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Existence of an unbounded branch of the set of solutions for equations of $p(x)$-Laplace type

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Abstract

We are concerned with the following nonlinear problem

$$-\operatorname{div}(\phi(x,|\nabla u|)\nabla u)=\mu|u|^{p(x)-2}u+f(\lambda,x,u,\nabla u)\quad \text{in } \Omega$$

subject to Dirichlet boundary conditions, provided that μ is not an eigenvalue of the $p(x)$-Laplacian. The purpose of this paper is to study the global behavior of the set of solutions for nonlinear equations of $p(x)$-Laplacian type by applying a bifurcation result for nonlinear operator equations.

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Keywords: $p(x)$-Laplacian; variable exponent Lebesgue-Sobolev spaces; weak solution; eigenvalue

1 Introduction

Rabinowitz [1] showed that the bifurcation occurring in the Krasnoselskii theorem is actually a global phenomenon by using the topological approach of Krasnoselskii [2]. As regards the p-Laplacian and generalized operators, nonlinear eigenvalue and bifurcation problems have been extensively studied by many researchers in various ways of approach; see [3–9]. While most of those results considered global branches bifurcating from the principal eigenvalue of the p-Laplacian, under suitable conditions, Väth [10] introduced another new approach to establish the existence of a global branch of solutions for the p-Laplacian problems by using nonlinear spectral theory for homogeneous operators. Recently, Kim and Väth [11] proposed a new approach. They observed the asymptotic behavior of an integral operator corresponding to the nonhomogeneous principal part at infinity and established the existence of an unbounded branch of solutions for equations involving nonhomogeneous operators of p-Laplace type.

In recent years, the study of differential equations and variational problems involving $p(x)$-growth conditions has received considerable attention since they can model physical phenomena which arise in the study of elastic mechanics, electro-rheological fluid dynamics and image processing, *etc.* We refer the readers to [12–15] and the references therein.

In this paper, we are concerned with the existence of an unbounded branch of the set of solutions for nonlinear elliptic equations of $p(x)$-Laplacian type subject to the Dirichlet boundary condition

$$\begin{cases} -\operatorname{div}(\phi(x,|\nabla u|)\nabla u) = \mu|u|^{p(x)-2}u + f(\lambda,x,u,\nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{B}$$

when μ is not an eigenvalue of

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = \mu|u|^{p(x)-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{E}$$

Here Ω is a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary $\partial\Omega$, the functions $\phi(x,t)$ are of type $|t|^{p(x)-2}$ with a continuous function $p:\overline{\Omega}\to(1,\infty)$ and $f:\mathbb{R}\times\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ satisfies a Carathéodory condition. When $p(x)$ is a constant function, the existence of an unbounded branch of the set of solutions for equations of p-Laplacian type operator is obtained in [11] (for generalizations to unbounded domains with weighted functions, see also [16, 17]). For the case of a variable function $p(x)$, the authors in [18] obtained the global bifurcation result for a class of degenerate elliptic equations by observing some properties of the corresponding integral operators in the weighted variable exponent Lebesgue-Sobolev spaces.

In the particular case when $\phi(x,t) = |t|^{p(x)-2}$, the operator involved in (B) is the $p(x)$-Laplacian. The studies for $p(x)$-Laplacian problems have been extensively considered by many researchers in various ways; see [18–23]. As far as we know, there are no papers concerned with the bifurcation theory for the nonlinear elliptic equations involving variable exponents except [18]. Noting that (B) has more complicated nonlinearities (it is nonhomogeneous) than the p-Laplacian equation, we need some more careful and new estimates. In particular, the fact that the principal eigenvalue for problem (E) is isolated plays a key role in obtaining the bifurcation result from the principal eigenvalue. Unfortunately, under some conditions on $p(x)$, the infimum of all positive eigenvalues for the $p(x)$-Laplacian might be zero; see [21]. This means that there is no principal eigenvalue for some variable exponent $p(x)$. Even if there exists a principal eigenvalue μ_*, this may not be isolated because μ_* is the infimum of all positive eigenvalues. Thus we cannot investigate the existence of global branches bifurcating from the principal eigenvalue of the $p(x)$-Laplacian. However, based on the work of Väth [10], global behavior of solutions for nonlinear problems involving the $p(x)$-Laplacian was considered in [18].

This paper is organized as follows. In Section 2, we state some basic results for the variable exponent Lebesgue-Sobolev spaces. In Section 3, some properties of the corresponding integral operators are presented. We will prove the main result on global bifurcation for problem (B) in Section 4. Finally, we give an example to illustrate our bifurcation result.

2 Preliminaries

In this section, we state some elementary properties for the variable exponent Lebesgue-Sobolev spaces which will be used in the next sections. The basic properties of the variable exponent Lebesgue-Sobolev spaces can be found in [24, 25].

To make a self-contained paper, we first recall some definitions and basic properties of the variable exponent Lebesgue spaces $L^{p(x)}(\Omega)$ and the variable exponent Lebesgue-Sobolev spaces $W^{1,p(x)}(\Omega)$.

Set

$$C_+(\overline{\Omega}) = \Big\{h \in C(\overline{\Omega}) : \min_{x\in\overline{\Omega}} h(x) > 1\Big\}.$$

For any $h \in C_+(\overline{\Omega})$, we define

$$h_+ = \sup_{x\in\Omega} h(x) \quad \text{and} \quad h_- = \inf_{x\in\Omega} h(x).$$

For any $p \in C_+(\overline{\Omega})$, we introduce the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) := \left\{u : u \text{ is a measurable real-valued function}, \int_\Omega \big|u(x)\big|^{p(x)}\,dx < \infty\right\},$$

endowed with the Luxemburg norm

$$\|u\|_{L^{p(x)}(\Omega)} = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \le 1\right\}.$$

The dual space of $L^{p(x)}(\Omega)$ is $L^{p'(x)}(\Omega)$, where $1/p(x) + 1/p'(x) = 1$. The variable exponent Lebesgue spaces are a special case of Orlicz-Musielak spaces treated by Musielak in [26].

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \big\{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\big\},$$

where the norm is

$$\|u\|_{W^{1,p(x)}(\Omega)} = \|u\|_{L^{p(x)}(\Omega)} + \|\nabla u\|_{L^{p(x)}(\Omega)}. \tag{2.1}$$

Definition 2.1 The exponent $p(\cdot)$ is said to be log-Hölder continuous if there is a constant C such that

$$\big|p(x) - p(y)\big| \le \frac{C}{-\log|x-y|} \tag{2.2}$$

for every $x, y \in \Omega$ with $|x-y| \le 1/2$.

Without additional assumptions on the exponent $p(x)$, smooth functions are not dense in the variable exponent Sobolev spaces. This was considered by Zhikov [27] in connection with Lavrentiev phenomenon. The importance of this above notion relies on the following fact: if $p(x)$ is log-Hölder continuous, then $C_0^\infty(\Omega)$ is dense in the variable exponent Sobolev spaces $W^{1,p(x)}(\Omega)$ (see [28, 29]).

Lemma 2.2 ([24, 25]) *The space $L^{p(x)}(\Omega)$ is a separable, uniformly convex Banach space, and its conjugate space is $L^{p'(x)}(\Omega)$, where $1/p(x) + 1/p'(x) = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have*

$$\left|\int_\Omega uv\,dx\right| \le \left(\frac{1}{p_-} + \frac{1}{(p')_-}\right)\|u\|_{L^{p(x)}(\Omega)}\|v\|_{L^{p'(x)}(\Omega)} \le 2\|u\|_{L^{p(x)}(\Omega)}\|v\|_{L^{p'(x)}(\Omega)}.$$

Lemma 2.3 ([24]) *Denote*

$$\rho(u) = \int_\Omega |u|^{p(x)}\,dx \quad \text{for all } u \in L^{p(x)}(\Omega).$$

Then

(1) $\rho(u) > 1$ $(= 1; < 1)$ *if and only if* $\|u\|_{L^{p(x)}(\Omega)} > 1$ $(= 1; < 1)$, *respectively*;
(2) *If* $\|u\|_{L^{p(x)}(\Omega)} > 1$, *then* $\|u\|^{p_-}_{L^{p(x)}(\Omega)} \le \rho(u) \le \|u\|^{p_+}_{L^{p(x)}(\Omega)}$;
(3) *If* $\|u\|_{L^{p(x)}(\Omega)} < 1$, *then* $\|u\|^{p_+}_{L^{p(x)}(\Omega)} \le \rho(u) \le \|u\|^{p_-}_{L^{p(x)}(\Omega)}$.

Lemma 2.4 ([23]) *Let* $q \in L^\infty(\Omega)$ *be such that* $1 \le p(x)q(x) \le \infty$ *for almost all* $x \in \Omega$. *If* $u \in L^{q(x)}(\Omega)$ *with* $u \neq 0$, *then*

(1) *If* $\|u\|_{L^{p(x)q(x)}(\Omega)} > 1$, *then* $\|u\|^{q_-}_{L^{p(x)q(x)}(\Omega)} \le \||u|^{q(x)}\|_{L^{p(x)}(\Omega)} \le \|u\|^{q_+}_{L^{p(x)q(x)}(\Omega)}$;
(2) *If* $\|u\|_{L^{p(x)q(x)}(\Omega)} < 1$, *then* $\|u\|^{q_+}_{L^{p(x)q(x)}(\Omega)} \le \||u|^{q(x)}\|_{L^{p(x)}(\Omega)} \le \|u\|^{q_-}_{L^{p(x)q(x)}(\Omega)}$.

Lemma 2.5 ([20]) *Let* $p \in C_+(\overline{\Omega})$ *satisfy the log-Hölder continuity condition* (2.2). *Then, for* $u \in W_0^{1,p(\cdot)}(\Omega)$, *the* $p(\cdot)$*-Poincaré inequality*

$$\|u\|_{L^{p(x)}(\Omega)} \le C\|\nabla u\|_{L^{p(x)}(\Omega)}$$

holds, where the positive constant C *depends on* p *and* Ω.

Lemma 2.6 ([28]) *Let* $\Omega \subset \mathbb{R}^N$ *be an open, bounded set with Lipschitz boundary and* $p \in C_+(\overline{\Omega})$ *with* $1 < p_- \le p_+ < \infty$ *satisfy the log-Hölder continuity condition* (2.2). *If* $q \in L^\infty(\Omega)$ *with* $q_- > 1$ *satisfies*

$$q(x) \le p^*(x) := \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } N > p(x), \\ +\infty & \text{if } N \le p(x), \end{cases} \tag{2.3}$$

for all $x \in \Omega$, *then we have*

$$W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$$

and the imbedding is compact if $\inf_{x\in\Omega}(p^*(x) - q(x)) > 0$.

3 Properties of the integral operators

In this section, we shall give some properties of the integral operators corresponding to problem (B) by applying the basic properties of the spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ which were given in the previous section.

Throughout this paper, let $p \in C_+(\overline{\Omega})$ satisfy the log-Hölder continuity condition (2.2) and $X := W_0^{1,p(x)}(\Omega)$ with the norm

$$\|u\|_X = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{\nabla u(x)}{\lambda}\right|^{p(x)} dx \le 1\right\},$$

which is equivalent to norm (2.1) due to Lemma 2.5.

Denote

$$\Omega_1 = \{x \in \Omega : 1 < p(x) < 2\}, \qquad \Omega_2 = \{x \in \Omega : p(x) \geq 2\}.$$

(We allow the case that one of these sets is empty.) Then it is obvious that $\Omega = \Omega_1 \cup \Omega_2$. We assume that

(HJ1) $\phi : \Omega \times [0,\infty) \to [0,\infty)$ satisfies the following conditions: $\phi(\cdot,\eta)$ is measurable on Ω for all $\eta \geq 0$ and $\phi(x,\cdot)$ is locally absolutely continuous on $[0,\infty)$ for almost all $x \in \Omega$.

(HJ2) There are a function $a \in L^{p'(x)}(\Omega)$ and a nonnegative constant b such that

$$\left|\phi(x,|v|)v\right| \leq a(x) + b|v|^{p(x)-1}$$

for almost all $x \in \Omega$ and for all $v \in \mathbb{R}^N$.

(HJ3) There exists a positive constant c such that the following conditions are satisfied for almost all $x \in \Omega$:

$$\phi(x,\eta) \geq c\eta^{p(x)-2} \quad \text{and} \quad \eta\frac{\partial\phi}{\partial\eta}(x,\eta) + \phi(x,\eta) \geq c\eta^{p(x)-2} \tag{3.1}$$

for almost all $\eta \in (0,1)$. If $x \in \Omega_2$, then condition (3.1) holds for almost all $\eta \in (1,\infty)$, and if $x \in \Omega_1$, then assume for almost all $\eta \in (1,\infty)$ instead

$$\phi(x,\eta) \geq c \quad \text{and} \quad \eta\frac{\partial\phi}{\partial\eta}(x,\eta) + \phi(x,\eta) \geq c. \tag{3.2}$$

Let $\langle\cdot,\cdot\rangle$ denote the usual of X and its dual X^* or the Euclidean scalar product on $\mathbb{R}^N$, respectively. Under hypotheses (HJ1) and (HJ2), we define an operator $J : X \to X^*$ by

$$\langle J(u),\varphi\rangle = \int_\Omega \langle \phi(x,|\nabla u(x)|)\nabla u(x), \nabla\varphi(x)\rangle\,dx \tag{3.3}$$

for all $\varphi \in X$.

The following estimate is a starting point for obtaining that the operator J is a homeomorphism. When $p(x)$ is constant, this is a particular form of Corollary 3.1 in [11] which is based on Lemma 3.1 in [6]; see [30, Lemma 1]. In fact, the special case that ϕ is independent of x is considered in [6]. The proof of the following proposition is essentially the same as that in [31]. For convenience, we give the proof.

Proposition 3.1 *Let* (HJ1) *and* (HJ3) *be satisfied. Then the following estimate*

$$\begin{aligned}&\langle \phi(x,|u|)u - \phi(x,|v|)v, u-v\rangle \\ &\quad\geq \begin{cases} c\min\{1,(|u|+|v|)^{p(x)-2}\}|u-v|^2 & \text{if } x \in \Omega_1 \text{ and } (u,v) \neq (0,0), \\ 4^{1-p_+}c|u-v|^{p(x)} & \text{if } x \in \Omega_2 \end{cases}\end{aligned} \tag{3.4}$$

holds for all $u, v \in \mathbb{R}^N$, where c is the positive constant from (HJ3).

Proof Let $u, v \in \mathbb{R}^N$ with $(u, v) \neq (0, 0)$. Let $\psi_i(x, u) = \phi(x, |u|)u_i$ for $i = 1, \dots, N$ and set $\eta = |u|$. Observe that

$$\begin{aligned}\sum_{i,j=1}^{N} \frac{\partial \psi_i(x,u)}{\partial u_j} w_i w_j &= \frac{1}{\eta}\frac{\partial \phi}{\partial \eta}(x,\eta)\sum_{i,j=1}^{N} w_i u_i w_j u_j + \phi(x,\eta)\sum_{i=1}^{N} w_i^2 \\ &= \frac{1}{\eta}\frac{\partial \phi}{\partial \eta}(x,\eta)\langle w, u\rangle^2 + \phi(x,\eta)|w|^2 \\ &= |w|^2\left(\eta\frac{\partial \phi}{\partial \eta}(x,\eta)\left\langle \frac{w}{|w|}, \frac{u}{\eta}\right\rangle^2 + \phi(x,\eta)\right)\end{aligned}$$

for all $u, w \in \mathbb{R}^N \setminus \{0\}$. We assume that $x \in \Omega_2$. Setting $\lambda = \langle w/|w|, u/\eta\rangle^2$, it follows from (3.1) that

$$\begin{aligned}\lambda\left(\eta\frac{\partial \phi}{\partial \eta}(x,\eta)\right) + \phi(x,\eta) &= (1-\lambda)\phi(x,\eta) + \lambda\left(\eta\frac{\partial \phi}{\partial \eta}(x,\eta) + \phi(x,\eta)\right) \\ &\geq c|u|^{p(x)-2},\end{aligned}$$

and so

$$\sum_{i,j=1}^{N} \frac{\partial \psi_i(x,u)}{\partial u_j} w_i w_j \geq c|u|^{p(x)-2}|w|^2. \tag{3.5}$$

Noticing that

$$\psi_i(x,u) - \psi_i(x,v) = \int_0^1 \sum_{j=1}^{N} \frac{\partial \psi_i(x,\overline{u})}{\partial \overline{u}_j}(u_j - v_j)\,dt, \tag{3.6}$$

where $\overline{u} = v + t(u - v)$, we have by (3.5) and (3.6) that

$$\begin{aligned}\big\langle \phi\big(x,|u|\big)u - \phi\big(x,|v|\big)v, u - v\big\rangle &= \sum_{i=1}^{N}\big(\psi_i(x,u) - \psi_i(x,v)\big)(u_i - v_i) \\ &= \int_0^1 \sum_{i,j=1}^{N} \frac{\partial \psi_i}{\partial \overline{u}_j}\big(x, v + t(u-v)\big)(u_j - v_j)(u_i - v_i)\,dt \\ &\geq \int_0^1 c\big|v + t(u-v)\big|^{p(x)-2}|u-v|^2\,dt.\end{aligned}$$

Without loss of generality, we may suppose that $|u| \leq |v|$. Then we obtain, for all $t \in [0, 1/4]$,

$$\big|v + t(u-v)\big| \geq |v| - \frac{1}{4}|u-v| \geq \frac{1}{4}|u-v|$$

and

$$\begin{aligned}\big\langle \phi\big(x,|u|\big)u - \phi\big(x,|v|\big)v, u - v\big\rangle &\geq \int_0^1 c\big|v + t(u-v)\big|^{p(x)-2}|u-v|^2\,dt \\ &\geq 4^{1-p_+}c|u-v|^{p(x)}.\end{aligned}$$

Now assume that $x \in \Omega_1$. As before, we obtain from (3.1) and (3.2) that

$$\sum_{i,j=1}^{N} \frac{\partial \psi_i(x,u)}{\partial u_j} w_i w_j \geq c \min\{1, |u|^{p(x)-2}\} |w|^2$$

for $u, w \in \mathbb{R}^N \setminus \{0\}$. Using the fact that $|tu + (1-t)v| \leq |u| + |v|$, we get

$$\begin{aligned}
\langle \phi(x,|u|)u - \phi(x,|v|)v, u - v \rangle &= \int_0^1 \sum_{i,j=1}^{N} \frac{\partial \psi_i}{\partial \overline{u}_j} \big(x, v + t(u-v)\big)(u_j - v_j)(u_i - v_i)\, dt \\
&\geq \int_0^1 c \min\{1, |v + t(u-v)|^{p(x)-2}\} |u-v|^2\, dt \\
&\geq c \min\{1, (|u| + |v|)^{p(x)-2}\} |u-v|^2.
\end{aligned}$$

This completes the proof. □

From Proposition 3.1, we can obtain the following result.

Theorem 3.2 *Assume that* (HJ1)-(HJ3) *hold. Then the operator* $J : X \to X^*$ *is a continuous, bounded, strictly monotone and coercive on* X.

Proof In view of (HJ1) and (HJ2), the superposition operator

$$\Lambda(u)(x) := \phi(x, |u(x)|) u(x)$$

acts from $L^{p(x)}(\Omega, \mathbb{R}^N)$ into $L^{p'(x)}(\Omega, \mathbb{R}^N)$ and is continuous; see Corollary 5.2.1 in [32]. Hence the continuity of J follows from the fact that J is the composition of the continuous map $\nabla : X \to L^{p(x)}(\Omega, \mathbb{R}^N)$, the map Λ and the bounded linear map $D : L^{p'(x)}(\Omega, \mathbb{R}^N) \to X^*$ given by

$$\langle Dv, w \rangle = \int_\Omega \langle v(x), \nabla w(x) \rangle\, dx.$$

Hence the operator J is bounded and continuous on X.

For any u in X with $\|u\|_X > 1$, it follows from (HJ3) that

$$\langle J(u), u \rangle \geq C \|u\|_X^{p_-}$$

for some positive constant C. Thus we get that

$$\frac{\langle J(u), u \rangle}{\|u\|_X} \to \infty$$

as $\|u\|_X \to \infty$ and therefore the operator J is coercive on X.

Next we will show that the operator J is strictly monotone on X. Set

$$p_0 = \inf_{x \in \Omega_1} p(x), \qquad p_1 = \sup_{x \in \Omega_1} p(x)$$

and

$$p_2 = \inf_{x\in\Omega_2} p(x), \qquad p_3 = \sup_{x\in\Omega_2} p(x).$$

(Of course, if the sets Ω_1 and Ω_2 are nonempty, then $p_1 = p_2 = 2$ by the continuity of $p(x)$.) It is clear that

$$\begin{aligned}
&\langle J(u) - J(v), u - v\rangle \\
&\quad= \int_\Omega \langle \phi(x, |\nabla u(x)|)\nabla u(x) - \phi(x, |\nabla v(x)|)\nabla v(x), \nabla u(x) - \nabla v(x)\rangle\,dx \\
&\quad= \int_{\Omega_1} \langle \phi(x, |\nabla u(x)|)\nabla u(x) - \phi(x, |\nabla v(x)|)\nabla v(x), \nabla u(x) - \nabla v(x)\rangle\,dx \\
&\qquad+ \int_{\Omega_2} \langle \phi(x, |\nabla u(x)|)\nabla u(x) - \phi(x, |\nabla v(x)|)\nabla v(x), \nabla u(x) - \nabla v(x)\rangle\,dx.
\end{aligned}$$

To get strict monotonicity of the operator J, without loss of generality, we divide the proof into two cases.

Case 1. Let u, v be in X with $\|\nabla u - \nabla v\|_{L^{p(x)}(\Omega_i)} > 1$ for $i = 1, 2$. By Proposition 3.1, we have

$$\langle \phi(x, |\nabla u(x)|)\nabla u(x) - \phi(x, |\nabla v(x)|)\nabla v(x), \nabla u(x) - \nabla v(x)\rangle \ge 4^{1-p_+} c|\nabla u(x) - \nabla v(x)|^{p(x)}$$

for almost all $x \in \Omega_2$. Integrating the above inequality over Ω and using Lemma 2.3, we assert that

$$\begin{aligned}
\langle J(u) - J(v), u - v\rangle &= \int_\Omega \langle \phi(x, |\nabla u(x)|)\nabla u(x) - \phi(x, |\nabla v(x)|)\nabla v(x), \nabla u(x) - \nabla v(x)\rangle\,dx \\
&\ge 4^{1-p_+} c\|\nabla u - \nabla v\|^{p_2}_{L^{p(x)}(\Omega_2)} \\
&\ge 4^{1-p_+} c\|\nabla u - \nabla v\|^{\frac{2p_0}{p_1}}_{L^{p(x)}(\Omega_2)}.
\end{aligned} \tag{3.7}$$

For almost all $x \in \Omega_1$, by Proposition 3.1, we have

$$\begin{aligned}
\langle J(u) - J(v), u - v\rangle &= \int_\Omega \langle \phi(x, |\nabla u(x)|)\nabla u(x) - \phi(x, |\nabla v(x)|)\nabla v(x), \nabla u(x) - \nabla v(x)\rangle\,dx \\
&\ge c\int_{\Omega_1} \ell^{p(x)-2}|\nabla u(x) - \nabla v(x)|^2\,dx,
\end{aligned} \tag{3.8}$$

where $\ell(x) = \min\{1, |\nabla u(x)| + |\nabla v(x)|\}$. From Hölder's inequality in Lemma 2.2, we obtain that

$$\begin{aligned}
&\int_{\Omega_1} |\nabla u(x) - \nabla v(x)|^{p(x)}\,dx \\
&\quad= \int_{\Omega_1} \ell^{\frac{p(x)(2-p(x))}{2}} \left(\ell^{\frac{p(x)(p(x)-2)}{2}} |\nabla u(x) - \nabla v(x)|^{p(x)}\right) dx \\
&\quad\le 2\left\|\ell^{\frac{p(x)(2-p(x))}{2}}\right\|_{L^{\frac{2}{2-p(x)}}(\Omega_1)} \left\|\ell^{\frac{p(x)(p(x)-2)}{2}} |\nabla u - \nabla v|^{p(x)}\right\|_{L^{\frac{2}{p(x)}}(\Omega_1)}.
\end{aligned} \tag{3.9}$$

The first term on the right-hand side in (3.9) is calculated by Lemma 2.4 as follows:

$$\begin{aligned}
&\left\| \ell^{\frac{p(x)(2-p(x))}{2}} \right\|_{L^{\frac{2}{2-p(x)}}(\Omega_1)} \\
&\quad \le \|\ell\|_{L^{p(x)}(\Omega_1)}^{\frac{p_0(2-p_1)}{2}} + \|\ell\|_{L^{p(x)}(\Omega_1)}^{\frac{p_1(2-p_0)}{2}} \\
&\quad \le \big\| |\nabla u| + |\nabla v| \big\|_{L^{p(x)}(\Omega_1)}^{\frac{p_0(2-p_1)}{2}} + \big\| |\nabla u| + |\nabla v| \big\|_{L^{p(x)}(\Omega_1)}^{\frac{p_1(2-p_0)}{2}} \\
&\quad \le \left(\|\nabla u\|_{L^{p(x)}(\Omega)} + \|\nabla v\|_{L^{p(x)}(\Omega)}\right)^{\frac{p_0(2-p_1)}{2}} + \left(\|\nabla u\|_{L^{p(x)}(\Omega)} + \|\nabla v\|_{L^{p(x)}(\Omega)}\right)^{\frac{p_1(2-p_0)}{2}}
\end{aligned}$$

for any $u, v \in X$. Since $\|\nabla u - \nabla v\|_{L^{p(x)}(\Omega_1)} > 1$, we deduce that

$$\left\| \ell^{\frac{p(x)(2-p(x))}{2}} \right\|_{L^{\frac{2}{2-p(x)}}(\Omega_1)} \le 2\left(\|u\|_X + \|v\|_X\right)^{\frac{p_1(2-p_0)}{2}}. \tag{3.10}$$

If

$$\int_{\Omega_1} \ell^{p(x)-2} \left|\nabla u(x) - \nabla v(x)\right|^2 dx > 1,$$

then it follows from (3.9), (3.10), Lemmas 2.3 and 2.4 that

$$\begin{aligned}
\|\nabla u - \nabla v\|_{L^{p(x)}(\Omega_1)}^{p_0} &\le 4\left(\|u\|_X + \|v\|_X\right)^{\frac{p_1(2-p_0)}{2}} \left\| \ell^{\frac{p(x)(p(x)-2)}{2}} |\nabla u - \nabla v|^{p(x)} \right\|_{L^{\frac{2}{p(x)}}(\Omega_1)} \\
&\le 4\left(\|u\|_X + \|v\|_X\right)^{\frac{p_1(2-p_0)}{2}} \left(\int_{\Omega_1} \ell^{p(x)-2} \left|\nabla u(x) - \nabla v(x)\right|^2 dx \right)^{\frac{p_1}{2}}.
\end{aligned}$$

Hence we deduce that

$$\begin{aligned}
\langle J(u) - J(v), u - v\rangle &\ge c4^{-\frac{2}{p_1}} \left(\|u\|_X + \|v\|_X\right)^{p_0-2} \|\nabla u - \nabla v\|_{L^{p(x)}(\Omega_1)}^{\frac{2p_0}{p_1}} \\
&\ge c4^{-\frac{2}{p_1}} \left(\|u\|_X + \|v\|_X\right)^{\frac{p_1(p_0-2)}{p_0}} \|\nabla u - \nabla v\|_{L^{p(x)}(\Omega_1)}^{\frac{2p_0}{p_1}}.
\end{aligned} \tag{3.11}$$

On the other hand, if

$$\int_{\Omega_1} \ell^{p(x)-2} \left|\nabla u(x) - \nabla v(x)\right|^2 dx < 1,$$

then the analogous argument implies that

$$\langle J(u) - J(v), u - v\rangle \ge C_1\left(\|u\|_X + \|v\|_X\right)^{\frac{p_1(p_0-2)}{p_0}} \|\nabla u - \nabla v\|_{L^{p(x)}(\Omega_1)}^{2} \tag{3.12}$$

for some positive constant C_1. From the previous inequalities (3.11) and (3.12), we have that

$$\langle J(u) - J(v), u - v\rangle \ge C_2\left(\|u\|_X + \|v\|_X\right)^{\frac{p_1(p_0-2)}{p_0}} \|\nabla u - \nabla v\|_{L^{p(x)}(\Omega_1)}^{\frac{2p_0}{p_1}}, \tag{3.13}$$

where C_2 is a positive constant. Consequently, we obtain by (3.7) and (3.13) that

$$\begin{aligned}
&\langle J(u) - J(v), u - v\rangle \\
&\quad = \int_\Omega \langle \phi(x, |\nabla u(x)|)\nabla u(x) - \phi(x, |\nabla v(x)|)\nabla v(x), \nabla u(x) - \nabla v(x)\rangle\,dx \\
&\quad \geq C_2\left(\|u\|_X + \|v\|_X\right)^{\frac{p_1(p_0-2)}{p_0}} \|\nabla u - \nabla v\|_{L^{p(x)}(\Omega_1)}^{\frac{2p_0}{p_1}} + 4^{1-p_+}c\|\nabla u - \nabla v\|_{L^{p(x)}(\Omega_2)}^{\frac{2p_0}{p_1}} \\
&\quad \geq C_3\left(\|u\|_X + \|v\|_X\right)^{\frac{p_1(p_0-2)}{p_0}} \|\nabla u - \nabla v\|_{L^{p(x)}(\Omega)}^{\frac{2p_0}{p_1}} \\
&\quad \geq C_4\left(\|u\|_X + \|v\|_X\right)^{\frac{p_1(p_0-2)}{p_0}} \|u - v\|_X^{\frac{2p_0}{p_1}}
\end{aligned}$$

for some positive constants C_3 and C_4.

Case 2. Let u, v be in X with $\|\nabla u - \nabla v\|_{L^{p(x)}(\Omega_i)} < 1$ for $i = 1, 2$ and $(u, v) \neq (0, 0)$. For almost all $x \in \Omega_1$, the following inequality holds:

$$\left(|\nabla u| + |\nabla v|\right)^{p(x)-2}|\nabla u - \nabla v|^2 \leq |\nabla u - \nabla v|^{p(x)}. \tag{3.14}$$

From the above relation (3.14) and Lemmas 2.2 and 2.4, we obtain that

$$\begin{aligned}
&\int_{\Omega_1} |\nabla u - \nabla v|^{p(x)}\,dx \\
&\quad = \int_{\Omega_1} \left(|\nabla u| + |\nabla v|\right)^{\frac{p(x)(2-p(x))}{2}} \left(\left(|\nabla u| + |\nabla v|\right)^{\frac{p(x)(p(x)-2)}{2}} |\nabla u - \nabla v|^{p(x)}\right)dx \\
&\quad \leq 2\left\|\left(|\nabla u| + |\nabla v|\right)^{\frac{p(x)(2-p(x))}{2}}\right\|_{L^{\frac{2}{2-p(x)}}(\Omega_1)} \left\|\left(|\nabla u| + |\nabla v|\right)^{\frac{p(x)(p(x)-2)}{2}} |\nabla u - \nabla v|^{p(x)}\right\|_{L^{\frac{2}{p(x)}}(\Omega_1)} \\
&\quad \leq 2\left(\|\nabla u\|_{L^{p(x)}(\Omega)} + \|\nabla v\|_{L^{p(x)}(\Omega)}\right)^\alpha \\
&\qquad \times \left(\int_{\Omega_1} \left(|\nabla u| + |\nabla v|\right)^{p(x)-2}|\nabla u - \nabla v|^2\,dx\right)^{\frac{p_0}{2}},
\end{aligned} \tag{3.15}$$

where α is either $\frac{p_0(2-p_1)}{2}$ or $\frac{p_1(2-p_0)}{2}$. Since $(|\nabla u| + |\nabla v|)^{p(x)-2}|\nabla u - \nabla v|^2 \leq \ell^{p(x)-2}|\nabla u - \nabla v|^2$, we assert by (3.15) and Proposition 3.1 that

$$\|\nabla u - \nabla v\|_{L^{p(x)}(\Omega_1)}^{p_1} \leq 2\left(\|u\|_X + \|v\|_X\right)^\alpha \left(\int_{\Omega_1} \ell^{p(x)-2}|\nabla u - \nabla v|^2\,dx\right)^{\frac{p_0}{2}},$$

and so

$$\langle J(u) - J(v), u - v\rangle \geq c2^{-\frac{2}{p_0}} \left(\|u\|_X + \|v\|_X\right)^{-\frac{2\alpha}{p_0}} \|\nabla u - \nabla v\|_{L^{p(x)}(\Omega_1)}^{\frac{2p_1}{p_0}} \tag{3.16}$$

for almost all $x \in \Omega_1$. For almost all $x \in \Omega_2$, Proposition 3.1 yields the following estimate:

$$\langle J(u) - J(v), u - v\rangle \geq 4^{1-p_+}c\|\nabla u - \nabla v\|_{L^{p(x)}(\Omega_2)}^{p_3}. \tag{3.17}$$

Consequently, it follows from (3.16) and (3.17) that

$$\begin{aligned}
&\langle J(u) - J(v), u - v\rangle \\
&\geq c2^{-\frac{2}{p_0}} \left(\|u\|_X + \|v\|_X\right)^{-\frac{2\alpha}{p_0}} \|\nabla u - \nabla v\|_{L^{p(x)}(\Omega_1)}^{\frac{2p_1}{p_0}} + 4^{1-p_+} c\|\nabla u - \nabla v\|_{L^{p(x)}(\Omega_2)}^{p_3} \\
&\geq C_5 \min\left\{c2^{-\frac{2}{p_0}} \left(\|u\|_X + \|v\|_X\right)^{-\frac{2\alpha}{p_0}}, 4^{1-p_+} c\right\} \|u - v\|_X^{\max\{\frac{2p_1}{p_0}, p_3\}} \qquad (3.18)
\end{aligned}$$

for some constant $C_5 > 0$. This completes the proof. □

Using the previous result, we show the topological property of the operator J which will be needed in the main result of the next section.

Lemma 3.3 *If* (HJ1), (HJ2) *and* (HJ3) *hold, then* $J : X \to X^*$ *is a homeomorphism onto* X^*.

Proof From Theorem 3.2, we see that $J : X \to X^*$ is strictly monotone and coercive. The Browder-Minty theorem hence implies that the inverse operator $J^{-1} : X^* \to X$ exists and is bounded; see Theorem 26.A. in [33]. For each $h \in X^*$, let (h_n) be any sequence in X^* that converges to h in X^*. Set $u_n = J^{-1}(h_n)$ and $u = J^{-1}(h)$ with $\|u_n - u\|_X < 1$. We obtain from (3.18) that

$$\|u_n - u\|_X \leq C_5^{-\frac{1}{\beta}} \min\left\{c2^{-\frac{2}{p_0}} \left(\|u_n\|_X + \|u\|_X\right)^{-\frac{2\alpha}{p_0}}, 4^{1-p_+} c\right\}^{-\frac{1}{\beta}} \left\|J(u_n) - J(u)\right\|_{X^*}^{\frac{1}{\beta}},$$

where $\beta = \max\{\frac{2p_1}{p_0}, p_3\}$. Since $\{u_n : n \in \mathbb{N}\}$ is bounded in X and $J(u_n) \to J(u)$ in X^* as $n \to \infty$, it follows that (u_n) converges to u in X. Thus, J^{-1} is continuous at each $h \in X^*$. This completes the proof. □

The main idea in obtaining our bifurcation result is to study the asymptotic behavior of the integral operator J and then to deduce a spectral result for operators that are not necessarily homogeneous. To do this, we consider a function $\phi_{p(\cdot)} : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ defined by

$$\phi_{p(x)}(x, v) := |v|^{p(x)-2} v$$

and an operator $J_{p(\cdot)} : X \to X^*$ defined by

$$\langle J_{p(\cdot)}(u), \varphi\rangle := \int_\Omega \left\langle \phi_{p(x)}\left(x, \nabla u(x)\right), \nabla\varphi(x)\right\rangle dx$$

for all $\varphi \in X$.

To discuss the asymptotic behavior of J, we require the following hypothesis.

(HJ4) For each $\varepsilon > 0$, there is a function $M \in L^{p(x)}(\Omega)$ such that for almost all $x \in \Omega$ the following holds:

$$\frac{|\phi(x, |v|)v - \phi_{p(x)}(x, v)|}{|v|^{p_- -1}} \leq \varepsilon$$

for all $v \in \mathbb{R}^N$ with $|v| > |M(x)|$.

Now we can show that the operators J and $J_{p(\cdot)}$ are asymptotic at infinity, as in Proposition 5.1 of [11].

Proposition 3.4 *Assume that* (HJ1), (HJ2) *and* (HJ4) *are fulfilled. Then we have*

$$\lim_{\|u\|_X \to \infty} \frac{\|J(u) - J_{p(\cdot)}(u)\|_{X^*}}{\|u\|_X^{p_- -1}} = 0.$$

Proof Given $\varepsilon > 0$, choose an $M \in L^{p(x)}(\Omega)$ such that for almost all $x \in \Omega$ the following holds:

$$\bigl|\phi\bigl(x, |v|\bigr)v - \phi_{p(x)}(x, v)\bigr| \le \varepsilon |v|^{p_- -1}$$

for all $v \in \mathbb{R}^N$ with $|v| > |M(x)|$. We have by (HJ2) that for almost all $x \in \Omega$ the estimate

$$\bigl|\phi\bigl(x, |v|\bigr)v - \phi_{p(x)}(x, v)\bigr| \le a(x) + (b+1)\bigl|M(x)\bigr|^{p(x)-1}$$

holds for all $v \in \mathbb{R}^N$ with $|v| \le |M(x)|$. Set

$$\alpha_M(x) = a(x) + (b+1)\bigl|M(x)\bigr|^{p(x)-1}.$$

Then α_M belongs to $L^{p'(x)}(\Omega)$ and for almost all $x \in \Omega$, the estimate

$$\bigl|\phi\bigl(x, |v|\bigr)v - \phi_{p(x)}(x, v)\bigr| \le \max\bigl\{\bigl|\alpha_M(x)\bigr|, \varepsilon |v|^{p_- -1}\bigr\}$$

holds for all $v \in \mathbb{R}^N$. From Hölder's inequality, we have that

$$\begin{aligned}
\bigl|\bigl\langle J(u) - J_{p(\cdot)}(u), \varphi\bigr\rangle\bigr| &= \left|\int_\Omega \bigl\langle \phi\bigl(x, \bigl|\nabla u(x)\bigr|\bigr)\nabla u(x) - \phi_{p(x)}\bigl(x, \nabla u(x)\bigr), \nabla\varphi(x)\bigr\rangle \, dx\right| \\
&\le 2\bigl\|\phi\bigl(x, |\nabla u|\bigr)\nabla u - \phi_{p(x)}(x, \nabla u)\bigr\|_{L^{p'(x)}(\Omega)} \|\nabla\varphi\|_{L^{p(x)}(\Omega)}
\end{aligned}$$

for all $\varphi \in X$, and hence for each $u \in X$, we obtain by Minkowski's inequality and the fact that $W_0^{1,p(x)}(\Omega) \hookrightarrow W_0^{1,p_-}(\Omega)$ that

$$\begin{aligned}
\bigl\|J(u) - J_{p(\cdot)}(u)\bigr\|_{X^*} &= \sup_{\|\varphi\|_X \le 1} \bigl|\bigl\langle J(u) - J_{p(\cdot)}(u), \varphi\bigr\rangle\bigr| \\
&\le 2\bigl(\bigl\|\phi\bigl(x, |\nabla u|\bigr)\nabla u - \phi_{p(x)}(x, \nabla u)\bigr\|_{L^{p'(x)}(\Omega)}\bigr) \\
&\le 2\bigl(\|\alpha_M\|_{L^{p'(x)}(\Omega)} + \varepsilon \bigl\||\nabla u|^{p_- -1}\bigr\|_{L^{p'(x)}(\Omega)}\bigr) \\
&\le 2\bigl(\|\alpha_M\|_{L^{p'(x)}(\Omega)} + \varepsilon C_1 \bigl\||\nabla u|^{p_- -1}\bigr\|_{L^{(p')_+}(\Omega)}\bigr) \\
&\le 2\left(\|\alpha_M\|_{L^{p'(x)}(\Omega)} + \varepsilon C_1 \left(\int_\Omega |\nabla u|^{p_-} \, dx\right)^{\frac{p_- -1}{p_-}}\right) \\
&\le 2\bigl(\|\alpha_M\|_{L^{p'(x)}(\Omega)} + \varepsilon C_1 \|u\|_{W_0^{1,p_-}(\Omega)}^{p_- -1}\bigr) \\
&\le 2\bigl(\|\alpha_M\|_{L^{p'(x)}(\Omega)} + \varepsilon C_2 \|u\|_X^{p_- -1}\bigr)
\end{aligned}$$

for some positive constants C_1 and C_2. From

$$\frac{\|J(u) - J_{p(\cdot)}(u)\|_{X^*}}{\|u\|_X^{p_- - 1}} \le 2\left(\frac{\|\alpha_M\|_{L^{p'(x)}(\Omega)}}{\|u\|_X^{p_- - 1}} + \varepsilon C_2 \right),$$

the conclusion follows, because the right-hand side of the inequality tends to ε as $\|u\|_X \to \infty$. This completes the proof. □

Next we deal with the properties for the superposition operator induced by the function f in (B). In particular, we give the compactness of this operator and the behavior of that at infinity, respectively. The ideas of the proof about these properties are completely the same as in [18]. We assume that the variable exponents are subject to the following restrictions:

$$\begin{cases} p^*(x) := \frac{Np(x)}{N-p(x)}, q(x) \in (\frac{Np(x)}{Np(x)-N+p(x)}, \infty) & \text{if } N > p(x), \\ p^*(x), q(x) \in (1, \infty) \text{ arbitrary} & \text{if } N \le p(x) \end{cases}$$

for almost all $x \in \Omega$. Assume that

(F1) $f : \mathbb{R} \times \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfies the Carathéodory condition in the sense that $f(\lambda, \cdot, u, v)$ is measurable for all $(\lambda, u, v) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N$ and $f(\cdot, x, \cdot, \cdot)$ is continuous for almost all $x \in \Omega$.

(F2) For each bounded interval $I \subset \mathbb{R}$, there are a function $a_I \in L^{q(x)}(\Omega)$ and a nonnegative constant b_I such that

$$\left|f(\lambda, x, u, v)\right| \le a_I(x) + b_I\left(|u|^{\frac{p^*(x)}{q(x)}} + |v|^{\frac{p(x)}{q(x)}}\right)$$

for almost all $x \in \Omega$ and all $(\lambda, u, v) \in I \times \mathbb{R} \times \mathbb{R}^N$.

(F3) There exist a function $a \in L^{p'(x)}(\Omega)$ and a locally bounded function $b : [0, \infty) \to \mathbb{R}$ with $\lim_{r\to\infty} b(r)/r = 0$ such that

$$\left|f(0, x, u, v)\right| \le a(x) + \left[b\left(|u| + |v|\right)\right]^{p_- - 1}$$

for almost all $x \in \Omega$ and all $(u, v) \in \mathbb{R} \times \mathbb{R}^N$.

Under assumptions (F1) and (F2), we can define an operator $F : \mathbb{R} \times X \to X^*$ by

$$\left\langle F(\lambda, u), v\right\rangle = \int_\Omega f\left(\lambda, x, u(x), \nabla u(x)\right) v(x)\, dx \tag{3.19}$$

and an operator $G : X \to X^*$ by

$$\left\langle G(u), v\right\rangle = \int_\Omega \left|u(x)\right|^{p(x)-2} u(x) v(x)\, dx \tag{3.20}$$

for all $v \in X$.

In proving the following result, a key idea is to use a continuity result on the superposition operators due to Väth [34]. For the case that $p(x)$ is a constant function, it has been proved in [11].

Lemma 3.5 *If* (F1) *and* (F2) *hold, then* $F : \mathbb{R} \times X \to X^*$ *is continuous and compact. Moreover, the operator* $G : X \to X^*$ *is continuous and compact.*

Proof A linear operator $I_1 : \mathbb{R} \times X \to \mathbb{R} \times L^{p^*(x)}(\Omega) \times L^{p(x)}(\Omega, \mathbb{R}^N)$ defined by

$$I_1(\lambda, u) := (\lambda, u, \nabla u) \quad \text{for } (\lambda, u) \in \mathbb{R} \times X$$

is clearly bounded because $\|u\|_{L^{p^*(x)}(\Omega)} \leq C\|u\|_X$ for some positive constant C. Set $Y := \mathbb{R} \times L^{p^*(x)}(\Omega) \times L^{p(x)}(\Omega, \mathbb{R}^N)$. Define the superposition operator $\Phi : Y \to L^{q(x)}(\Omega)$ by

$$\Phi(\lambda, u, v)(x) := f\bigl(\lambda, x, u(x), v(x)\bigr).$$

If I is a bounded interval in $\mathbb{R}$ and $a_I \in L^{q(x)}(\Omega)$ and $b_I \in [0, \infty)$ are chosen from (F2), then Φ is bounded because

$$\begin{aligned}
&\int_\Omega \bigl|\Phi(\lambda, u, v)\bigr|^{q(x)}\,dx \\
&\quad \leq \int_\Omega \Bigl(3 \max\Bigl\{|a_I|, b_I|u|^{\frac{p^*(x)}{q(x)}}, b_I|v|^{\frac{p(x)}{q(x)}}\Bigr\}\Bigr)^{q(x)}\,dx \\
&\quad \leq 3^{q_+}\biggl(\int_\Omega \Bigl[|a_I| + b_I|u|^{\frac{p^*(x)}{q(x)}} + b_I|v|^{\frac{p(x)}{q(x)}}\Bigr]^{q(x)}\,dx\biggr) \\
&\quad \leq 12^{q_+}\biggl(\int_\Omega |a_I|^{q(x)}\,dx + (1 + b_I)^{q_+}\int_\Omega |u|^{p^*(x)}\,dx + (1 + b_I)^{q_+}\int_\Omega |v|^{p(x)}\,dx\biggr).
\end{aligned}$$

Since Y is a generalized ideal space and $L^{q(x)}(\Omega)$ is a regular ideal space (since $L^{q(x)}(\Omega)$ satisfies Δ_2-condition), Theorem 6.4 of [34] implies that Φ is continuous on Y. Recalling the fact that the conjugate function of $q(x)$ is strictly less than $p^*(x)$, we know by Lemma 2.6 that the embedding $I_2 : X \hookrightarrow L^{q'(x)}(\Omega)$ is continuous and compact and so is the adjoint operator $I_2^* : L^{q(x)}(\Omega) \to X^*$ given by

$$\bigl\langle I_2^*(u), \varphi\bigr\rangle = \int_\Omega u\varphi\,dx$$

for any $\varphi \in X$. From the relation $F = I_2^* \circ \Phi \circ I_1$, it follows that F is continuous and compact. In particular, if we set $f(\lambda, x, u, v) = |u|^{p(x)-2}u$, then G is continuous and compact. This completes the proof. □

We observe the behavior of $F(0, \cdot)$ at infinity.

Lemma 3.6 *Under assumptions* (F1) *and* (F3), *the operator* $F(0, \cdot) : X \to X^*$ *has the following property*:

$$\lim_{\|u\|_X \to \infty} \frac{\|F(0, u)\|_{X^*}}{\|u\|_X^{p_- - 1}} = 0.$$

Proof Let $0 < \varepsilon < 1$ be arbitrary. Choose a positive constant R such that $|b(r)| \leq \varepsilon r$ for all $r \geq R$. Since b is locally bounded, there is a nonnegative constant C_R such that $|b(r)| \leq C_R$

for all $r \in [0, R]$. Let $u \in X$. Set $\Omega_R = \{x \in \Omega : |u(x)| + |\nabla u(x)| \leq R\}$. By assumption (F3), Minkowski's inequality and the fact that $W_0^{1,p(x)}(\Omega) \hookrightarrow W_0^{1,p_-}(\Omega)$, we obtain

$$\begin{aligned}
&\big\| f\big(0, x, u(x), \nabla u(x)\big) \big\|_{L^{p'(x)}(\Omega)} \\
&\quad \leq \|a\|_{L^{p'(x)}(\Omega)} + \big\| b\big(|u| + |\nabla u|\big)^{p_- -1} \big\|_{L^{p'(x)}(\Omega)} \\
&\quad \leq \|a\|_{L^{p'(x)}(\Omega)} + C\big\| b\big(|u| + |\nabla u|\big)^{p_- -1} \big\|_{L^{(p')_+}(\Omega)} \\
&\quad \leq \|a\|_{L^{p'(x)}(\Omega)} + C\bigg(\int_\Omega \big| b\big(\big|u(x)\big| + \big|\nabla u(x)\big|\big)\big|^{p_-} \,dx \bigg)^{\frac{1}{(p')_+}} \\
&\quad \leq \|a\|_{L^{p'(x)}(\Omega)} + C\bigg(\int_{\Omega_R} (C_R)^{p_-} \,dx \bigg)^{\frac{p_- -1}{p_-}} + C\bigg(\int_{\Omega \setminus \Omega_R} \varepsilon^{p_-} \big(\big|u(x)\big| + \big|\nabla u(x)\big|\big)^{p_-} \,dx \bigg)^{\frac{p_- -1}{p_-}} \\
&\quad \leq \|a\|_{L^{p'(x)}(\Omega)} + C\big(C_R^{p_-} \operatorname{meas}(\Omega_R)\big)^{\frac{p_- -1}{p_-}} + 2^{p_- -1} \varepsilon^{p_- -1} C \|u\|_{W_0^{1,p_-}(\Omega)}^{p_- -1} \\
&\quad \leq \|a\|_{L^{p'(x)}(\Omega)} + C\big(C_R^{p_-} \operatorname{meas}(\Omega_R)\big)^{\frac{p_- -1}{p_-}} + 2^{p_- -1} \varepsilon^{p_- -1} C \|u\|_X^{p_- -1}
\end{aligned}$$

for all $u \in X$, where C are some positive constants. It follows from Hölder's inequality that

$$\begin{aligned}
\big|\big\langle F(0, u), \varphi \big\rangle\big| &\leq 2 \big\| f\big(0, x, u(x), \nabla u(x)\big) \big\|_{L^{p'(x)}(\Omega)} \|\varphi\|_{L^{p(x)}(\Omega)} \\
&\leq C\Big(\|a\|_{L^{p'(x)}(\Omega)} + \big(C_R^{p_-} \operatorname{meas}(\Omega_R)\big)^{\frac{p_- -1}{p_-}} + 2^{p_- -1} \varepsilon^{p_- -1} \|u\|_X^{p_- -1} \Big) \|\varphi\|_X
\end{aligned}$$

for all $u, \varphi \in X$. Therefore, we get

$$\lim_{\|u\|_X \to \infty} \frac{\|F(0, u)\|_{X^*}}{\|u\|_X^{p_- -1}} = 0. \qquad \square$$

Recall that a real number μ is called an *eigenvalue of* (E) if the equation

$$J_{p(\cdot)}(u) = \mu G(u)$$

has a solution u_0 in X that is different from the origin.

Now we consider the following spectral result for nonhomogeneous operators. When $p(x)$ is a constant function, the following assertion has been shown to hold by virtue of the Furi-Martelli-Vignoli spectrum; see Theorem 4 of [35] or Lemma 27 of [10].

Lemma 3.7 *If μ is not an eigenvalue of* (E), *we have*

$$\liminf_{\|u\|_X \to \infty} \frac{\|J(u) - \mu G(u)\|_{X^*}}{\|u\|_X^{p_- -1}} > 0.$$

Proof Suppose that

$$\liminf_{\|u\|_X \to \infty} \frac{\|J(u) - \mu G(u)\|_{X^*}}{\|u\|_X^{p_- -1}} = 0.$$

Choose an unbounded sequence (u_n) in X with $\|u_n\|_X > 1$ such that

$$\frac{\|J(u_n) - \mu G(u_n)\|_{X^*}}{\|u_n\|_X^{p_- - 1}} \to 0 \quad \text{as } n \to \infty. \tag{3.21}$$

Set $v_n = u_n / \|u_n\|_X$ for $n \in \mathbb{N}$. Then we have

$$\left\|J_{p(\cdot)}(v_n) - \mu G(v_n)\right\|_{X^*} \le \frac{\|J_{p(\cdot)}(u_n) - J(u_n)\|_{X^*}}{\|u_n\|_X^{p_- - 1}} + \frac{\|J(u_n) - \mu G(u_n)\|_{X^*}}{\|u_n\|_X^{p_- - 1}}.$$

Hence it follows from Proposition 3.4 and (3.21) that

$$\left\|J_{p(\cdot)}(v_n) - \mu G(v_n)\right\|_{X^*} \to 0 \quad \text{as } n \to \infty. \tag{3.22}$$

By the compactness of G, we may assume that $G(v_n)$ converges to some point $w \in X^*$. From (3.22) it follows that $J_{p(\cdot)}(v_n) \to \mu w$ as $n \to \infty$. Put $v := J_{p(\cdot)}^{-1}(\mu w)$. Since $J_{p(\cdot)}$ is a homeomorphism (see Theorem 3.2 in [18]), we get that $v \neq 0$ and $v_n \to v$ as $n \to \infty$ and so

$$\begin{aligned}\left\|J_{p(\cdot)}(v) - \mu G(v)\right\|_{X^*} &\le \left\|J_{p(\cdot)}(v) - J_{p(\cdot)}(v_n)\right\|_{X^*} + \left\|J_{p(\cdot)}(v_n) - \mu G(v_n)\right\|_{X^*} \\ &\quad + \left\|\mu G(v_n) - \mu G(v)\right\|_{X^*} \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

We conclude that μ is an eigenvalue of (E). This completes the proof. □

4 Main result

In this section, we are preparing to prove our main result. First we give the definition of weak solutions for our problem.

Definition 4.1 A weak solution of (B) is a pair (λ, u) in $\mathbb{R} \times X$ such that

$$J(u) - \mu G(u) = F(\lambda, u) \quad \text{in } X^*,$$

where J, F and G are defined by (3.3), (3.19) and (3.20), respectively.

The following result about the existence of an unbounded branch of solutions for nonlinear operator equations is taken from Theorem 2.2 of [11] (see also [10]) as a key tool in obtaining our bifurcation result.

Lemma 4.2 *Let X be a Banach space and Y be a normed space. Suppose that $J : X \to Y$ is a homeomorphism and $G : X \to Y$ is a continuous and compact operator such that the composition $J^{-1} \circ (-G)$ is odd. Let $F : \mathbb{R} \times X \to Y$ be a continuous and compact operator. If the set*

$$\bigcup_{t \in [0,1]} \left\{u \in X : J(u) + G(u) = tF(0, u)\right\}$$

is bounded, then the set

$$\left\{(\lambda, u) \in \mathbb{R} \times X : J(u) + G(u) = F(\lambda, u)\right\}$$

has an unbounded connected set $C \subseteq (\mathbb{R} \setminus \{0\}) \times X$ such that $\overline{C}$ intersects $\{0\} \times X$.

Proof Since $J^{-1} \circ (-G)$ is odd, Borsuk's theorem implies that the condition

$$\deg\bigl(I - \bigl(J^{-1} \circ (-G)\bigr), B_r, 0\bigr) \neq 0$$

is satisfied for all sufficiently large $r > 0$, where I is the identity operator on X and B_r is the open ball in X centered at 0 of radius r, respectively. In view of Theorem 2.2 of [11], the conclusion holds. □

Based on the above lemma, we now can prove the main result on bifurcation result for problem (B).

Theorem 4.3 *Suppose that conditions* (HJ1)-(HJ4) *and* (F1)-(F3) *are satisfied. If μ is not an eigenvalue of* (E), *then there is an unbounded connected set $C \subseteq (\mathbb{R} \setminus \{0\}) \times X$ such that every point (λ, u) in C is a weak solution of the above problem* (B) *and $\overline{C}$ intersects $\{0\} \times X$.*

Proof Apply Lemma 4.2 with $X = W_0^{1,p(x)}(\Omega)$ and $Y = X^*$. From Lemmas 3.3 and 3.5 we know that $J : X \to X^*$ is a homeomorphism, the operators G and F are continuous and compact, and $J^{-1} \circ (\mu G)$ is odd. Since μ is not an eigenvalue of (E), Lemmas 3.6 and 3.7 imply that for some $\beta > 0$, there is a positive constant $R > 1$ such that

$$\bigl\|J(u) - \mu G(u)\bigr\|_{X^*} > \beta \|u\|_X^{p_- -1} > \bigl\|F(0,u)\bigr\|_{X^*} \geq \bigl\|tF(0,u)\bigr\|_{X^*}$$

for all $u \in X$ with $\|u\|_X \geq R$ and for all $t \in [0,1]$. Therefore, the set

$$S = \bigcup_{t\in[0,1]} \bigl\{u \in X : J(u) - \mu G(u) = tF(0,u)\bigr\}$$

is bounded. By Lemma 4.2, the set

$$\bigl\{(\lambda, u) \in \mathbb{R} \times X : J(u) - \mu G(u) = F(\lambda, u)\bigr\}$$

contains an unbounded connected set C which $\overline{C}$ intersects $\{0\} \times X$. This completes the proof. □

Finally, we give an example which illustrates an application of our bifurcation result.

Example 4.4 Let $\beta \in (p_- - 1, p_+]$, $w \in L^\infty(\Omega)$ and $\alpha \in L^{p(x)/\beta}(\Omega) \cap L^\infty(\Omega)$. Assume that $w(x) \geq \varepsilon > 0$ and there is a real number δ in $(0,1]$ such that

$$\delta - 1 \leq \alpha(x) \leq \max\left\{\frac{2(p_- - 1)}{\beta - (p_- - 1)}, \frac{4\beta(p_- - 1)}{(\beta - (p_- - 1))^2} - \delta\right\}$$

for almost all $x \in \Omega$. Let

$$\phi\bigl(x, |v|\bigr) = \omega(x)\left(1 + \frac{\alpha(x)}{1 + |v|^\beta}\right)|v|^{p(x)-2}$$

for all $v \in \mathbb{R}^N$. If μ is not an eigenvalue of (E) and assumptions (F1)-(F3) are fulfilled, then there is an unbounded connected set C intersecting $\{0\} \times X$ such that every point (λ, u)

in C is a weak solution of the nonlinear equation

$$\begin{cases} -\operatorname{div}(\phi(x,|\nabla u|)\nabla u) = \mu|u|^{p-2}u + f(\lambda,x,u,\nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Proof Putting $\lambda(x) := (p(x)-1)/\beta$, we claim that

$$\frac{|\alpha(x)|}{1+|v|^{\beta}}|v|^{p(x)-1} \le \frac{|\alpha(x)|^{\lambda(x)}}{\lambda_-} + \frac{|v|^{p(x)-1}}{1-\lambda_+} \quad \bigl(v \in \mathbb{R}^N\bigr).$$

If $v = 0$, the inequality is clear. Now let $v \neq 0$. It follows from Young's inequality that

$$\begin{aligned} \frac{|\alpha(x)|}{1+|v|^{\beta}} &= \frac{|\alpha(x)|}{|v|^{\beta}} \cdot \frac{|v|^{\beta}}{(1+|v|^{\beta})} \\ &\le \frac{|\alpha(x)|^{\lambda(x)}}{\lambda_-|v|^{p(x)-1}} + \frac{|v|^{\beta(1-\lambda(x))}}{(1-\lambda_+)(1+|v|^{\beta})^{1-\lambda(x)}} \end{aligned}$$

and hence

$$\begin{aligned} \frac{|\alpha(x)|}{1+|v|^{\beta}}|v|^{p(x)-1} &\le \frac{|\alpha(x)|^{\lambda(x)}}{\lambda_-} + \frac{|v|^{\beta}}{(1-\lambda_+)|v|^{\beta(1-\lambda(x))}} \\ &= \frac{|\alpha(x)|^{\lambda(x)}}{\lambda_-} + \frac{|v|^{p(x)-1}}{1-\lambda_+}. \end{aligned}$$

Set $a(x) = |w(x)|(\lambda_-)^{-1}|\alpha(x)|^{\lambda(x)}$ and choose $c_0 > 0$ such that $|w(x)| \le c_0$ for almost everywhere $x \in \Omega$. Then

$$\begin{aligned} \int_{\Omega} \bigl|a(x)\bigr|^{p'(x)}\,dx &= \int_{\Omega} \bigl|w(x)\bigr|^{p'(x)} \cdot \frac{1}{\lambda_-^{p'(x)}} \cdot \bigl|\alpha(x)\bigr|^{\lambda(x)p'(x)}\,dx \\ &\le \left(\frac{c_0}{\lambda_-}\right)^{(p')_+} \int_{\Omega} \bigl|\alpha(x)\bigr|^{\frac{p(x)}{\beta}}\,dx < \infty, \quad \textit{i.e.},\ a \in L^{p'(x)}(\Omega). \end{aligned}$$

Thus (HJ1) and (HJ2) are satisfied. Removing some null set from Ω if necessary, we may suppose that the hypotheses are satisfied for all $x \in \Omega$. If we put

$$\phi(x,\eta) = w(x)\left(1 + \frac{\alpha(x)}{1+\eta^{\beta}}\right)\eta^{p(x)-2} \quad (\eta \ge 0),$$

we observe that the first relation in (3.1) holds, because $\alpha(x) \ge \varepsilon - 1$ and $w(x) \ge \varepsilon > 0$. Moreover, a straightforward calculation shows that for all $u > 0$,

$$\begin{aligned} &\frac{1}{w(x)\eta^{p(x)-2}}\left(\eta\frac{\partial\phi}{\partial\eta}(x,\eta) + \phi(x,\eta)\right) \\ &\quad \ge (p_- - 1)\left(1 + \frac{\alpha(x)}{1+\eta^{\beta}}\right) - \frac{\alpha(x)\beta\eta^{\beta}}{(1+\eta^{\beta})^2} \\ &\quad = \frac{p_- - 1}{(1+\eta^{\beta})^2}\left(\eta^{2\beta} + \left[2 - \left(\frac{\beta}{p_- - 1} - 1\right)\alpha(x)\right]\eta^{\beta} + \bigl[\alpha(x) + 1\bigr]\right). \end{aligned}$$

From an analogous argument in the proof of Corollary 3.2 in [11], we can show that this expression is bounded from below by a positive constant which is independent of $\eta > 0$ and $x \in \Omega$. Therefore (HJ3) is satisfied. Finally, (HJ4) holds if for each $\varepsilon > 0$ we choose

$$M(x) = \left|\varepsilon^{-1} w(x)\alpha(x)\right|^{1/\beta}$$

for almost all $x \in \Omega$. □

Competing interests
The author declares that he has no competing interests.

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