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Nontrivial solutions for a quasilinear elliptic system

Xiyou Cheng[*] and Lu Yang

[*]Correspondence:
chengxy@lzu.edu.cn
School of Mathematics and
Statistics, Lanzhou University,
Lanzhou, 730000, P.R. China

Abstract

In this paper, we deal with the existence of three nontrivial solutions for the potential system of p-Laplacian equations with homogeneous Dirichlet boundary conditions. Applying the Nehari procedure and abstract linking theorem on a product space, we give a linking structure for our variational problem, and then, combining with the classical minimax principle, we obtain three nontrivial critical values for the relevant energy functional.

MSC: 35J20; 35J25

Keywords: elliptic equations; Nehari manifold; linking theorem; minimax principle

1 Introduction

In this paper, we are concerned with the multiplicity of nontrivial solutions for the following system of quasilinear equations:

$$\begin{cases} -\Delta_{p_1} u = f_1(x, u) + \lambda h_1(x, u, v), & \text{in } \Omega, \\ -\Delta_{p_2} v = f_2(x, v) + \lambda h_2(x, u, v), & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega, \end{cases} \qquad (1.1)$$

where $-\Delta_{p_i} u = \operatorname{div}(|\nabla u|^{p_i-2}\nabla u)$ $(p_i > 1)$, Ω is a bounded smooth domain in $\mathbb{R}^N$ $(N \geq 3)$, $f_i : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$, $h_i : \Omega \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ are Carathéodory functions $(i = 1, 2)$, and there exists a function $H \in C^1(\Omega \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ such that

$$\nabla H(x, u, v) = \big(H_u(x, u, v), H_v(x, u, v)\big) = \big(h_1(x, u, v), h_2(x, u, v)\big).$$

In recent years, many authors have studied the existence of nontrivial solutions for Laplacian systems and p-Laplacian systems, see [1–5] and the references therein. Usually the authors change the problem into the critical point problem of the corresponding energy functional and then apply the critical point theory or the variational method, or they change it into the fixed point problem of the corresponding compactly continuous mapping and then apply topological degree theory or the method of lower and upper solutions. For instance, in [2] Costa and Magalhaes unified the cooperative and noncooperative Laplacian systems, and they got the existence of nontrivial solutions via the variational approach; in [1] Conti *et al.* dealt with the competitive Laplacian system, and they established the existence of positive solutions by the Nehari procedure, critical point theory, and topological degree theory; in [3, 4] the authors studied the sublinear p-Laplacian systems, and

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they obtained the existence of positive solutions by the method of lower and upper solutions and Leray-Schauder degree theory, respectively; in [5] Zhang and Zhang considered the existence of nontrivial solutions for nonlinear Laplacian systems and p-Laplacian systems applying the direct variational method. More recently, some authors discussed the multiplicity of nontrivial solutions for Laplacian systems and p-Laplacian systems, see [6–12] and references therein. In [6, 11, 12], the authors provided the existence results of three nontrivial solutions for system (1.1) with one parameter for the case $p = 2$, where the main methods used are the Nehari procedure and the linking theorem on product space. In [8] Motreanu and Zhang consider a general noncoercive quasilinear elliptic system, they establish the existence of two opposite constant sign solutions; in the case where the system has a variational structure, under the proper hypotheses, they obtain a third nontrivial solution, which is sign changing in the sense that one cannot have both components of the new solution of the same constant sign; their approach relies on a suitable method of sub-supersolutions combined with truncation and variational arguments that do not require a subcritical growth condition. In [9] Shen and Zhang established the existence of two positive solutions for multi-parameter p-Laplacian systems with critical exponents by use of the Nehari procedure and the variational approach. In [7, 10], the present author and coauthors dealt with a class of Laplacian systems with superlinear and sublinear terms applying the fixed point index formula on a product cone, and they obtained the existence and multiplicity of positive solutions.

Motivated by some ideas in [1, 7, 9, 12], we shall deal with the existence of nonnegative solutions (especially, positive solutions) for system (1.1) with superlinear and subcritical nonlinear terms. It is well known that the Ambrosetti-Rabinowitz type result (see [13, 14]) can be extended to system (1.1) with superlinear and subcritical nonlinear terms by imposing the Ambrosetti-Rabinowitz conditions and other proper conditions on nonlinear terms. Now, we have a natural question of when system (1.1) has multiple nonnegative solutions. In order to obtain the multiplicity of nonnegative solutions for system (1.1), we need only to construct the multiple critical values of the corresponding energy functional. For this matter, first we get two ground states in view of the Nehari procedure, establish a linking structure by the abstract linking theorem on the product space, and then construct the third critical value by the classical minimax principle.

The present paper is organized as following. In Section 2, we provide a linking structure for our variational problem (see Theorem 2.2); in Section 3, we verify that the energy functional satisfies the (P.S.) condition (see Theorem 3.1); in Section 4, we prove the existence of three nonnegative solutions for system (1.1) as λ is small enough, and, in addition, one of them is positive if the equations excluding coupled terms have both a unique positive solution (see Theorem 4.1).

2 Linking structure

In this section, we list some preliminaries, including the abstract linking theorem on the product space and the concept of Nehari manifold, and then we give a linking structure, which is useful for constructing the critical value of the functional associated with system (1.1).

Definition 2.1 [12] Let f be a real functional on Banach space X and c be a real constant, we say that $S_f^c = \{u \in X \mid f(u) = c, u \neq 0\}$ has the sphere property, if the following hypotheses are satisfied:

(S_1) f is continuous on X;

(S_2) there is a homeomorphic mapping between S_f^c and unit sphere of X;

(S_3) for any fixed $u \in X \backslash \{0\}$, the equation $f(tu) = c$ has a unique solution $t \in (0, \infty)$;

(S_4) X is separated into two open connected subsets by S_f^c and the origin is contained in one of the subsets.

Theorem 2.1 [12] *Let X, Y be Banach spaces with the following direct sum decomposition*:

$$X = X_1 \oplus X_2, \qquad Y = Y_1 \oplus Y_2,$$

where X_1, Y_1 are finite dimensional subspaces of X, Y, respectively. Let f, g be the real functionals on X, Y, respectively, c, d be two real constants and S_f^c, S_g^d have the sphere property. Take $(e_x, e_y) \in X_2 \times Y_2$ such that $f(e_x) > c$ and $g(e_y) > d$. Denote

$$Q^X = \big\{ u + t e_x \mid u \in X_1 \cap B_{R_1}, t \in [0, 1] \big\}, \qquad Q^Y = \big\{ v + s e_y \mid v \in Y_1 \cap B_{R_2}, s \in [0, 1] \big\};$$

$$S_1^{X_2} = S_f^c \cap X_2, \qquad S_2^{Y_2} = S_g^d \cap Y_2; \qquad Q = Q^X \times Q^Y \text{ and } S = S_1 \times S_2,$$

then ∂Q links S.

Definition 2.2 [15] Assume that $\varphi \in C^1(X, \mathbb{R})$ is such that $\varphi'(0) = 0$, then the constraint set $N = \{ u \in X \mid \langle \varphi'(u), u \rangle = 0, u \neq 0 \}$ is called a Nehari manifold of X.

Let $F_i(x, u) = \int_0^u f_i(x, s) \, ds$, and

$$\varphi_i(u) = \frac{1}{p_i} \int_\Omega |\nabla u|^{p_i} \, dx - \int_\Omega F_i(x, u) \, dx, \quad u \in W_0^{1, p_i}(\Omega), i = 1, 2.$$

Denote by N_i the Nehari manifold of φ_i on $X_i = W_0^{1, p_i}(\Omega)$; we have the following.

Lemma 2.1 *Assume that the following conditions are satisfied*:

(F_1) *$f_i \in C(\overline{\Omega} \times \mathbb{R}^+, \mathbb{R}^+)$, and there exist constants $q_i \in (p_i, p_i^*)$ and $C_i > 0$ such that*

$$\big| f_i(x, u) \big| \leq C_i \big(|u|^{p_i - 1} + |u|^{q_i - 1} \big) \quad \text{for all } (x, u) \in \Omega \times \mathbb{R}^+;$$

(F_2) *there exist constants $\alpha_i > \max\{ p_1, p_2 \}$ such that*

$$0 < \alpha_i F_i(x, u) \leq u f_i(x, u) \quad \text{for any } x \in \Omega, u > 0;$$

(F_3) *$\limsup_{u \to 0} \frac{f_i(x, u)}{|u|^{p_i - 2} u} < \delta_{1, p_i}$ uniformly w.r.t. $x \in \Omega$, here δ_{1, p_i} is the first eigenvalue of the operator $-\Delta_{p_i}$ on $W_0^{1, p_i}(\Omega)$;*

(F_4) *$f_i(x, u) / u^{p_i - 1}$ is an increasing function of u on $\mathbb{R}^+ / \{0\}$.*

Then the Nehari manifolds N_1, N_2 have the sphere property.

Proof It is very similar to the proof of Lemma 4.1 on page 72 in [15], so we omit it. $\square$

Theorem 2.2 *Assume that f_1, f_2 satisfy conditions (F_1)-(F_4). Let $S = N_1 \times N_2$ and $Q = \{ t e_1 \mid t \in [0, 1] \} \times \{ s e_2 \mid s \in [0, 1] \}$, here $e_i \in W_0^{1, p_i}(\Omega)$. If $\varphi_i(e_i) < 0$, then ∂Q links S.*

Proof Let $g_i(u) = \langle -\varphi_i'(u), u \rangle$, then $S_{g_i}^0 = N_i$. Let $h_i(t) = -\varphi_i(te_i)$, then $h_i(0) = 0$ and $h_i(1) > 0$, and by conditions (F_1), (F_3) and (F_4) it is easy to verify that there exists a unique $t_i^0 \in (0,1)$ such that $h_i'(t_i^0) = 0$, and $h_i'(t) < 0$ for $t \in (0, t_i^0)$, $h_i'(t) > 0$ for $t \in (t_i^0, +\infty)$. Thus

$$g_i(e_i) = \langle -\varphi_i'(e_i), e_i \rangle = h_i'(1) > 0.$$

From Lemma 2.1 and Theorem 2.1, ∂Q links S. $\qquad\square$

3 Palais-Smale condition

It is well known that the nontrivial critical points of the functional

$$\psi(u,v) = \varphi_1(u) + \varphi_2(v) - \lambda \int_\Omega H(x,u,v)\,dx, \quad (u,v) \in W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega)$$

correspond to nontrivial solutions of system (1.1). In the following, we show that the functional ψ satisfies the (P.S.) condition.

Theorem 3.1 *Assume that f_1, f_2 satisfy conditions (F_1)-(F_2), and h_1, h_2 satisfy the conditions*:

(H_1) *there exist constants $q \in (1, \min\{p_1^*, p_2^*\})$ and $C_3 > 0$ such that*

$$\big|h_1(x,u,v)\big| + \big|h_2(x,u,v)\big| \leq C_3\big(|u|^{q-1} + |v|^{q-1} + 1\big),$$

(H_2) *there exist constants $\beta_i \in (1, \min\{\alpha_1, \alpha_2\})$, $C_4 > 0$ and $R > 0$ such that*

$$uh_1(x,u,v) + vh_2(x,u,v) \leq C_4\big(|u|^{\beta_1} + |v|^{\beta_2}\big) \quad if\ |u| + |v| \geq R,$$

then the functional ψ satisfies (P.S.) *condition.*

Proof Let $\{(u_n, v_n)\} \subset W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega)$ be a sequence such that

$$\big\{\psi(u_n, v_n)\big\} \text{ is bounded, and } \psi'(u_n, v_n) \to 0 \text{ as } n \to \infty, \tag{3.1}$$

we need to prove that $\{(u_n, v_n)\}$ has a strongly convergent subsequence.

Claim I. $\{(u_n, v_n)\}$ is bounded in $W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega)$.

In fact, from (3.1) there exists a constant $C_5 > 0$ such that

$$\varphi_1(u_n) + \varphi_2(v_n) - \lambda \int_\Omega H(x, u_n, v_n)\,dx \leq C_5, \tag{3.2}$$

and for n large enough

$$\begin{cases} \int_\Omega |\nabla u_n|^{p_1}\,dx - \int_\Omega u_n f_1(x, u_n)\,dx - \lambda \int_\Omega u_n h_1(x, u_n, v_n)\,dx \\ \quad \geq -\big(\int_\Omega |\nabla u_n|^{p_1}\,dx\big)^{1/p_1}, \\ \int_\Omega |\nabla v_n|^{p_2}\,dx - \int_\Omega v_n f_2(x, v_n)\,dx - \lambda \int_\Omega v_n h_2(x, u_n, v_n)\,dx \\ \quad \geq -\big(\int_\Omega |\nabla v_n|^{p_2}\,dx\big)^{1/p_2}. \end{cases} \tag{3.3}$$

Taking $r \in (\max\{p_1, p_2\}, \min\{\alpha_1, \alpha_2\})$, combining with (3.2) and (3.3), we have

$$
\begin{aligned}
C_5 &+ \left(\int_\Omega |\nabla u_n|^{p_1} \, dx \right)^{1/p_1} + \left(\int_\Omega |\nabla v_n|^{p_2} \, dx \right)^{1/p_2} \\
&\geq \left(\frac{1}{p_1} - \frac{1}{r} \right) \int_\Omega |\nabla u_n|^{p_1} \, dx + \left(\frac{1}{p_2} - \frac{1}{r} \right) \int_\Omega |\nabla v_n|^{p_2} \, dx \\
&\quad - \int_\Omega \left(F_1(x, u_n) - \frac{1}{r} u_n f_1(x, u_n) \right) dx - \int_\Omega \left(F_2(x, v_n) - \frac{1}{r} v_n f_2(x, v_n) \right) dx \\
&\quad + \lambda \left[\int_\Omega \frac{1}{r} \left(u_n h_1(x, u_n, v_n) + v_n h_2(x, u_n, v_n) \right) dx - \int_\Omega H(x, u_n, v_n) \, dx \right] \\
&\geq \left(\frac{1}{p_1} - \frac{1}{r} \right) \int_\Omega |\nabla u_n|^{p_1} \, dx + \left(\frac{1}{p_2} - \frac{1}{r} \right) \int_\Omega |\nabla v_n|^{p_2} \, dx \\
&\quad + \frac{\alpha_1 - r}{r} \int_\Omega F_1(x, u_n) \, dx + \frac{\alpha_2 - r}{r} \int_\Omega F_2(x, v_n) \, dx \\
&\quad + \lambda \left[\int_\Omega \frac{1}{r} \left(u_n h_1(x, u_n, v_n) + v_n h_2(x, u_n, v_n) \right) dx - \int_\Omega H(x, u_n, v_n) \, dx \right] - C_6 \\
&\geq \left(\frac{1}{p_1} - \frac{1}{r} \right) \int_\Omega |\nabla u_n|^{p_1} \, dx + \left(\frac{1}{p_2} - \frac{1}{r} \right) \int_\Omega |\nabla v_n|^{p_2} \, dx \\
&\quad + C_7 \int_\Omega \left(|u_n|^{\alpha_1} + |v_n|^{\alpha_2} \right) dx - \lambda C_8 \int_\Omega \left(|u_n|^{\beta_1} + |v_n|^{\beta_2} \right) dx - C_9 \\
&\geq \left(\frac{1}{p_1} - \frac{1}{r} \right) \int_\Omega |\nabla u_n|^{p_1} \, dx + \left(\frac{1}{p_2} - \frac{1}{r} \right) \int_\Omega |\nabla v_n|^{p_2} \, dx - C_{10},
\end{aligned}
$$

as n large enough, where C_i's are positive constants, which implies that $\{(u_n, v_n)\}$ is bounded.

Claim II. $\{(u_n, v_n)\}$ has a strongly convergent subsequence.

From Claim I, there exists a subsequence, relabel it as $\{(u_n, v_n)\}$, such that

$$
u_n \rightharpoonup u \quad \text{in } W_0^{1, p_1}(\Omega), \qquad v_n \rightharpoonup v \quad \text{in } W_0^{1, p_2}(\Omega), \quad \text{as } n \to \infty.
$$

In what follows, we show that

$$
u_n \to u \quad \text{in } W_0^{1, p_1}(\Omega), \qquad v_n \to v \quad \text{in } W_0^{1, p_2}(\Omega), \quad \text{as } n \to \infty.
$$

In fact, by (3.1) we get

$$
\langle \psi'(u_n, v_n), u_n - u \rangle \to 0, \quad \text{as } n \to \infty, \tag{3.4}
$$

that is,

$$
\langle -\Delta_{p_1} u_n - f_1(x, u_n) - \lambda h_1(x, u_n, v_n), u_n - u \rangle \to 0, \quad \text{as } n \to \infty. \tag{3.5}
$$

From conditions (F_1), (H_1), and the compact embedding theorems (see [16]), we have

$$
\begin{cases}
|\langle f_1(x, u_n), u_n - u \rangle| \leq \| f_1(x, u_n) \|_{L^{q_1'}} \| u_n - u \|_{L^{q_1}} \to 0, & \text{as } n \to \infty, \\
|\langle h_1(x, u_n, v_n), u_n - u \rangle| \leq \| h_1(x, u_n, v_n) \|_{L^{q'}} \| u_n - u \|_{L^q} \to 0, & \text{as } n \to \infty,
\end{cases} \tag{3.6}
$$

where q'_1, q' are the conjugate numbers of q_1, q, respectively. In combination with (3.5) and (3.6), we obtain

$$\langle -\Delta_{p_1} u_n, u_n - u \rangle \to 0, \quad \text{as } n \to \infty. \tag{3.7}$$

By the property (S_+) of p-Laplacian operator (see [14, 17]), we know that $u_n \to u$ in $W_0^{1,p_1}(\Omega)$ as $n \to \infty$. Similarly, we can prove that $v_n \to v$ in $W_0^{1,p_2}(\Omega)$ as $n \to \infty$. □

4 Nontrivial solutions

First, we consider the least energy critical point of the functional φ_i, which is very useful for the multiplicity of nontrivial solutions to system (1.1).

For that purpose, we define

$$c_i^* = \inf_{N_i} \varphi_i, \qquad c_i^{**} = \inf_{u \in W_0^{1,p_i}(\Omega) \setminus \{0\}} \max_{t \geq 0} \varphi_i(tu) \quad \text{and} \quad c_i = \inf_{\gamma \in \Gamma_i} \max_{t \in [0,1]} \varphi_i(\gamma(t)),$$

where

$$\Gamma_i = \left\{ \gamma \in C\left([0,1], W_0^{1,p_i}(\Omega)\right) \mid \gamma(0) = 0, \varphi_i(\gamma(1)) < 0 \right\}, \quad i = 1, 2.$$

Similar to [15], we have the following.

Lemma 4.1 *Under conditions* (F_1)-(F_4), $c_i^* = c_i^{**} = c_i > 0$, *and* c_i *is a critical value of* φ_i.

Proof It is very similar to the proof of Theorem 4.2 on page 73 in [15], so we omit it. □

Lemma 4.2 *Let* f_i *satisfy conditions* (F_1)-(F_4), g_i *be of* C^2 *class and satisfy* $g_i(x,u) \leq Cf_i(x,u)$ *for all* $u > 0$ *with* $C > 0$ *a constant* $(i = 1, 2)$. *Denote*

$$G_i(x,t) = \int_0^t g_i(x,s)\,ds, \qquad \psi_i(\lambda, u) = \varphi_i(u) - \lambda \int_\Omega G_i(x,u)\,dx,$$

then, on N_i, *we have*

$$\psi_i(\lambda, u) \geq (1 - \lambda D)\varphi_i(u)$$

for all $\lambda > 0$ *with* $D > 0$ *a constant.*

Proof By the assumptions on g_i, we have

$$G_i(x,u) \leq CF_i(x,u)$$

for all $u \in \mathbb{R}^+$, then

$$\psi_i(\lambda, u) \geq \varphi_i(u) - \lambda C \int_\Omega F_i(x,u)\,dx$$

$$= \frac{1}{p_i} \int_\Omega |\nabla u|^{p_i}\,dx - (1 + \lambda C) \int_\Omega F_i(x,u)\,dx.$$

Let $D = \frac{p_i C}{\alpha_i - p_i}$, then

$$\frac{p_i(C+D)}{D} \int_\Omega F_i(x,u)\,dx = \alpha_i \int_\Omega F_i(x,u)\,dx \le \int_\Omega u f_i(x,u)\,dx = \int_\Omega |\nabla u|^{p_i}\,dx$$

due to $u \in N_i$, which means that

$$\frac{1}{p_i} \int_\Omega |\nabla u|^{p_i}\,dx - (1 + \lambda C) \int_\Omega F_i(x,u)\,dx \ge (1 - \lambda D) \int_\Omega \left[\frac{1}{p_i}|\nabla u|^{p_i} - F_i(x,u)\right] dx.$$

The proof is complete. $\qquad\square$

Theorem 4.1 *Assume that f_1, f_2 satisfy conditions* (F$_1$)-(F$_4$) *and that h_1, h_2 satisfy conditions* (H$_1$)-(H$_2$) *and the following condition*:

(H$_3$) *for a.a. $x \in \Omega$ and all $u, v \in \mathbb{R}^+$,*

$$h_i(x,u,0) = h_i(x,0,v) = 0, \quad i = 1,2.$$

Then, there exists a constant $\Lambda > 0$ such that system (1.1) *has at least three nonnegative solutions for any $\lambda \in (0, \Lambda)$. Furthermore, if the problems*

$$-\Delta_{p_i} u = f_i(x,u) \quad \text{in } \Omega, \text{ with } u = 0 \text{ on } \partial\Omega \ (i = 1,2)$$

have both a unique positive solution, then for any $\lambda \in (0, \Lambda)$, system (1.1) *has at least three nonnegative solutions in which there is a nontrivial positive solution.*

Proof From Lemma 4.1, there exists $w_i \in N_i$ such that $\varphi_i(w_i) = c_i$ and $\varphi_i'(w_i) = 0$ $(i = 1,2)$, which implies that $(w_1, 0)$ and $(0, w_2)$ are nonnegative solutions of the following system:

$$\begin{cases} -\Delta_{p_1} u = f_1(x,u), & \text{in } \Omega, \\ -\Delta_{p_2} v = f_2(x,v), & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega. \end{cases} \qquad (4.1)$$

Combining with condition (H$_3$), $(w_1, 0)$ and $(0, w_2)$ are nonnegative solutions of system (1.1) with $\psi(w_1, 0) = \varphi_1(w_1) = c_1$ and $\psi(0, w_2) = \varphi_2(w_2) = c_2$.

Now, we prove the existence of the third nonnegative solution.

First, we consider $\psi(tw_1, sw_2)$, $\forall t, s \ge 1$, as follows:

$$\begin{aligned}
\psi(tw_1, sw_2) &= \frac{1}{p_1} \int_\Omega |\nabla(tw_1)|^{p_1}\,dx + \frac{1}{p_2} \int_\Omega |\nabla(sw_2)|^{p_2}\,dx - \int_\Omega F_1(x, tw_1)\,dx \\
&\quad - \int_\Omega F_2(x, sw_2)\,dx - \lambda \int_\Omega H(x, tw_1, sw_2)\,dx \\
&= \frac{1}{p_1} \int_\Omega |\nabla(tw_1)|^{p_1}\,dx + \frac{1}{p_2} \int_\Omega |\nabla(sw_2)|^{p_2}\,dx \\
&\quad - \int_\Omega F_1(x, tw_1)\,dx - \int_\Omega F_2(x, sw_2)\,dx
\end{aligned}$$

$$- \lambda \int_{\Omega} \int_0^1 \left[tw_1 h_1(x, \tau tw_1, \tau sw_2) + sw_2 h_2(x, \tau tw_1, \tau sw_2) \right] d\tau \, dx$$

$$\leq \frac{1}{p_1} \int_{\Omega} \left| \nabla(tw_1) \right|^{p_1} dx + \frac{1}{p_2} \int_{\Omega} \left| \nabla(sw_2) \right|^{p_2} dx$$

$$- C_{11} \left(\int_{\Omega} |tw_1|^{\alpha_1} dx + \int_{\Omega} |sw_2|^{\alpha_2} dx \right)$$

$$\leq C_{12}(t+s)^{\max\{p_1, p_2\}} - C_{13}(t+s)^{\min\{\alpha_1, \alpha_2\}} + C_{14},$$

where C_i's are positive constants, which implies that there exists an $M > 0$ such that

$$\psi(tw_1, sw_2) < 0, \quad \text{as } t + s \geq M. \tag{4.2}$$

Let $e_i = Mw_i$ $(i = 1, 2)$, and

$$Q = \{ te_1 \mid t \in [0,1] \} \times \{ se_2 \mid s \in [0,1] \} \quad \text{and} \quad S = N_1 \times N_2,$$

then $\varphi_1(e_1) = \psi(Mw_1, 0) < 0$ and $\varphi_2(e_2) = \psi(0, Mw_2) < 0$, thus ∂Q links S by Theorem 2.2.

Second, we claim that $\inf_S \psi(u, v) > \sup_{\partial Q} \psi(u, v)$ as λ is small enough. In fact, by Lemma 4.1,

$$c_i = c_i^{**} = \inf_{u \in W_0^{1, p_i}(\Omega) \backslash \{0\}} \max_{t \geq 0} \varphi_i(tu) \geq \sup_{t \in [0,1]} \varphi_i(te_i),$$

which, in connection with (4.2), implies that

$$\sup_{(u,v) \in \partial Q} \psi(u, v) \leq \max\{c_1, c_2\} \tag{4.3}$$

for all $\lambda > 0$.

On the other hand, combining with Lemma 4.2, for any $(u, v) \in S$ we have

$$\psi(u, v) = \varphi_1(u) + \varphi_2(v) - \lambda \int_{\Omega} H(x, u, v) \, dx$$

$$\geq (1 - \lambda D_1)\varphi_1(u) + (1 - \lambda D_2)\varphi_2(v) - \lambda D_3,$$

here D_i's are proper positive constants. Now let $\Lambda = \min\{D_1^{-1}, D_2^{-1}, \frac{c_1 + c_2 - \max\{c_1, c_2\}}{c_1 D_1 + c_2 D_2 + D_3}\}$, then for all $\lambda \in (0, \Lambda)$,

$$\inf_{(u,v) \in S} \psi(u, v) \geq (1 - \lambda D_1)\varphi_1(u) + (1 - \lambda D_2)\varphi_2(v) - \lambda D_3 > \max\{c_1, c_2\}. \tag{4.4}$$

Hence, $\inf_S \psi(u, v) > \sup_{\partial Q} \psi(u, v)$ is inferred from (4.3) and (4.4).

Finally, we define

$$c = \inf_{\gamma \in \Gamma} \sup_Q \psi(\gamma(u, v)), \quad \text{here } \Gamma = \left\{ \gamma \in C\left(Q, W_0^{1, p_1}(\Omega) \times W_0^{1, p_2}(\Omega) \right) \mid \gamma|_{\partial Q} = \text{id}|_{\partial Q} \right\}.$$

In combination with Theorem 3.1 and the classical minimax principle (see [15, 18]), $c \geq \inf_S \psi > \max\{c_1, c_2\}$ is the third critical value of ψ.

Therefore, system (1.1) has at least three nonnegative solutions for $\lambda \in (0, \Lambda)$.

In addition, if the equations

$$\begin{cases} -\Delta_{p_i} u = f_i(x, u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega \end{cases}$$

have both a unique positive solution ($i = 1, 2$), the third nonnegative solution of system (1.1) is actually positive. In fact, if one component of the third solution equals zero, its functional value equals c_1 or c_2, which is a contradiction. □

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