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Positive solution for a class of coupled (p,q)-Laplacian nonlinear systems

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Abstract

In this article, we prove the existence of a nontrivial positive solution for the elliptic system

$$\begin{cases} -\Delta_p u = \omega(x)f(v) & \text{in } \Omega, \\ -\Delta_q v = \rho(x)g(u) & \text{in } \Omega, \\ (u,v) = (0,0) & \text{on } \partial\Omega, \end{cases}$$

where Δ_p denotes the p-Laplacian operator, $p,q > 1$ and Ω is a smooth bounded domain in $\mathbb{R}^N$ $(N \geq 2)$. The *weight* functions ω and ρ are continuous, nonnegative and not identically null in Ω, and the nonlinearities f and g are continuous and satisfy simple hypotheses of *local* behavior, without involving monotonicity hypotheses or conditions at ∞. We apply the fixed point theorem in a cone to obtain our result.

MSC: 35B09; 35J47; 58J20

1 Introduction

Coupled systems involving quasilinear operators as the p-Laplacian have been a theme of interest for researchers of partial differential equations. In this paper we prove the existence of a nontrivial positive solution for the elliptic system

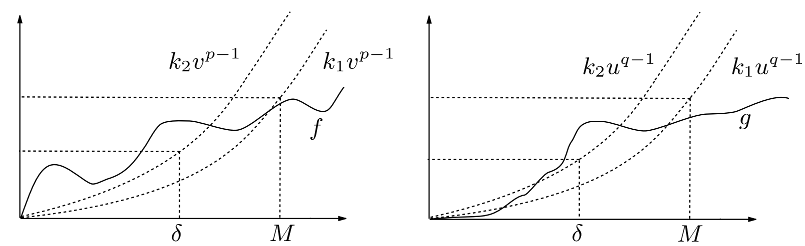
$$\begin{cases} -\Delta_p u = \omega(x)f(v) & \text{in } \Omega, \\ -\Delta_q v = \rho(x)g(u) & \text{in } \Omega, \\ (u,v) = (0,0) & \text{on } \partial\Omega, \end{cases} \tag{P}$$

where Δ_p denotes the p-Laplacian operator defined by $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $p,q > 1$ and Ω denotes a smooth bounded domain in $\mathbb{R}^N$ $(N \geq 2)$. In other words, we will prove the existence of a pair $(u,v) \in C^{1,\alpha}(\Omega) \times C^{1,\alpha}(\Omega)$ such that (u,v) satisfies (P), with u and v strictly positive in Ω. The *weight* functions $\omega, \rho : \overline{\Omega} \to \mathbb{R}$ are continuous, nonnegative in Ω and positive in $\Omega_\varepsilon = \{x \in \Omega : 0 < \operatorname{dist}(x, \partial\Omega) < \varepsilon\}$ for some $\varepsilon > 0$. The nonlinearities $f, g : [0,\infty) \times [0,\infty) \to [0,\infty)$ are continuous, g is positive in $(0,\lambda)$ for some $\lambda > 0$, and both satisfy simple hypotheses of *local* behavior.

We suppose that the nonlinearity f is superlinear at origin and f, g are allowed to be sub- or superlinear at ∞. Moreover, there is no monotonicity hypotheses on these nonlinearities. We suppose the existence of positive constants $0 < \delta < M$ such that

(H1) $0 \leq f(v) \leq k_1 M^{p-1}$, $0 \leq g(u) \leq k_1 M^{q-1}$ if $0 \leq u, v \leq M$,

(H2) $f(v) \geq k_2 v^{p-1}$ if $0 \leq v \leq \delta$,



(a) The graph of f stays above $k_2 v^{p-1}$ if $0 \le v \le \delta$ and below $k_1 M^{p-1}$ if $0 \le v \le M$.

(b) The graph of g stays below $k_1 M^{q-1}$ if $0 \le u \le M$.

Figure 1 The nonlinearities *f* and *g* satisfy (H1) and (H2).

where the constants k_1 and k_2 depend only on ω, ρ and Ω (see Figure 1). These constants will be defined later on in this paper and, as proved in [1], $k_1 < \lambda_p < k_2$, where λ_p is the first eigenvalue of the p-Laplacian operator.

Elliptic problems concerning the existence of positive solutions for equations and systems of equations related to Dirichlet problems have been studied in several papers during the last decades. In this way, many existence results for systems involving the p-Laplacian operator in general bounded domains in $\mathbb{R}^N$ have been considered in recent articles. In particular, systems as (P) have been studied in articles in [2–6] for example.

The main interest of this paper is studying systems whose nonlinearities present some kind of coupling as (P). A paper which deals with this sort of problem is [2], where problem (P) is considered under the assumptions $p = q = 2$, $\omega = \rho \equiv 1$. In this paper, among other hypotheses, the nonlinearities f and g are supposed to be at least continuous if $N = 1$ and locally Holder continuous with exponent $\beta \in (0;1]$ if $N \geq 2$. Moreover, both are supposed to be nondecreasing in $[0,\infty)$, nonnegative at origin and satisfying (for $p = 2$) the fundamental condition

$$\lim_{x\to+\infty} \frac{f(Mg(x)^{\frac{1}{p-1}})}{x^{p-1}} = 0 \quad \text{for every } M > 0. \tag{1}$$

Schauder's fixed point theorem, the Leray-Schauder degree and a variant of Krasnoselskii's method are applied to guarantee the existence of a positive solution for (P).

The studies of [2] were extended by Hai and Shivaji in [4] (for $p = q > 1$), [5] (for $p = q = 2$) and by Hai in [3] (for $p, q > 1$). In these papers, the authors deal with problem (P), $\omega \equiv \lambda$ and $\rho \equiv \mu$ (in [4] and [5], $\lambda = \mu$), with no sign conditions on $f(0)$ or $g(0)$ and without monotonicity conditions on f or g. In this way, semipositone cases were also considered in these papers (for more details about semipositone problems, see [7] and the references therein).

In [4], the nonlinearities f and g in (P) are supposed to be C^1, monotone and satisfying the following conditions at ∞

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} g(x) = \infty \tag{2}$$

in addition to condition (1). The existence of a positive solution is guaranteed for large λ by applying the sub-supersolution method.

The paper [5] deals with problem (P) in the particular case $p = 2$ and a positive solution is guaranteed by applying the sub- and supersolution method and Schauder's fixed point theorem. The nonlinearities $f, g : [0; \infty) \to \mathbb{R}$ considered are continuous and there exist positive numbers L, K such that $f(x) \geq L$ and $g(x) \geq L$ for $x \geq K$. Moreover, the authors considered, as it has been done in [2], condition (1) with $p = 2$.

In [3], the author obtains necessary and sufficient conditions for the existence of positive solutions for problem (P). The nonlinearities f and g are positive, continuous and nondecreasing in $[0, +\infty)$, with g strictly positive for $x > 0$, and

$$f^{\frac{1}{p-1}}\left(cg^{\frac{1}{q-1}}(x)\right) \tag{3}$$

sublinear at 0 and ∞. In this paper, the maximum principle and fixed point arguments are applied to guarantee the existence of a solution.

Another paper dealing with the existence of a positive solution for a class of coupled systems is [6]. In this paper, the authors studied problem (P), with $w(x) = \lambda a(x)$ and $\rho(x) = \lambda b(x)$, in which λ is a positive parameter. The existence of a solution is guaranteed via the method of sub- and supersolution if, among other assumptions, the functions a and b considered are C^1 sign-changing functions that may be negative near the boundary and the positive nonlinearities f and g are supposed to be C^1 and nondecreasing.

Recently, many articles have applied fixed point results to prove the existence of positive solutions of partial differential equations or systems (see, for example, [1, 8–12]). In this paper we study problem (P) in general domains, assuming that (H1) and (H2) hold. As system (P) has no variational structure, our main arguments are based on fixed-point index and comparison theorems, following the ideas of [8, 10] and [11]. In particular, our assumptions on the nonlinearities do not involve monotonicity hypotheses or sublinearity conditions at ∞.

Our strategy is as follows. At first, we show an existence result for the radial case when $\Omega = B_1 := \{x \in \mathbb{R}^N : |x| = 1\}$, applying a fixed point theorem in a cone. Afterwards, we utilize this result to prove our main existence result for (P), when $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain. In this case, we do a symmetrization of *weigh* functions and combine comparison theorems with a new application of the fixed point theorem.

For completeness, we will consider concrete examples of coupled systems for which it is possible to apply our method to guarantee the existence of at least one positive solution. It will be clear in some of these examples that conditions (1), (2) and (3) are not required in our method.

2 The radial case

Let us consider the radial version of problem (P)

$$\begin{cases} -\Delta_p u = w(x)f(v) & \text{in } B_1, \\ -\Delta_q v = \rho(x)g(u) & \text{in } B_1, \\ (u, v) = (0, 0) & \text{on } \partial B_1, \end{cases} \tag{P$_\mathrm{r}$}$$

where $B_1 = \{x \in \mathbb{R}^N : |x| < 1\}$ and the weight functions $w, \rho : \overline{B}_1 \to \mathbb{R}$ are radial, continuous, nonnegative and not identically null functions. The positive functions f and g are supposed to be continuous and satisfying local conditions that depend on the positive constants defined in (6) and (7).

Let $\psi_{p'}(t) = t^{\frac{1}{p-1}}$ (the inverse of the well-known function $\psi_p(t) = |t|^{p-2}t$), and let us define

$$\zeta_p(r) = \int_r^1 \psi_{p'}\left(\int_0^r \left(\frac{s}{\theta}\right)^{N-1} w(s)\,ds\right) d\theta. \tag{4}$$

Consider $\xi > 0$ the real number such that

$$\zeta_p(\xi) = \max_{0\le r\le 1} \zeta_p(r). \tag{5}$$

Moreover, define the positive constants k_1 and k_2 by

$$\begin{aligned} k_1 = \min\Bigg\{&\left(\int_0^1 \psi_{p'}\left(\int_0^\theta \left(\frac{s}{\theta}\right)^{N-1} w(s)\,ds\right) d\theta\right)^{1-p}, \\ &\left(\int_0^1 \psi_{q'}\left(\int_0^\theta \left(\frac{s}{\theta}\right)^{N-1} \rho(s)\,ds\right) d\theta\right)^{1-q}\Bigg\} \end{aligned} \tag{6}$$

and

$$k_2 = \left(\int_\xi^1 \psi_{p'}\left(\int_0^{\frac{\xi}{2}} \left(\frac{s}{\theta}\right)^{N-1} w(s)\,ds\right) d\theta\right)^{1-p}. \tag{7}$$

It is easy to see that $0 < k_1 < k_2$. In fact,

$$\begin{aligned} \int_0^1 \psi_{p'}\left(\int_0^\theta \left(\frac{s}{\theta}\right)^{N-1} w(s)\,ds\right) d\theta &> \int_\xi^1 \psi_{p'}\left(\int_0^\xi \left(\frac{s}{\theta}\right)^{N-1} w(s)\,ds\right) d\theta \\ &> \int_\xi^1 \psi_{p'}\left(\int_0^{\frac{\xi}{2}} \left(\frac{s}{\theta}\right)^{N-1} w(s)\,ds\right) d\theta. \end{aligned}$$

Remark 1 If $B_1 \subset \mathbb{R}^N$ $(N > 2)$ is the unitary ball and $\rho(x) = w(x) = 1$ are the weight functions in problem (P), it is easy to see that k_1 and k_2 are given by

$$k_1 := \min\left\{N\left(p'\right)^{p-1}, N\left(q'\right)^{q-1}\right\} \tag{8}$$

and

$$k_2 := \begin{cases} 2^N N(ep')^{p-1} & \text{if } N = p, \\ 2^N N(p')^{p-1}\left(\frac{N}{p}\right)^{\frac{N}{(p-N)(1-p)}} & \text{if } N \neq p, \end{cases} \tag{9}$$

in which p' and q' are the conjugate exponents of p and q, respectively.

Finally we assume that the nonlinearities f and g satisfy the local conditions.
(H1) $0 \le f(v) \le k_1 M^{p-1}$, $0 \le g(u) \le k_1 M^{q-1}$ if $0 < u, v \le M$ for some $M > 0$.
(H2) $f(v) \ge k_2 v^{p-1}$ if $0 < v \le \delta$ for some $0 < \delta < M$.
Now, we are in a position to state the main result of this section: the existence of a positive radial solution for $(\mathrm{P_r})$.

Theorem 2 *Suppose that $f, g : [0, +\infty) \to [0, +\infty)$ are positive, continuous nonlinearities satisfying* (H1) *and* (H2) (*with constants k_1 and k_2 defined in* (6) *and* (7)), *and let the radial weight functions $w, \rho : \overline{B}_1 \to \mathbb{R}$ be continuous, nonnegative and not identically null. Then problem* ($\mathrm{P_r}$) *has at least a nontrivial positive solution. Moreover, if (u, v) is a positive solution for* ($\mathrm{P_r}$), *then*

$$\delta \leq \left\|(u,v)\right\|_\infty = \max\left\{\|u\|_\infty, \|v\|_\infty\right\} \leq M.$$

To prove the last theorem, we will apply a well-known result of the fixed-point index theory, known as a fixed point cone theorem (see, for example, [13]).

Lemma 3 *Let E be a Banach space and $\|\cdot\|$ be its norm. Let K be a cone in E. For $R > 0$, define $K_R = \{x \in K : \|x\| \leq R\}$ and denote its boundary by ∂K_R, that is, $\partial K_R = \{x \in K : \|x\| = R\}$. Suppose that*

$$T : \overline{K}_R \to K$$

is completely continuous.

(i) *If there exists $x_0 \in K - \{0\}$ such that*

$$x - Tx \neq tx_0 \quad \text{for all } x \in \partial K_R \text{ and } t \geq 0,$$

then

$$i(T, K_R, K) = 0.$$

(ii) *If $\|Tx\| \leq \|x\|$ for $x \in \partial K_R$ and $Tx \neq x$ for $x \in \partial K_R$, then*

$$i(T, K_R, K) = 1.$$

In what follows, we will consider $X := C[0,1] \times C[0,1]$ with the norm

$$\left\|(u,v)\right\|_\infty = \max\left\{\|u\|_\infty, \|v\|_\infty\right\}$$

and the cone $K = \{(u,v) \in X : u, v \geq 0\} \subset X$.

Proof of Theorem 2 It is easy to see that (u, v) is a solution of ($\mathrm{P_r}$) if and only if (u, v) is a fixed point of $T : K \to K$ given by

$$T(u,v) = \left(T_1(u,v), T_2(u,v)\right), \tag{10}$$

where

$$T_1\left(u(r), v(r)\right) = \int_r^1 \psi_{p'}\left(\int_0^\theta \left(\frac{s}{\theta}\right)^{N-1} w(s) f\left(v(s)\right) ds\right) d\theta$$

and

$$T_2\big(u(r),v(r)\big) = \int_r^1 \psi_{q'}\left(\int_0^\theta \left(\frac{s}{\theta}\right)^{N-1} \rho(s)g\big(u(s)\big)\,ds\right)d\theta.$$

Moreover, it is straightforward that the operator T is completely continuous.

If $(u,v) \in K$ is such that $\|(u,v)\|_\infty = M$, we have

$$\big\|T(u,v)\big\|_\infty \le M = \big\|(u,v)\big\|_\infty.$$

In fact, by (H1) we obtain

$$\big\|T_1(u,v)\big\|_\infty \le k_1^{\frac{1}{p-1}} M \int_0^1 \psi_{p'}\left(\int_0^\theta \left(\frac{s}{\theta}\right)^{N-1} w(s)\,ds\right)d\theta = M$$

and

$$\big\|T_2(u,v)\big\|_\infty \le k_1^{\frac{1}{q-1}} M \int_0^1 \psi_{p'}\left(\int_0^\theta \left(\frac{s}{\theta}\right)^{N-1} \rho(s)\,ds\right)d\theta = M.$$

As a consequence of Lemma 3, it follows that

$$i(T,K_M,K) = 1.$$

Now, we will prove that $i(T,K_\delta,K) = 0$. In order to prove that, we will show that exists $(u_0,v_0) \in K_\delta$ such that

$$(u,v) - T(u,v) \neq t(u_0,v_0) \quad \text{for all } (u,v) \in \partial K_\delta \text{ and } t > 0.$$

Let $\Theta \in C[0,1]$, $0 \le \Theta \le 1$, be defined by

$$\Theta(r) = \begin{cases} 1 & \text{if } r \in [0,\frac{\xi}{2}], \\ 0 & \text{if } r \in [\xi,1], \end{cases}$$

and let

$$\big(u_0(r),v_0(r)\big) = \big(\Theta(r),\Theta(r)\big).$$

We claim that

$$(u,v) - T(u,v) \neq t\big(\Theta(r),\Theta(r)\big) \quad \text{for all } (u,v) \in \partial K_\delta \text{ and } t > 0.$$

In fact, let us suppose that there are $(u^*,v^*) \in K_\delta$ and $t_0 \ge 0$ such that

$$\big(u^*,v^*\big) - T\big(u^*,v^*\big) = t_0\big(\Theta(r),\Theta(r)\big). \tag{11}$$

Since $T(u^*,v^*) \in K$, we obtain

$$\begin{cases} u^*(r) \ge t_0\Theta(r), \\ v^*(r) \ge t_0\Theta(r). \end{cases}$$

Let $t^* \in \mathbb{R}$ be such that

$$t^* = \sup\{t : u^*(r) \geq t\Theta(r) \text{ and } v^*(r) \geq t\Theta(r) \text{ for all } r \in [0,1]\}. \tag{12}$$

(Note that it is immediate that $t^* < \infty$. In fact, if $t^* = \infty$ and if $\Theta(r) \neq 0$, we have $u^*(r) = \infty$, which contradicts $u^* \in K_\delta$.)

As $\|(u^*, u^*)\|_\infty = \delta$, we have $0 \leq v^*(r) \leq \delta$ for all $r \in [0,1]$. If $r \in [0,\xi]$, by (11) and (H2) we obtain

$$\begin{aligned}
u^*(r) &= \int_r^1 \psi_{p'}\left(\int_0^\theta \left(\frac{s}{\theta}\right)^{N-1} w(s) f\left(v^*(s)\right) ds\right) d\theta + t_0\Theta(r) \\
&\geq k_2^{\frac{1}{p-1}} \int_r^1 \psi_{p'}\left(\int_0^\theta \left(\frac{s}{\theta}\right)^{N-1} w(s)\left(v^*(s)\right)^{p-1} ds\right) d\theta + t_0\Theta(r) \\
&\geq k_2^{\frac{1}{p-1}} t^* \int_r^1 \psi_{p'}\left(\int_0^\theta \left(\frac{s}{\theta}\right)^{N-1} w(s)\left(\Theta(s)\right)^{p-1} ds\right) d\theta + t_0\Theta(r) \\
&\geq k_2^{\frac{1}{p-1}} t^* \int_\xi^1 \psi_{p'}\left(\int_0^\xi \left(\frac{s}{\theta}\right)^{N-1} w(s)\left(\Theta(s)\right)^{p-1} ds\right) d\theta + t_0\Theta(r) \\
&\geq k_2^{\frac{1}{p-1}} t^* \int_\xi^1 \psi_{p'}\left(\int_0^{\frac{\xi}{2}} \left(\frac{s}{\theta}\right)^{N-1} w(s)\left(\Theta(s)\right)^{p-1} ds\right) d\theta + t_0\Theta(r) \\
&= k_2^{\frac{1}{p-1}} t^* \int_\xi^1 \psi_{p'}\left(\int_0^{\frac{\xi}{2}} \left(\frac{s}{\theta}\right)^{N-1} w(s)\, ds\right) d\theta + t_0\Theta(r) \\
&\geq t^* + t_0\Theta(r) \\
&\geq \left(t^* + t_0\right)\Theta(r).
\end{aligned}$$

Since $\Theta(r) = 0$ for $r \in [\xi,1]$, it follows that

$$u^*(r) \geq \left(t^* + t_0\right)\Theta(r) \quad \text{for all } r \in [0,1],$$

which contradicts (12).

Then

$$i(T, K_\delta, K) = 0$$

and, consequently,

$$i(T, K_M \backslash K_\delta, K) = i(T, K_M, K) - i(T, K_\delta, K) = 1.$$

Thus, T has a nontrivial fixed point in $K_M \backslash K_\delta$. The regularity of the solution follows from classical results of Lieberman and Tolksdorf (see [14] and [15]). □

3 The general case

Now we will establish the main result of this paper: the existence of a nontrivial positive solution for (P) when $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain.

3.1 The constants k_1 and k_2 in Ω

For the general problem

$$\begin{cases} -\Delta_p u = w(x)f(v) & \text{in } \Omega, \\ -\Delta_q v = \rho(x)g(u) & \text{in } \Omega, \\ (u,v) = (0,0) & \text{on } \partial\Omega, \end{cases} \tag{P}$$

we define $T : K \to K$ by

$$T(u,v) = \bigl(T_1(u,v), T_2(u,v)\bigr) = (z_1, z_2),$$

where

$$\begin{cases} -\Delta_p z_1 = w(x)f(v) & \text{in } \Omega, \\ -\Delta_q z_2 = \rho(x)g(u) & \text{in } \Omega, \\ (z_1,z_2) = (0,0) & \text{on } \partial\Omega. \end{cases} \tag{13}$$

It is well known in p-Laplacian operator theory that T is completely continuous. Moreover, a simple maximum principle argument guarantees that $z_i > 0$ in Ω for $i = 1, 2$.

As it has been done in the radial case, in order to obtain a result of existence for (P), we will apply Lemma 3.

Let us denote by $\phi_{s,h} \in C^{1,\alpha}(\Omega)$ the solution of

$$\begin{cases} -\Delta_s u = h(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $h \in L^{s'}(\Omega)$ and s' is such that $\frac{1}{s} + \frac{1}{s'} = 1$. As ρ and w satisfy the same hypotheses as h, we can define

$$k_1 = \min\bigl\{ \|\phi_{p,w}\|_\infty^{1-p}, \|\phi_{q,\rho}\|_\infty^{1-q} \bigr\}. \tag{14}$$

Fix $x_0 \in \Omega$ such that $w(x_0) \not\equiv 0$, let $R > 0$ be such that $\overline{B(x_0,R)} \subset \Omega$ and

$$w^*(s) = \begin{cases} \min\{w(y) : |y - x_0| = s\} & \text{if } 0 < s \le R, \\ w(x_0) & \text{if } s = 0. \end{cases} \tag{15}$$

Furthermore, let us define k_2 by

$$k_2 = \left(\int_\xi^R \psi_{p'} \left(\int_0^{\frac{\xi}{2}} \left(\frac{s}{r}\right)^{N-1} w^*(s)\, ds \right) d\theta \right)^{1-p} \tag{16}$$

(where ξ is defined similarly as it has been done in (4) and (5)).

3.2 Main theorem

Theorem 4 *Suppose that* $f, g : [0,+\infty) \to [0,+\infty)$ *are positive, continuous nonlinearities satisfying* (H1) *and* (H2) (*with constants* k_1 *and* k_2 *defined in* (14) *and* (16), *respectively*), *and let the weight functions* $w, \rho : \overline{\Omega} \to \mathbb{R}$ *be continuous, nonnegative and not identically* *null. Then problem* (P) *has at least a nontrivial positive solution. Moreover, if* (u,v) *is a positive solution for* (P), *then*

$$\delta \le \left\|(u,v)\right\|_\infty = \max\left\{\|u\|_\infty, \|v\|_\infty\right\} \le M.$$

Proof Let $(u,v) \in K$ be such that $\|(u,v)\|_\infty = M$. It follows from (H1) and (13) that

$$-\Delta_p z_1 = w(x)f(v) \le w(x)k_1 M^{p-1} = -\Delta_p\left(k_1^{\frac{1}{p-1}} M\phi_{p,w}\right) \quad \text{in } \Omega$$

and

$$z_1 = 0 = k_1^{\frac{1}{p-1}} M\phi_{p,w} \quad \text{in } \partial\Omega.$$

Therefore, by the maximum principle, we conclude that

$$z_1 \le k_1^{\frac{1}{p-1}} M\phi_{p,w} \le k_1^{\frac{1}{p-1}} M\|\phi_{p,w}\|_\infty \le M$$

and, consequently,

$$\|z_1\|_\infty \le M.$$

In the same way, we obtain

$$\|z_2\|_\infty \le M,$$

and, as a consequence of the last two inequalities, we have

$$\left\|T(u,v)\right\|_\infty \le M = \left\|(u,v)\right\|_\infty.$$

It follows from Lemma 3 that

$$i(T, K_M, K) = 1.$$

We claim that $i(T, K_\delta, K) = 0$. To prove our claim, we will show, as it has been done in the radial case, that there exists $(u_0, v_0) \in K_\delta$ such that

$$(u,v) - T(u,v) \ne t(u_0, v_0) \quad \text{for all } (u,v) \in \partial K_\delta \text{ and } t > 0.$$

For fixed $x_0 \in \Omega$, let $\Theta \in C(\overline{\Omega})$ be such that $\Theta(x) \in [0,1]$ and

$$\Theta(x) = \Theta\left(|x - x_0|\right) = \begin{cases} 1 & \text{if } 0 \le |x - x_0| \le \frac{\xi}{2}, \\ 0 & \text{if } \xi \le |x - x_0| \le R. \end{cases}$$

Consider

$$\left(u_0(r), v_0(r)\right) = \left(\Theta(r), \Theta(r)\right),$$

where $r = |x - x_0|$.

We claim that

$$(u,v) - T(u,v) \neq t(\Theta,\Theta) \quad \text{for all } (u,v) \in \partial K_\delta \text{ and } t > 0.$$

In fact, suppose that there exist $(u^*,v^*) \in K_\delta$ and $t_0 \geq 0$ such that

$$\left(u^*,v^*\right) - \left(z_1^*,z_2^*\right) = t_0(\Theta,\Theta), \tag{17}$$

where $(z_1^*,z_2^*) = T(u^*,v^*)$.

Since $T(u^*,v^*) \in K$, it follows that

$$\begin{cases} u^*(x) \geq t_0\Theta(x), \\ v^*(x) \geq t_0\Theta(x). \end{cases}$$

Let z_{1R}^* be the solution of

$$\begin{cases} -\Delta_p z_{1R}^* = w^*(r)f^*(r) \quad \text{in } B(x_0,R), \\ z_{1R}^* = 0 \quad \text{on } \partial B(x_0,R), \end{cases}$$

where

$$f^*(s) = \begin{cases} \min\{f(v^*(y)) : |y-x_0| = s\} & \text{if } 0 < s \leq R, \\ f(v^*(x_0)) & \text{if } s = 0. \end{cases} \tag{18}$$

Then

$$z_{1R}^*(x) = z_{1R}^*\left(|x|\right) = \int_{|x-x_0|}^R \psi_{p'}\left(\int_0^\theta \left(\frac{s}{\theta}\right)^{N-1} w^*(s)f^*(s)\,ds\right)d\theta.$$

Note that given $x \in B(x_0,R)$, we obtain

$$\begin{cases} -\Delta_p z_{1R}^*(|x|) = w^*(s)f^*(s) \leq w(x)f(v^*(x)) = -\Delta_p z_1^* \quad \text{in } B(x_0,R), \\ z_{1R}^* = 0 < z_1^* \quad \text{on } \partial B(x_0,R), \end{cases}$$

and by the maximum principle, we have

$$z_{1R}^* \leq z_1^* \quad \text{for every } x \in B(x_0,R).$$

Thus, by (17) we obtain

$$u^*(x) = z_1^*(x) + t_0\Theta(x) \geq z_{1R}^*(x) + t_0\Theta(x) \quad \text{for all } x \in B(x_0,R).$$

Moreover, if $x \in B(x_0,\xi) \subset B(x_0,R)$, we have

$$\begin{aligned} u^*(x) &\geq \int_{|x-x_0|}^R \psi_{p'}\left(\int_0^\theta \left(\frac{s}{\theta}\right)^{N-1} w^*(s)f^*(s)\,ds\right)d\theta + t_0\Theta(r) \\ &\geq \int_\xi^R \psi_{p'}\left(\int_0^{\frac{\xi}{2}} \left(\frac{s}{\theta}\right)^{N-1} w^*(s)f^*(s)\,ds\right)d\theta + t_0\Theta(r). \end{aligned}$$

As $s \in [0, \frac{\xi}{2}]$, we conclude from (18) that there is $y \in B(x_0, \xi/2)$ such that

$$f^*(s) = f\bigl(v^*(y)\bigr).$$

Noting that $f(v^*(y)) \geq k_2(v^*)^{p-1}(y)$ and considering

$$t^* := \sup\bigl\{t : u^*(x) \geq t\Theta(x) \text{ and } v^*(x) \geq t\Theta(x) \text{ for all } x \in \Omega\bigr\}, \tag{19}$$

we have

$$\begin{aligned}
u^*(x) &\geq \int_\xi^R \psi_{p'}\left(\int_0^{\frac{\xi}{2}} \left(\frac{s}{\theta}\right)^{N-1} w^*(s) f^*(s)\,ds\right) d\theta + t_0\Theta(x) \\
&= \int_\xi^R \psi_{p'}\left(\int_0^{\frac{\xi}{2}} \left(\frac{s}{\theta}\right)^{N-1} w^*(s) f\bigl(v^*(y)\bigr)\,ds\right) d\theta + t_0\Theta(x) \\
&\geq \int_\xi^R \psi_{p'}\left(\int_0^{\frac{\xi}{2}} \left(\frac{s}{\theta}\right)^{N-1} w^*(s) k_2\bigl(v^*\bigr)^{p-1}(y)\,ds\right) d\theta + t_0\Theta(x) \\
&\geq k_2^{\frac{1}{p-1}} t^* \int_\xi^R \psi_{p'}\left(\int_0^{\frac{\xi}{2}} \left(\frac{s}{\theta}\right)^{N-1} w^*(s) \Theta^{p-1}(y)\,ds\right) d\theta + t_0\Theta(x) \\
&= k_2^{\frac{1}{p-1}} t^* \int_\xi^R \psi_{p'}\left(\int_0^{\frac{\xi}{2}} \left(\frac{s}{\theta}\right)^{N-1} w^*(s)\,ds\right) d\theta + t_0\Theta(x)
\end{aligned}$$

by the fact that $\Theta(y) = 1$ if $y \in B(x_0, \xi/2)$. Repeating the same ideas of the radial case, we conclude that

$$u^*(x) \geq \bigl(t_0 + t^*\bigr)\Theta(x),$$

which contradicts (19). By the additivity of index, it follows that

$$i(T, K_M \backslash K_\delta, K) = 1$$

proving the theorem. As in the radial case, the regularity follows from [14] and [15]. □

Remark 5 Due to the hypotheses on the nonlinearities and on the weight functions, simple applications of the maximum principle allow us to guarantee that if (u, v) is a solution of problem (P), then both u and v are strictly positive in Ω.

4 Examples

Example 6 Let us consider the problem

$$\begin{cases} -\Delta_p u = \alpha v^\beta & \text{in } \Omega, \\ -\Delta_q v = \gamma u^\mu & \text{in } \Omega, \\ (u, v) = (0, 0) & \text{on } \partial\Omega, \end{cases} \tag{20}$$

where $\alpha, \beta, \gamma, \mu > 0$ and $\Omega \subset \mathbb{R}^N$ $(N > 2)$ is a smooth domain. If $p - 1 > \beta$ and $q - 1 \geq \mu$, problem (20) has at least one positive solution.

Considering constants (8) and (9), it is enough to consider $0 < \delta < M$ such that

$$M \geq \max\left\{\left(\frac{\alpha}{k_1}\right)^{\frac{1}{(p-1)-\beta}}, \left(\frac{\gamma}{k_1}\right)^{\frac{1}{(q-1)-\mu}}\right\} \tag{21}$$

and

$$0 < \delta \leq \left(\frac{\alpha}{k_2}\right)^{\frac{1}{(p-1)-\beta}}. \tag{22}$$

Since $k_1 < k_2$, we have $\delta < M$.

Choosing δ and M as above, hypotheses (H1) and (H2) are verified and, as a consequence of Theorem 2, we guarantee the existence of a positive solution for coupled system (20). Furthermore, according to Theorem 4, it is easy to see that if (u,v) is the considered positive solution, we have $\|(u,v)\|_\infty$ as large as α is.

One of the advantages of our method is that conditions (1), (2) and (3) are not required. Let us see examples of these situations.

Consider the problem

$$\begin{cases} -\Delta_p u = f_1(v) & \text{in } \Omega, \\ -\Delta_q v = \gamma u^\mu & \text{in } \Omega, \\ (u,v) = (0,0) & \text{on } \partial\Omega, \end{cases} \tag{23}$$

where $\gamma, \mu > 0$, $\Omega \subset \mathbb{R}^N$ $(N > 2)$ is a smooth domain and f_1 is a nonlinearity satisfying (H1) and (H2). Examples of f_1 will be presented in the following examples.

Example 7 Consider the nonlinearity f_1 in problem (23) given by

$$f_1(v) := \begin{cases} \alpha v^\beta & \text{if } 0 \leq v \leq M, \\ \alpha(v^k - M^k + M^\beta) & \text{if } v > M, \end{cases} \tag{24}$$

where $\alpha > 0$, $k > 0$ and M is the positive constant whose existence is guaranteed in Example 6. If $p-1 > \beta$ and $q-1 > \mu$, the same arguments as those applied in Example 6 can guarantee the existence of a positive solution to (23).

It is clear that according to the constant k, condition (1) is not satisfied by the nonlinearities of problem (23). In fact, if $p = q$ and $k > \frac{(p-1)^2}{\mu}$, simple calculations show that

$$\lim_{x\to+\infty} \frac{f(Mg(x)^{\frac{1}{p-1}})}{x^{p-1}} = \alpha M^k \gamma^{\frac{k}{p-1}} \lim_{x\to+\infty} u^{\frac{\mu k-(p-1)^2}{p-1}} = +\infty$$

and condition (1) does not hold.

Example 8 Now, let us consider f_1 given by (24) as the nonlinearity of problem (23). In this case, we have

$$\lim_{x\to+\infty} \frac{(f(cg(x)^{\frac{1}{q-1}}))^{\frac{1}{p-1}}}{x^{p-1}} = \left(\alpha c^k \gamma^{\frac{k}{q-1}}\right)^{\frac{1}{p-1}} \lim_{x\to+\infty} u^{\frac{\mu k-(p-1)^2(q-1)}{(p-1)(q-1)}}.$$

In this way, $(f(cg(x)^{\frac{1}{q-1}}))^{\frac{1}{p-1}}$ can be either sub- or superlinear at $+\infty$ according to the constant k. Therefore, we have an example in which we guarantee the existence of a positive solution even if condition (3) is not satisfied.

Example 9 Finally, let us consider problem (23), with the nonlinearity f_1 given by

$$f_1(v) := \begin{cases} \alpha v^\beta & \text{if } 0 \le v \le M, \\ \alpha M^\beta & \text{if } v > M \end{cases}$$

in which M is the positive constant whose existence is guaranteed in Example 6, $p - 1 > \beta$ and $q - 1 > \mu$. As a consequence of previous examples, it is straightforward to guarantee the existence of a positive solution to this problem. Furthermore, it is clear that condition (2) does not hold.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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