 4.2 gives the exact solution $x(t) = \frac{88{,}179\sqrt{\pi}t^{\frac{19}{8}}}{262{,}144} \doteq 0.596211t^{2.375}$ of equation (23). Next, by changing $x(t) = z(t)t$ we get

$$z^2(t) = \frac{(\frac{1}{4})^{\frac{1}{2}}}{\Gamma(\frac{1}{2})}\int_0^t \left[\frac{s^{\frac{1}{2}}}{(t^{\frac{1}{4}} - s^{\frac{1}{4}})^{\frac{1}{2}}}\right] z(s)\,ds, \quad t \in [0,1]. \tag{24}$$

Of course, we get a solution $z(t) = \frac{88{,}179\sqrt{\pi}t^{\frac{11}{8}}}{262{,}144}$. In equation (5) for (24), we set $K(t,s) = 1$, $\alpha = \frac{1}{2}$, $\gamma = \frac{3}{2}$, $\beta = \frac{1}{4}$, $n = 0$, $T = 1$, $p = 2$, and $q = 1$. After some computation, we find that

$$F(t) = G(t) = \frac{2}{\Gamma(\frac{1}{2})}\mathbb{B}\left(\frac{23}{2}, \frac{1}{2}\right)t^{\frac{11}{8}} = \frac{88{,}179\sqrt{\pi}t^{\frac{11}{8}}}{262{,}144}.$$

Obviously, all the assumptions in Theorem 3.7 are satisfied. Numerical result is given in Figure 1.

Example 5.2 In equation (5), we set $K(t,s) = e^{4t}$, $\alpha = \frac{3}{4}$, $\beta = \gamma = \frac{1}{2}$, $n = 0$, $T = 1$, $p = 2$, and $q = 1$. Now, we turn to consider the following homogeneous Abel-type integral equation with weakly singular kernels and power-law nonlinearity:

$$x^2(t) = \frac{(\frac{1}{2})^{\frac{1}{4}}}{\Gamma(\frac{3}{4})}\int_0^t \left[\frac{e^{4t}s^{-\frac{1}{2}}}{(t^{\frac{1}{2}} - s^{\frac{1}{2}})^{\frac{1}{4}}}\right] x(s)\,ds, \quad t \in [0,1]. \tag{25}$$

After some computation, we find that

$$F(t) = \frac{(\frac{1}{2})^{-\frac{3}{4}}}{\Gamma(\frac{3}{4})}\mathbb{B}\left(\frac{7}{4}, \frac{3}{4}\right)t^{\frac{3}{8}} = \frac{42^{\frac{3}{4}}\Gamma(\frac{7}{4})}{3\sqrt{\pi}}t^{\frac{3}{8}} \doteq 1.16274t^{0.375},$$

$$G(t) = \frac{e^4(\frac{1}{2})^{-\frac{3}{4}}}{\Gamma(\frac{3}{4})}\mathbb{B}\left(1, \frac{3}{4}\right)t^{\frac{3}{8}} = \frac{42^{\frac{3}{4}}e^4}{3\Gamma(\frac{3}{4})}t^{\frac{3}{8}} \doteq 99.9092t^{0.375}.$$

Obviously, all the assumptions in Theorem 3.7 are satisfied. Then, the problem (5.2) has a unique solution in $[F, G]$. Numerical results are given in Figure 2.

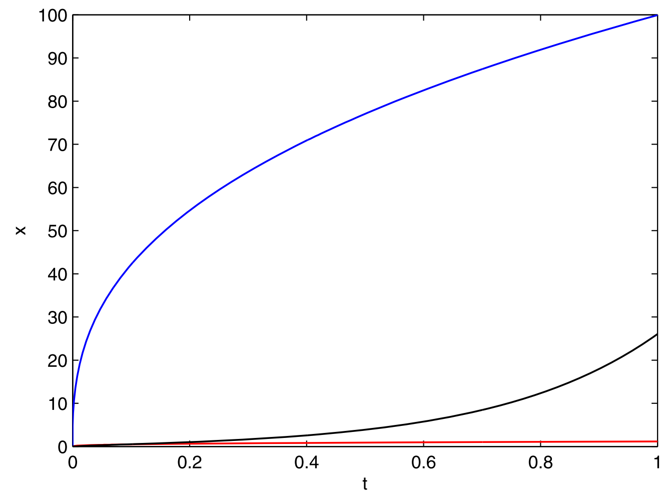

Figure 2 Solution (black line) of equation (25) and the boundaries *F* (red line) and *G* (blue line) for Example 5.2.

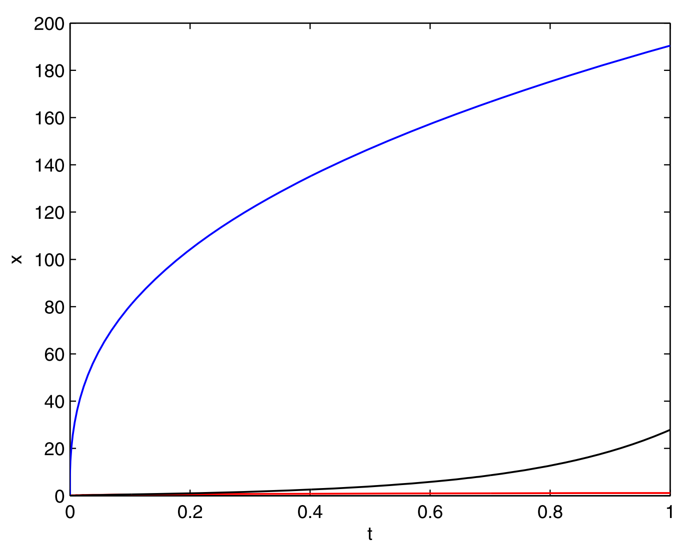

Figure 3 Solution (black line) of equation (26) and the boundaries *F* (red line) and *G* (blue line) for Example 5.3.

Example 5.3 In equation (2), we set $K(t,s) = e^{4t}$, $\alpha = \frac{3}{4}$, $\beta = \gamma = \frac{1}{2}$, $n = 0$, $T = 1$, $p_\pm = 2$, $q_\pm = 1$, $h(x) = x^2(1 - \frac{1}{300}x)$, $g(x) = x$, and $M = 200$. Now, we turn to considering the following homogeneous Abel-type integral equation with weakly singular kernels and polynomial law nonlinearity:

$$x^2(t)\left(1 - \frac{1}{300}x(t)\right) = \frac{(\frac{1}{2})^{\frac{1}{4}}}{\Gamma(\frac{3}{4})} \int_0^t \left[\frac{e^{4t}s^{-\frac{1}{2}}}{(t^{\frac{1}{2}} - s^{\frac{1}{2}})^{\frac{1}{4}}}\right] x(s)\, ds, \quad t \in [0,1]. \tag{26}$$

It is clear that now $a_+ = 1$, $a_- = \frac{1}{3}$, and $b_\pm = 1$, so equation (4) holds. After some computation, we find that

$$F(t) = \frac{(\frac{1}{2})^{-\frac{3}{4}}}{\Gamma(\frac{3}{4})}\mathbb{B}\left(\frac{7}{4}, \frac{3}{4}\right)t^{\frac{3}{8}} = \frac{42^{\frac{3}{4}}\Gamma(\frac{7}{4})}{3\sqrt{\pi}}t^{\frac{3}{8}} \doteq 1.16274t^{0.375},$$

$$G(t) = \frac{3e^4(\frac{1}{2})^{-\frac{3}{4}}}{\Gamma(\frac{3}{4})}\mathbb{B}\left(\frac{7}{4}, \frac{3}{4}\right)t^{\frac{3}{8}} = \frac{42^{\frac{3}{4}}e^4\Gamma(\frac{7}{4})}{\sqrt{\pi}}t^{\frac{3}{8}} \doteq 190.45t^{0.375}.$$

Since now $A \doteq 1.16274 < 190.45 \doteq B < M = 200$, obviously, all the assumptions in Theorem 3.6 are satisfied. Then, the problem (5.3) has a solution in $[F, G]$. Numerical results are given in Figure 3.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The work presented here was carried out in collaboration between the authors. The authors contributed to every part of this study equally and read and approved the final version of the manuscript.

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Acknowledgements
The first and second authors acknowledge the support by National Natural Science Foundation of China (11201091) and Key Projects of Science and Technology Research in the Chinese Ministry of Education (211169), Key Support Subject (Applied Mathematics), Key Project on the Reforms of Teaching Contents and Course System of Guizhou Normal College and Doctor Project of Guizhou Normal College (13BS010). The third author acknowledges the support by Grants VEGA-MS 1/0071/14, VEGA-SAV 2/0029/13 and APVV-0134-10.

Received: 14 October 2013 Accepted: 19 December 2013 Published: 17 Jan 2014

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10.1186/1687-2770-2014-20
Cite this article as: Wang et al.: Analysis of Abel-type nonlinear integral equations with weakly singular kernels. *Boundary Value Problems* 2014, 2014:20