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Global existence of timelike minimal surface of general co-dimension in Minkowski space time

Yinxia Wang[1*] and Jingzi Liu[2]

*Correspondence: yinxia117@126.com
[1]School of Mathematics and Information Sciences, North China University of Water Resources and Electric Power, Zhengzhou, 450011, China
Full list of author information is available at the end of the article

Abstract
In this paper, we prove that the global existence of solutions to timelike minimal surface equations having arbitrary co-dimension with slow decay initial data in two space dimensions and three space dimensions, provided that the initial value is suitably small.

MSC: 35L70

Keywords: timelike minimal surface; global existence; slow decay initial value

1 Introduction

The theory of minimal surfaces has a long history, originating with the papers of Lagrange (1760) and the famous Plateau problem; we refer to the classical papers by Calabi [1] and by Cheng and Yau [2]. Timelike minimal submanifolds may be viewed as simple but nontrivial examples of D-branes, which play an important role in string theory, and the system under consideration here thus has natural generalizations motivated by string theory. The case of timelike surfaces has been investigated by several authors (see [3–5] and [6]). Huang and Kong [7] studied the motion of a relativistic torus in the Minkowski space $\mathbb{R}^{1+n}$ ($n \geq 3$). They derived the equations for the motion of relativistic torus in the Minkowski space $\mathbb{R}^{1+n}$ ($n \geq 3$). This kind of equation also describes the three dimensional timelike extremal submanifolds in the Minkowski space $\mathbb{R}^{1+n}$. They showed that these equations can be reduced to a $(1+2)$ dimensional quasilinear symmetric hyperbolic system and the system possesses some interesting properties, such as nonstrict hyperbolicity, constant multiplicity of eigenvalues, linear degeneracy of all characteristic fields, and the strong null condition (see [8] and [9]). They also found and proved the interesting fact that all plane wave solutions to these equations are lightlike extremal submanifolds and *vice versa*, except for a type of special solution. For small initial data with compact support, the global existence problem for timelike minimal hypersurfaces has been considered by Brendle [10] and Lindblad [11].

Paul *et al.* [12] investigated timelike minimal submanifolds of dimension $1+n$, $n \geq 2$, of Minkowski spacetimes of dimension $1+n+q$, $q \geq 1$. The authors considered an embedding of $\mathbb{R}^{1+n}$ into Minkowski spacetime $\mathbb{R}^{1+n+q}$ given by the graph of a map $f : \mathbb{R}^{1+n} \longrightarrow \mathbb{R}^q$. Let Greek indices $\alpha, \beta, \ldots$ take values in $0, 1, \ldots, n$ and let uppercase Latin indices $I, J, \ldots$ take values in $1, \ldots, q$. Introduce cartesian coordinates x_α on $\mathbb{R}^{1+n}$ and x^I on $\mathbb{R}^q$. The induced metric $\mathbb{R}^{1+n}$ is

$$h_{\alpha\beta} = \eta_{\alpha\beta} + f^I_\alpha f^J_\beta \delta_{IJ}, \tag{1.1}$$

where $f_I = x^I \circ f$, $f^\alpha = \partial_\alpha f^I$ and $\eta = \operatorname{diag}(1,1,\dots,1)$ is the Minkowski metric. By variational principles (see [13]), they derived the Euler-Lagrange equations

$$\partial_\mu\big[\sqrt{-\det h}\,h^{\mu\nu} f^I_\nu\big] = 0, \quad I = 1,\dots,q. \tag{1.2}$$

Moreover for a small initial value with compact support, they also proved the global existence of classical solutions for (1.2).

In this paper, we consider (1.2) with the initial data

$$t = 0: \quad f^I = \varepsilon f^I_0(x), \qquad f^I_t = \varepsilon f^I_1(x), \tag{1.3}$$

where $f^I_0(x), f^I_1(x) \in C^\infty(\mathbb{R}^n)$ $(n = 2, 3)$ satisfying

$$\big|f^I_0(x)\big| \le \frac{A}{(1+|x|)^k}, \qquad \big|f^I_1(x)\big| \le \frac{A}{(1+|x|)^{k+1}} \quad (k > 0, I = 1,\dots,q),$$

where $A > 0$ is a constant and $\varepsilon > 0$ is a small parameter. The aim of this paper is to prove that the Cauchy problem (1.2), (1.3) has a global classical solution, provided that the initial value $f^I_0(x), f^I_1(x) \in C^\infty(\mathbb{R}^n)$ is sufficiently small and satisfy $|f^I_0(x)| \le \frac{A}{(1+|x|)^k}$, $|f^I_1(x)| \le \frac{A}{(1+|x|)^{k+1}}$ $(k > \frac{n}{2}$, $I = 1,\dots,q)$. We reduce the restriction on compact support of the initial data to some decay. In other words, we show the global existence of solutions to timelike minimal surface in two space dimensions and three space dimensions, provided that the initial value is suitably small.

To study (1.2), we note that (1.2) can be written in divergence form

$$\Box f^I = \partial_\mu\big[F^{\mu\nu} f^I_\nu\big], \tag{1.4}$$

where $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$ is the Minkowski wave operator and $F^{\mu\nu} = \eta^{\mu\nu} - \sqrt{-\det h}\,h^{\mu\nu}$, as well as in the form

$$H^{\mu\nu}_{JL}(\partial f)\partial_\mu\partial_\nu f^I = 0, \quad I = 1,\dots,q, \tag{1.5}$$

where

$$H^{\mu\nu}_{JL} = \sqrt{-\det h}\big[\delta_{JL} h^{\mu\nu} - \delta_{IJ}\delta_{KL}\big(h^{\mu\nu} h^{\alpha\beta} f^K_\beta f^I_\beta + h^{\mu\alpha} h^{\nu\beta} f^I_\alpha f^K_\beta + h^{\mu\alpha} h^{\nu\beta} f^K_\alpha f^I_\beta\big)\big]. \tag{1.6}$$

We raise and lower Greek (intrinsic) indices using $h_{\mu\nu}$ and its inverse, while Latin (extrinsic) indices are raised and lowered using the identity δ_{IJ} and its inverse. From (1.6), it follows that $H^{\mu\nu}_{IJ}$ has the symmetries

$$H^{\mu\nu}_{JL} = H^{\mu\nu}_{LJ} = H^{\nu\mu}_{JL}. \tag{1.7}$$

Due to the symmetries, an energy estimate and local well posedness holds for the system (1.5).

The plan of this paper is as follows. In Section 2, we cite some estimates and prove some estimates on the solution of linear wave equations. The global existence of solutions to timelike minimal surface equations with slow decay initial value in two space dimensions and three space dimensions will be proved in Section 3 and Section 4, respectively.

2 Preliminaries

Following Klainerman [14], we introduce a set of partial differential operators

$$Z = \big((\partial_i), i = 0,1,\dots,n; L_0; (\Omega_{ij}), 1 \le i < j \le n; (\Omega_{0i}), i = 1,\dots,n\big), \tag{2.1}$$

where

$$\partial_0 = \frac{\partial}{\partial t}, \qquad \partial_i = \frac{\partial}{\partial x_i} \quad (i = 1,\dots,n), \tag{2.2}$$

$$L_0 = t\partial_0 + \sum_{i=1}^{n} x_i \partial_i, \tag{2.3}$$

$$\Omega_{ij} = x_i\partial_j - x_j\partial_i \quad (1 \le i < j \le n) \tag{2.4}$$

and

$$\Omega_{0i} = t\partial_i + x_i\partial_0 \quad (i = 1,\dots,n). \tag{2.5}$$

Z^α denotes a product of $|\alpha|$ of the vector fields (2.2), (2.3), (2.4), and (2.5). $\alpha = (\alpha_1,\dots,\alpha_\sigma)$ is a multi-index, $|\alpha| = \alpha_1 + \cdots + \alpha_\sigma$, σ is the number of partial differential operators in $Z : Z = (Z_1,\dots,Z_\sigma)$ and

$$Z^\alpha = Z_1^{\alpha_1} \cdots Z_\sigma^{\alpha_\sigma}. \tag{2.6}$$

It is easy to prove Lemma 2.1 (see [15]).

Lemma 2.1 *For any multi-index $\alpha = (\alpha_1,\dots,\alpha_\sigma)$, we have*

$$\big[\Box, Z^\alpha\big] = \sum_{|\beta| \le |\alpha|-1} A_{\alpha\beta} Z^\beta \Box \tag{2.7}$$

and

$$\big[\partial_i, Z^\alpha\big] = \sum_{|\beta| \le |\alpha|-1} B_{\alpha\beta} Z^\beta \partial = \sum_{|\beta| \le |\alpha|-1} \tilde{B}_{\alpha\beta} \partial Z^\beta \quad (i = 0,1,\dots,n), \tag{2.8}$$

where $[\,,]$ stands for the Poisson bracket, $\beta = (\beta_1,\dots,\beta_\sigma)$ are multi-indices, $\Box$ is the wave operator, $\partial = (\partial_0,\partial_1,\dots,\partial_n)$ and $A_{\alpha\beta}$, $B_{\alpha\beta}$, and $\tilde{B}_{\alpha\beta}$ are constants.

We need the following lemma that is basically established in [16] and [17]. For completeness, the proof will also be sketched here.

Lemma 2.2 *Let $\phi_0(x), \phi_1(x) \in C^\infty(\mathbb{R}^2)$ and satisfy*

$$\left|\phi_0(x)\right| \le \frac{A}{(1+|x|)^k}, \qquad \left|\phi_1(x)\right| \le \frac{A}{(1+|x|)^{k+1}} \quad (k>1).$$

Assume that $\phi = \phi(t,x)$ is a solution to the following Cauchy problem:

$$\begin{cases} \phi_{tt} - \triangle\phi = 0, \quad x \in \mathbb{R}^2, t > 0, \\ t = 0: \quad \phi = \phi_0(x), \qquad \phi_t = \phi_1(x), \quad x \in \mathbb{R}^2. \end{cases} \tag{2.9}$$

Then we have

$$\left|\phi(t,x)\right| \le \begin{cases} \frac{CA}{\sqrt{1+t+|x|}(1+|t-|x||)^{k-\frac{1}{2}}} \quad (|x| \ge t), \\ \frac{CA}{\sqrt{1+t+|x|}\sqrt{1+|t-|x||}} \quad (|x| \le t). \end{cases} \tag{2.10}$$

Remark 2.1 Under the condition that

$$\left|\phi_0(x)\right| \le \frac{A}{(1+|x|)^{k+1}}, \qquad \left|\phi_1(x)\right| \le \frac{A}{(1+|x|)^{k+1}} \quad (k>1),$$

Tsutaya [18] has showed that the solution of the Cauchy problem (2.9) satisfies

$$\left|\phi(t,x)\right| \le \frac{CA}{\sqrt{1+t+|x|}\sqrt{1+|t-|x||}}.$$

Obviously, Lemma 2.2 improves the result in [18].

Proof The solution of (2.9) is given

$$\phi(t,x) = \frac{1}{2\pi t^2}\int_{|x-y|\le t} \frac{t\phi_0(y) + t^2\phi_1(y) + t\nabla\phi_0(y)\cdot(y-x)}{(t^2-|y-x|^2)^{\frac{1}{2}}}\,dy. \tag{2.11}$$

First, we make an estimate for $|\frac{1}{2\pi t}\int_{|x-y|\le t} \frac{\phi_0(y)}{(t^2-|y-x|^2)^{\frac{1}{2}}}\,dy|$; switching to polar coordinates, we have

$$\begin{aligned} &\left|\frac{1}{2\pi t}\int_{|x-y|\le t} \frac{\phi_0(y)}{(t^2-|y-x|^2)^{\frac{1}{2}}}\,dy\right| \\ &\quad \le \frac{A}{2\pi t}\int_{|x-y|\le t} \frac{1}{\sqrt{t^2-|y-x|^2}(1+|y|)^k}\,dy \\ &\quad \le \frac{A}{2\pi t}\left(\int_{|t-|x||}^{t+|x|} \frac{r}{(1+r)^k}\int_{-\varphi}^{\varphi} \frac{1}{\sqrt{t^2-|x|^2-r^2+2r|x|\cos\psi}}\,d\psi\,dr\right. \\ &\qquad \left. + \chi\left(t-|x|\right)\int_0^{t-|x|} \frac{r}{(1+r)^k}\int_{-\pi}^{\pi} \frac{1}{\sqrt{t^2-|x|^2-r^2+2r|x|\cos\psi}}\,d\psi\,dr\right), \end{aligned} \tag{2.12}$$

$$\varphi = \arccos\frac{|x|^2+r^2-t^2}{2|x|r},$$

where $x = (|x|\cos\theta, |x|\sin\theta)$ and $y = (r\cos(\theta+\psi), r\sin(\theta+\psi))$, and χ is the characteristic function of positive numbers.

Let $h(y)$ be a continuous function on $\mathbb{R}^2$ and $y = (r\cos(\theta+\psi), r\sin(\theta+\psi))$. Define

$$H\big(t,|x|,r,\theta,h\big) = \begin{cases} \int_{-\varphi}^{\varphi} \frac{h(r,\theta+\psi)}{\sqrt{t^2-|x|^2-r^2+2|x|r\cos\psi}}\,d\psi, & |\frac{|x|^2+r^2-t^2}{2|x|r}| \le 1, \\ \int_{-\pi}^{\pi} \frac{h(r,\theta+\psi)}{\sqrt{t^2-|x|^2-r^2+2|x|r\cos\psi}}\,d\psi, & |\frac{|x|^2+r^2-t^2}{2|x|r}| \ge 1, \end{cases}$$

and

$$H\big(t,|x|,r\big) = H\big(t,|x|,r,\theta,1\big),$$

where, as before, φ is given by

$$\varphi = \arccos\frac{|x|^2+r^2-t^2}{2|x|r}.$$

We will use the following proposition, which is proved in Kovalyov [19].

Proposition 2.1 (I) *If $t \ge |x| + r$ and $|\frac{|x|^2+r^2-t^2}{2|x|r}| \ge 1$, then $H(t,|x|,r)$ satisfies*

$$H\big(t,|x|,r\big) \le C\frac{\ln(2+\frac{r|x|}{t^2-(r+|x|)^2})}{\sqrt{t^2-|x|^2-r^2}} \le \frac{C}{\sqrt{t^2-(r+|x|)^2}}. \tag{2.13}$$

(II) *If $t \le |x| + r$ and $|\frac{|x|^2+r^2-t^2}{2|x|r}| \le 1$, then*

$$H\big(t,|x|,r\big) \le \frac{C}{\sqrt{r|x|}}\ln\left(2+\frac{r|x|\chi(t-|x|)}{(r+|x|)^2-t^2}\right), \tag{2.14}$$

where χ is the characteristic function of positive numbers.

We next continue to make an estimate for (2.12); we make an estimate for the right-hand side of (2.12) by dividing into two cases.

Case 1. $|x| \ge t$.

By (2.14), we get

$$\left|\frac{1}{2\pi t}\int_{|x-y|\le t}\frac{\phi_0(y)}{(t^2-|y-x|^2)^{\frac{1}{2}}}\,dy\right| \le \frac{CA}{t\sqrt{|x|}}\int_{|x|-t}^{t+|x|}\frac{1}{(1+r)^{k-\frac{1}{2}}}\,dr. \tag{2.15}$$

We subdivide into three cases again.

(i) $k > \frac{3}{2}$.

$$\frac{CA}{t\sqrt{|x|}}\int_{|x|-t}^{t+|x|}\frac{1}{(1+r)^{k-\frac{1}{2}}}\,dr = \frac{CA}{t\sqrt{|x|}(1+|x|-t)^{k-\frac{3}{2}}}\left[1-\left(\frac{1+|x|-t}{1+|x|+t}\right)^{k-\frac{3}{2}}\right].$$

Note that

$$1-s^{k-\frac{3}{2}} \le C(1-s), \quad \forall 0 \le s \le 1$$

and

$$1-\frac{1+|x|-t}{1+|x|+t}=\frac{2t}{1+|x|+t}.$$

Thus,

$$\begin{aligned}\left|\frac{1}{2\pi t}\int_{|x-y|\le t}\frac{\phi_0(y)}{(t^2-|y-x|^2)^{\frac12}}\,dy\right| &\le \frac{CA}{\sqrt{|x|}(1+|x|-t)^{k-\frac32}(1+|x|+t)}\\ &\le \frac{CA}{\sqrt{|x|+t}(1+|x|-t)^{k-\frac12}}.\end{aligned}\tag{2.16}$$

(ii) $k=\frac32$.
From (2.15), we have

$$\begin{aligned}&\left|\frac{1}{2\pi t}\int_{|x-y|\le t}\frac{\phi_0(y)}{(t^2-|y-x|^2)^{\frac12}}\,dy\right|\\ &\quad\le \frac{CA}{t\sqrt{|x|}}\int_{|x|-t}^{t+|x|}\frac{1}{(1+r)}\,dr\\ &\quad= \frac{CA}{t\sqrt{|x|}}\ln\left(1+\frac{2t}{1+|x|-t}\right)\le \frac{CA}{\sqrt{|x|}(1+|x|-t)}\\ &\quad\le \frac{CA}{\sqrt{|x|+t}(1+|x|-t)}.\end{aligned}\tag{2.17}$$

(iii) $1<k<\frac32$.
It follows from (2.15) that

$$\begin{aligned}\left|\frac{1}{2\pi t}\int_{|x-y|\le t}\frac{\phi_0(y)}{(t^2-|y-x|^2)^{\frac12}}\,dy\right| &\le \frac{CA}{t\sqrt{|x|}}\left[\left(1+t+|x|\right)^{\frac32-k}-\left(1+|x|-t\right)^{\frac32-k}\right]\\ &= \frac{CA}{t\sqrt{|x|}(1+|x|-t)^{k-\frac32}}\left[\left(\frac{1+t+|x|}{1+|x|-t}\right)^{\frac32-k}-1\right].\end{aligned}$$

Note that $1<k<\frac32$; we get

$$\left(\frac{1+t+|x|}{1+|x|-t}\right)^{\frac32-k}-1\le\frac{Ct}{1+|x|-t}.$$

Hence

$$\left|\frac{1}{2\pi t}\int_{|x-y|\le t}\frac{\phi_0(y)}{(t^2-|y-x|^2)^{\frac12}}\,dy\right|\le\frac{CA}{\sqrt{|x|+t}(1+|x|-t)^{k-\frac12}}.\tag{2.18}$$

In other words, if $|x|\ge t$ and $t\ge 1$, from (2.16)-(2.18), we get

$$\left|\frac{1}{2\pi t}\int_{|x-y|\le t}\frac{\phi_0(y)}{(t^2-|y-x|^2)^{\frac12}}\,dy\right|\le\frac{CA}{\sqrt{1+|x|+t}(1+|x|-t)^{k-\frac12}}\quad(k>1).\tag{2.19}$$

In what follows, we prove that (2.19) also holds if $0<t<1$ and $|x|\ge t$. In this case, we also subdivide into two cases.

(1) $|t-|x|| \leq 1$.

By changing variables, $r = |x-y|$, we obtain

$$\begin{aligned}\left|\frac{1}{2\pi t}\int_{|x-y|\leq t}\frac{\phi_0(y)}{(t^2-|y-x|^2)^{\frac{1}{2}}}\,dy\right| &\leq \frac{1}{2\pi t}\int_{|x-y|\leq t}\frac{CA}{(t^2-|y-x|^2)^{\frac{1}{2}}(1+|y|)^k}\,dy\\ &\leq \frac{CA}{\sqrt{1+t+|x|}(1+|x|-t)^{k-\frac{1}{2}}}.\end{aligned} \tag{2.20}$$

(2) $|t-|x|| > 1$.

Note that $0 < t < 1$ and $|x| \geq t$, thus we obtain $|x| > t+1$. From Cases (i)-(iii), we get

$$\left|\frac{1}{2\pi t}\int_{|x-y|\leq t}\frac{\phi_0(y)}{(t^2-|y-x|^2)^{\frac{1}{2}}}\,dy\right| \leq \frac{CA}{\sqrt{1+t+|x|}(1+|x|-t)^{k-\frac{1}{2}}}. \tag{2.21}$$

In other words, when $|x| \geq t$, from (2.19)-(2.21), we have

$$\left|\frac{1}{2\pi t}\int_{|x-y|\leq t}\frac{\phi_0(y)}{(t^2-|y-x|^2)^{\frac{1}{2}}}\,dy\right| \leq \frac{CA}{\sqrt{1+t+|x|}(1+|x|-t)^{k-\frac{1}{2}}}. \tag{2.22}$$

Case 2. $|x| \leq t$.

From (2.12), we get

$$\left|\frac{1}{2\pi t}\int_{|x-y|\leq t}\frac{\phi_0(y)}{(t^2-|y-x|^2)^{\frac{1}{2}}}\,dy\right| \leq I + II, \tag{2.23}$$

where

$$I = \frac{A}{2\pi t}\int_{t-|x|}^{t+|x|}\frac{H(t,|x|,r)r}{(1+r)^k}$$

and

$$II = \frac{A}{2\pi t}\int_{0}^{t-|x|}\frac{H(t,|x|,r)r}{(1+r)^k}.$$

In what follows, we make an estimate for I and II, respectively, when $t+|x| \geq 1$.

It follows from (2.14) that

$$I \leq \frac{CA}{t\sqrt{|x|}}\int_{t-|x|}^{t+|x|}\ln\left(2+\frac{|x|}{|x|+r-t}\right)\frac{1}{(1+r)^{k-\frac{1}{2}}}\,dr. \tag{2.24}$$

By changing variables $\xi = |x|+r-t$,

$$\begin{aligned} I &\leq \frac{CA}{t\sqrt{|x|}}\int_0^{2|x|}\ln\left(2+\frac{|x|}{\xi}\right)\frac{1}{(1+\xi+t-|x|)^{k-\frac{1}{2}}}\,d\xi\\ &\leq \frac{CA}{t\sqrt{|x|}(1+t-|x|)^{k-\frac{1}{2}}}\int_0^{2|x|}\ln\frac{3|x|}{\xi}\,d\xi\\ &\leq \frac{CA}{\sqrt{1+t+|x|}(1+t-|x|)^{k-\frac{1}{2}}}.\end{aligned} \tag{2.25}$$

By (2.13), we get

$$\begin{aligned} II &\le \frac{CA}{t} \int_0^{t-|x|} \frac{1}{\sqrt{t^2-(|x|+r)^2}(1+r)^{k-1}}\,dr \\ &\le \frac{CA}{t\sqrt{t+|x|}} \int_0^{t-|x|} \frac{1}{\sqrt{t-|x|-r}(1+r)^{k-1}}\,dr. \end{aligned}$$

Let $\rho = \sqrt{t-|x|-r}$, then

$$\begin{aligned} II &\le \frac{CA}{t\sqrt{t+|x|}} \int_0^{\sqrt{t-|x|}} \frac{1}{(1+t-|x|-\rho^2)^{k-1}}\,d\rho \\ &\le \frac{CA}{t\sqrt{t+|x|}(1+t-|x|)^{\frac{k-1}{2}}} \int_0^{\sqrt{t-|x|}} \frac{1}{(\sqrt{1+t-|x|}-\rho)^{k-1}}\,d\rho. \end{aligned}$$

In what follows, we make estimate II by dividing into three cases.

(i) $k > 2$.

$$\begin{aligned} II &\le \frac{CA}{t\sqrt{t+|x|}(1+t-|x|)^{\frac{k-1}{2}}} \left[\frac{1}{(\sqrt{1+t-|x|}-\sqrt{t-|x|})^{k-2}} - \frac{1}{(1+t-|x|)^{\frac{k-2}{2}}}\right] \\ &\le \frac{CA}{t\sqrt{t+|x|}(1+t-|x|)^{\frac{k-1}{2}}} \left(\sqrt{1+t-|x|}+\sqrt{t-|x|}\right)^{k-2} \\ &\le \frac{CA}{\sqrt{1+t+|x|}\sqrt{1+t-|x|}}. \end{aligned}$$

(ii) $k = 2$.

$$II \le \frac{CA}{t\sqrt{t+|x|}\sqrt{1+t-|x|}} \ln \frac{\sqrt{1+t-|x|}}{\sqrt{1+t-|x|}-\sqrt{t-|x|}} \le \frac{CA}{\sqrt{1+t+|x|}\sqrt{1+t-|x|}}.$$

(iii) $1 < k < 2$.

$$\begin{aligned} II &\le \frac{CA}{t\sqrt{t+|x|}(1+t-|x|)^{\frac{k-1}{2}}} \left[\left(1+t-|x|\right)^{\frac{-k+2}{2}} - \left(\sqrt{1+t-|x|}-\sqrt{t-|x|}\right)^{-k+2}\right] \\ &\le \frac{CA}{\sqrt{1+t+|x|}\sqrt{1+t-|x|}}. \end{aligned}$$

In other words, when $t + |x| \ge 1$, we get

$$II \le \frac{CA}{\sqrt{1+t+|x|}\sqrt{1+t-|x|}}. \tag{2.26}$$

For $0 < t + |x| < 1$, by changing variables $r = |x-y|$, we obtain

$$\begin{aligned} \left|\frac{1}{2\pi t} \int_{|x-y|\le t} \frac{\phi_0(y)}{(t^2-|y-x|^2)^{\frac{1}{2}}}\,dy\right| &\le \frac{1}{2\pi t} \int_{|x-y|\le t} \frac{CA}{(t^2-|y-x|^2)^{\frac{1}{2}}(1+|y|)^k}\,dy \\ &\le \frac{CA}{\sqrt{1+t+|x|}\sqrt{1+t-|x|}}. \end{aligned} \tag{2.27}$$

Combining (2.25)-(2.27) gives

$$\left|\frac{1}{2\pi t}\int_{|x-y|\le t}\frac{\phi_0(y)}{(t^2-|y-x|^2)^{\frac{1}{2}}}\,dy\right| \le \frac{CA}{\sqrt{1+t+|x|}\sqrt{1+t-|x|}} \quad \bigl(|x|\le t\bigr). \tag{2.28}$$

Thus (2.22) and (2.28) imply that

$$\left|\frac{1}{2\pi t}\int_{|x-y|\le t}\frac{\phi_0(y)}{(t^2-|y-x|^2)^{\frac{1}{2}}}\,dy\right| \le \begin{cases}\frac{CA}{\sqrt{1+t+|x|}(1+|t-|x||)^{k-\frac{1}{2}}} & (|x|\ge t),\\ \frac{CA}{\sqrt{1+t+|x|}\sqrt{1+|t-|x||}} & (|x|\le t).\end{cases} \tag{2.29}$$

By Tsutaya [18], we obtain

$$\left|\frac{1}{2\pi}\int_{|x-y|\le t}\frac{\phi_1(y)}{(t^2-|y-x|^2)^{\frac{1}{2}}}\,dy\right| \le \begin{cases}\frac{CA}{\sqrt{1+t+|x|}(1+|t-|x||)^{k-\frac{1}{2}}} & (|x|\ge t),\\ \frac{CA}{\sqrt{1+t+|x|}\sqrt{1+|t-|x||}} & (|x|\le t),\end{cases} \tag{2.30}$$

and

$$\left|\frac{1}{2\pi t}\int_{|x-y|\le t}\frac{\nabla\phi_0(y)\cdot(y-x)}{(t^2-|y-x|^2)^{\frac{1}{2}}}\,dy\right| \le \begin{cases}\frac{CA}{\sqrt{1+t+|x|}(1+|t-|x||)^{k-\frac{1}{2}}} & (|x|\ge t),\\ \frac{CA}{\sqrt{1+t+|x|}\sqrt{1+|t-|x||}} & (|x|\le t).\end{cases} \tag{2.31}$$

Equation (2.10) follows from (2.29)-(2.31), and (2.11) immediately. Then we have completed the proof of lemma. □

The following lemma plays a key role in our main results. It is basically established in [20] and [21].

Lemma 2.3 *Let* $\phi_0(x), \phi_1(x) \in C^\infty(\mathbb{R}^3)$ *and satisfy*

$$\bigl|\phi_0(x)\bigr| \le \frac{A}{(1+|x|)^k}, \qquad \bigl|\phi_1(x)\bigr| \le \frac{A}{(1+|x|)^{k+1}} \quad (k>1).$$

Assume that u is a solution to the following Cauchy problem:

$$\begin{cases}\phi_{tt}-\triangle\phi = 0,\\ t=0: \quad \phi=\phi_0(x), \qquad \phi_t=\phi_1(x).\end{cases} \tag{2.32}$$

Then

$$\bigl|\phi(t,x)\bigr| \le \frac{CA}{(1+t+|x|)(1+|t-|x||)^{k-1}} \quad (k>1). \tag{2.33}$$

Lemma 2.4 *Let* $\phi=\phi(t,x)\in C^2$ *satisfy*

$$\Box\phi + \sum_{j,k=0}^{n}\gamma^{jk}(t,x)\partial_j\partial_k\phi = F, \quad 0\le t\le T$$

and assume that ϕ *decays to* 0 *at infinity. If*

$$|\gamma| = \sum\bigl|\gamma^{jk}\bigr| \le \frac{1}{2}, \quad 0\le t\le T.$$

It follows for $0 \le t \le T$ *that*

$$\begin{aligned}\|\partial\phi(t,\cdot)\|_{L^2} &\le 2\exp\left(\int_0^t 2|\dot{\gamma}(\tau)|\,d\tau\right)\|\partial\phi(0,\cdot)\|_{L^2} \\ &\quad + 2\int_0^t \exp\left(\int_s^t 2|\dot{\gamma}(\tau)|\,d\tau\right)\|F(s,\cdot)\|_{L^2}\,ds,\end{aligned} \tag{2.34}$$

where $|\dot{\gamma}(t)| = \sup|\partial_i\gamma^{jk}(t,\cdot)|$.

For the proof of Lemma 2.4, see Klainerman [22].

Using Lemmas 2.2, 2.3 and the $L^1 - L^\infty$ estimate of the linear wave equation with zero initial data, it is not difficulty to prove the following.

Lemma 2.5 *Suppose that* $n = 2, 3$. *Let* $\phi = \phi(t,x)$ *be the solution to the Cauchy problem*

$$\begin{cases}\phi_{tt} - \triangle\phi = g, & x \in \mathbb{R}^n, t > 0, \\ t = 0: \quad \phi = \varepsilon\phi_0(x), & \phi_t = \varepsilon\phi_1(x), \quad x \in \mathbb{R}^n.\end{cases} \tag{2.35}$$

Then

$$\begin{aligned}&(1+t)^{\frac{n-1}{2}}\|\phi(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \\ &\quad \le C(\phi_0,\phi_1)\varepsilon + C\sum_{|I|\le n-1}\int_0^t \left\|\left(Z^I g\right)(\tau,\cdot)/\left(1+\tau+|\cdot|\right)^{\frac{n-1}{2}}\right\|_{L^1}d\tau.\end{aligned} \tag{2.36}$$

By Lemmas 2.2 and 2.3, we can prove the following lemma.

Lemma 2.6 *Assume that* $n \ge 2$ *and* $\phi = \phi(t,x)$ *is the solution to the Cauchy problem*

$$\begin{cases}\phi_{tt} - \triangle\phi = \sum_{j=0}^n a_j\partial_j G_j, & x \in \mathbb{R}^n, t > 0, \\ t = 0: \quad \phi = \varepsilon\phi_0(x), & \phi_t = \varepsilon\phi_1(x), \quad x \in \mathbb{R}^n,\end{cases} \tag{2.37}$$

where the coefficients a_j $(j = 0,\ldots,n)$ *are constants. Then we have*

$$\|\phi(t,\cdot)\|_{L^2} \le C\left(\phi_0,\phi_1,G_0(0,\cdot)\right)m(t)\varepsilon + C\sum_{j=0}^n\int_0^t \|f_j(\tau,\cdot)\|_{L^2}\,d\tau, \tag{2.38}$$

where

$$m(t) = \begin{cases}\ln(2+t), & n = 2, \\ 1, & n = 3,\end{cases}$$

and $C(\phi_0,\phi_1,G_0(0,\cdot))$ *depends on* ϕ_0, ϕ_1 *and* $G_0(0,\cdot)$.

3 Global existence in three space dimensions

Theorem 3.1 *Suppose that* $f_0^I(x), f_1^I(x) \in C^\infty(\mathbb{R}^3)$ *and satisfy*

$$\left|f_0^I(x)\right| \le \frac{A}{(1+|x|)^k}, \qquad \left|f_1^I(x)\right| \le \frac{A}{(1+|x|)^{k+1}} \quad \left(k > \frac{3}{2}, I = 1,\ldots,q\right),$$

where $A > 0$ is a constant. Then there exists ε_0 such that for $0 < \varepsilon \leq \varepsilon_0$ the Cauchy problem (1.2), (1.3) *has a global classical solution for all $t \geq 0$.*

Proof The local existence argument follows from the method of Picard iteration [23] (see also [24] and [12]). In what follows, we will prove the global existence of the classical solutions by a continuous induction, or a bootstrap argument. Let $N \geq 7$, we set

$$\begin{aligned} M_1(t) &= \sum_{|\alpha| \leq N} \left\| \partial Z^\alpha f(t,\cdot) \right\|_{L^2}, \\ M_2(t) &= \sum_{|\alpha| \leq N} \left\| Z^\alpha f(t,\cdot) \right\|_{L^2}, \\ N_1(t) &= \sum_{|\alpha| \leq \frac{N+1}{2}} \left\| \partial Z^\alpha f(t,\cdot) \right\|_{L^\infty}, \end{aligned} \tag{3.1}$$

and

$$N_2(t) = \sum_{|\alpha| \leq \frac{N+1}{2}+1} \left\| Z^\alpha f(t,\cdot) \right\|_{L^\infty}.$$

To set up the bootstrap argument, we assume that there is a positive constant K so that on $[0,T)$ we have the following estimates for the norms defined in (3.1):

$$\begin{aligned} & M_1(t) \leq K\varepsilon, \\ & M_2(t) \leq K\varepsilon, \\ & (1+t)N_1(t) \leq K\varepsilon, \end{aligned} \tag{3.2}$$

and

$$(1+t)N_2(t) \leq K\varepsilon.$$

To close the bootstrap, we can prove that we can in fact choose K sufficiently large and ε suitably small so that the above inequalities hold independent of T with K replaced by $\frac{1}{2}K$.

From Lemma 2.1 and (1.5), we obtain

$$H^{\mu\nu}_{IJ} \partial_\mu \partial_\nu \left(Z^\alpha f^I \right) = \sum_{k \geq 3, \sum |\alpha_i| \leq |\alpha|+1} H_{I,I_1 \cdots I_k, \gamma_1 \cdots \gamma_k, \alpha_1 \cdots \alpha_k} \left(\partial_{\gamma_1} Z^{\alpha_1} f^{I_1} \right) \cdots \left(\partial_{\gamma_k} Z^{\alpha_k} f^{I_k} \right). \tag{3.3}$$

It follows from Lemma 2.4 and (3.2), (3.3) for $|\alpha| \leq N$ that

$$\begin{aligned} M_1(t) &\leq C\left(\varepsilon + C\int_0^t N_1^2(\tau) M_1(\tau)\, d\tau \right) \exp\left(C\int_0^t N_1^2(\tau)\, d\tau \right) \\ &\leq \frac{1}{2} K\varepsilon, \end{aligned} \tag{3.4}$$

if K is sufficiently large and ε_0 is suitably small.

Note that $\sqrt{\det h}h^{\mu\nu} = \eta^{\mu\nu} + O(|\partial f|^2)$; from (1.4) and Lemma 2.1, we have

$$\Box(Z^\alpha f^I) = \partial_\mu \Bigg[\sum_{k\geq 3, \sum|\alpha_i|\leq|\alpha|+1} F^{\mu,I}_{I,I_1\cdots I_k,\gamma_1\cdots\gamma_k,\alpha_1\cdots\alpha_k} (\partial_{\gamma_1} Z^{\alpha_1} f^{I_1}) \cdots (\partial_{\gamma_k} Z^{\alpha_k} f^{I_k}) \Bigg], \tag{3.5}$$

where again at most one of the α_i can satisfy $|\alpha_i| > \frac{1}{2}|\alpha|$.

Applying Lemma 2.6 to (3.5), we obtain

$$\begin{aligned} M_2(t) &\leq C\varepsilon + C\int_0^t N_1^2(t) M_1(t)\,d\tau \\ &\leq \frac{1}{2}K\varepsilon, \end{aligned} \tag{3.6}$$

if K is sufficiently large and ε_0 is suitably small.

Since $h^{\mu\nu} = \eta^{\mu\nu} + O(|\partial f|^2)$, (3.3) may also be written as

$$\Box(Z^\alpha f^I) = \sum_{k\geq 3, \sum|\alpha_i|\leq|\alpha|+1} \hat{H}_{I,I_1\cdots I_k,\gamma_1\cdots\gamma_k,\alpha_1\cdots\alpha_k} (\partial_{\gamma_1} Z^{\alpha_1} f^{I_1}) \cdots (\partial_{\gamma_k} Z^{\alpha_k} f^{I_k}), \tag{3.7}$$

where $\hat{H} = O(|\partial f|^2)$.

Using Lemma 2.5, (3.2), and (3.7), we get

$$\begin{aligned} N_2(t) &\leq C(1+t)^{-1}\left(\varepsilon + \int_0^t \frac{N_1(\tau)+N_2(\tau)}{1+\tau}\big(M_1(\tau)+M_2(\tau)\big)^2\,d\tau\right) \\ &\leq \frac{1}{2}K\varepsilon(1+t)^{-1}, \end{aligned}$$

if K is sufficiently large and ε_0 is suitably small.

So

$$(1+t)N_2(t) \leq \frac{1}{2}K\varepsilon, \tag{3.8}$$

if K is sufficiently large and ε_0 is suitably small.

From (3.1), we know that the estimate for $N_2(t)$ implies the desired estimate for $N_1(t)$. We have completed the proof of Theorem 3.1. □

4 Global existence in two space dimensions

Theorem 4.1 *Suppose that* $f_0^I(x), f_1^I(x) \in C^\infty(\mathbb{R}^2)$ *and satisfy*

$$\big|f_0^I(x)\big| \leq \frac{A}{(1+|x|)^k}, \qquad \big|f_1^I(x)\big| \leq \frac{A}{(1+|x|)^{k+1}} \quad (k > 1, I = 1, \ldots, q),$$

where $A > 0$ *is a constant. Then there exists* ε_0 *such that for* $0 < \varepsilon \leq \varepsilon_0$ *the Cauchy problem* (1.2), (1.3) *has a global classical solution for all* $t \geq 0$.

Proof The local existence argument follows from the method of Picard iteration [23] (see also [24] and [12]). In what follows, we will prove global existence of classical solutions by a continuous induction, or bootstrap argument. Let $N \geq 5$, we set

$$\begin{aligned} M_1(t) &= \sum_{|\alpha| \leq N} \|\partial Z^\alpha f(t,\cdot)\|_{L^2}, \\ M_2(t) &= \sum_{|\alpha| \leq N} \|Z^\alpha f(t,\cdot)\|_{L^2}, \\ N_1(t) &= \sum_{|\alpha| \leq \frac{N+1}{2}} \|\partial Z^\alpha f(t,\cdot)\|_{L^\infty}, \end{aligned} \tag{4.1}$$

and

$$N_2(t) = \sum_{|\alpha| \leq \frac{N+1}{2}+1} \|Z^\alpha f(t,\cdot)\|_{L^\infty}.$$

To set up the bootstrap argument, we assume that there is a positive constant K so that on $[0,T)$ we have following estimates for the norms defined in (4.1),

$$\begin{aligned} &M_1(t) \leq K\varepsilon(1+t)^\iota, \\ &M_2(t) \leq K\varepsilon(1+t)^\iota, \\ &(1+t)^{\frac{1}{2}} N_1(t) \leq K\varepsilon \end{aligned} \tag{4.2}$$

and

$$(1+t)^{\frac{1}{2}} N_2(t) \leq K\varepsilon,$$

where $0 < \iota < \frac{1}{2}$ is a fixed, arbitrary constant.

To close the bootstrap, we can prove that we can in fact choose K sufficiently large and ε suitably small so that the above inequalities hold independent of T with K replaced by $\frac{1}{2}K$.

It follows from Lemma 2.4 and (3.3) for $|\alpha| \leq N$ that

$$\begin{aligned} M_1(t) &\leq C\varepsilon \exp\left(C\int_0^t N_1^2(\tau)\,d\tau\right) + C\int_0^t \exp\left(C\int_\tau^t N_1^2(s)\,ds\right) N_1^2(\tau) M_1(\tau)\,d\tau \\ &\leq C\varepsilon \exp\left(C\int_0^t \frac{(K\varepsilon)^2}{1+\tau}\,d\tau\right) + C\int_0^t \exp\left(C\int_\tau^t \frac{(K\varepsilon)^2}{1+s}\,ds\right) \frac{(K\varepsilon)^3}{(1+\tau)^{1-\iota}}\,d\tau \\ &\leq K\varepsilon(1+t)^\iota, \end{aligned} \tag{4.3}$$

if K is sufficiently large and ε_0 is suitably small.

Applying Lemma 2.6 to (3.5), we obtain

$$\begin{aligned} M_2(t) &\leq C\varepsilon \ln(1+t) + C\int_0^t N_1^2(t) M_1(t)\,d\tau \\ &\leq \frac{1}{2} K\varepsilon(1+t)^\iota, \end{aligned} \tag{4.4}$$

if K is sufficiently large and ε_0 is suitably small.

In what follows, we make an estimate for $N_2(t)$. In order to make this estimate for $N_2(t)$, define the following null forms:

$$Q_{00}(u,v) = u_t v_t - \sum_{i=1}^{2} u_i v_i \tag{4.5}$$

and

$$Q_{ij}(u,v) = u_i v_j - u_j v_i, \quad 0 \leq i,j \leq 2, i \neq j. \tag{4.6}$$

Let Q symbolically stand for any of the full forms (4.5) and (4.6). Then

$$ZQ(u,v) = Q(Zu,v) + Q(u,Zv) + \sum a_{ij} Q_{ij}(u,v) \tag{4.7}$$

for some constants a_{ij}.

Let Q be one of null form in (4.5)-(4.7), we have

$$\left|Q(u,v)(t,x)\right| \leq C\bigl(1+t+|x|\bigr)^{-1} \sum_{|\alpha|=1} \left|Z^\alpha u(t,x)\right| \sum_{|\alpha|=1} \left|Z^\alpha v(t,x)\right|. \tag{4.8}$$

Note that the Lagrangian associated to the volume element of the induced metric is $\sqrt{-\det h}$. For small $|\partial f|$, we have

$$-\det h = 1 + \eta^{\mu\nu} \delta_{IJ} f_\mu^I f_\nu^J + O\bigl(|\partial f|^4\bigr) = 1 + \delta_{IJ} Q_{00}\bigl(f^I,f^J\bigr) + O\bigl(|\partial f|^4\bigr)$$

and thus the Euler-Lagrange equations take the form

$$\bigl(1+\delta_{KL} Q_{00}\bigl(f^K,f^L\bigr)\bigr)\Box f^I = \frac{1}{2}\eta^{\mu\nu} f_\mu^I \partial_\nu\bigl[\delta_{AB} Q_{00}\bigl(f^A,f^B\bigr)\bigr] + O\bigl(\left|\partial^2 f\right|\left|\partial f\right|^4\bigr).$$

For small $|\partial f|$, we obtain

$$\bigl(1+\delta_{KL} Q_{00}\bigl(f^K,f^L\bigr)\bigr)^{-1} = 1 + O\bigl(|\partial f|^2\bigr).$$

So we have

$$\Box f^I = \frac{1}{2} Q_{00}\bigl(f^I, \delta_{AB} Q_{00}\bigl(f^A,f^B\bigr)\bigr) + O\bigl(\left|\partial^2 f\right|\left|\partial f\right|^4\bigr). \tag{4.9}$$

By Lemma 2.1, (4.9), we have

$$\Box Z^\alpha f^I = \frac{1}{2} Q_{00}\bigl(Z^{\alpha_1} f^I, \delta_{AB} Q_{00}\bigl(Z^{\alpha_2} f^A, Z^{\alpha_3} f^B\bigr)\bigr) + O\bigl(\left|Z^{\beta_1}\partial^2 f\right|\left|Z^{\beta_2}\partial f\right|^4\bigr), \tag{4.10}$$

where $|\alpha_1| + |\beta_2| \leq |\alpha|$.

From Lemma 2.5 and (4.1), (4.2), (4.8), (4.10) when $\frac{N+1}{2} + 2 \leq N$, *i.e.*, $N \geq 5$, we get

$$\begin{aligned}(1+t)^{\frac{1}{2}} N_2(t) &\leq C\varepsilon + C\int_0^t \frac{(N_1(\tau)+N_2(\tau))}{(1+\tau)^{\frac{3}{2}}}\bigl(M_1(\tau)+M_2(\tau)\bigr)^2\,d\tau \\ &\leq C\varepsilon + C\int_0^t K^3\varepsilon^3(1+t)^{2\delta-2}\,d\tau \leq \frac{1}{2}K\varepsilon,\end{aligned} \tag{4.11}$$

if K is sufficiently large and ε_0 is suitably small and since $0 < \delta < \frac{1}{2}$.

From (4.1), we know that the estimate for $N_2(t)$ implies the desired estimate for $N_1(t)$. We have completed the proof of the theorem. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed to each part of this work equally and read and approved the final manuscript.

Author details
[1]School of Mathematics and Information Sciences, North China University of Water Resources and Electric Power, Zhengzhou, 450011, China. [2]College of Electric Power, North China University of Water Resources and Electric Power, Zhengzhou, 450011, China.

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