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Periodic BVPs for fractional order impulsive evolution equations

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Abstract

In this paper, we study periodic BVPs for fractional order impulsive evolution equations. The existence and boundedness of piecewise continuous mild solutions and design parameter drift for periodic motion of linear problems are presented. Furthermore, existence results of piecewise continuous mild solutions for semilinear impulsive periodic problems are showed. Finally, an example is given to illustrate the results.

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1 Introduction

In order to describe dynamics of populations subject to abrupt changes as well as other evolution processes such as harvesting, diseases, and so forth, many researchers have used impulsive differential systems to describe the model since the last century. For a wide-ranging bibliography and exposition on this important object see for instance the monographs of [1–4] and the papers [5–12].

Fractional differential equations appear naturally in fields such as viscoelasticity, electrical circuits, nonlinear oscillation of earthquakes *etc.* In particular, impulsive fractional evolution equations are used to describe many practical dynamical systems in many evolutionary processes models. Recently, Wang *et al.* [13] discussed Cauchy problems and nonlocal problems for impulsive fractional evolution equations involving the Caputo fractional derivative. However, periodic boundary value problems (BVPs for short) for impulsive fractional evolution equations have not been studied extensively.

In this paper we study the periodic BVPs for impulsive fractional evolution equations. Firstly, we discuss periodic BVPs for impulsive fractional evolution equations:

$$\begin{cases} {}^c D_{0,t}^q x(t) = Ax(t) + f(t), & q \in (0,1), t \in J = [0, T], t \neq t_k, \\ x(0) = x(T), \\ x(t_k^+) = x(t_k^-) + y_k, & k = 1, 2, \ldots, \delta, \end{cases} \tag{1}$$

in Banach space X, where ${}^c D_{0,t}^q$ is the Caputo fractional derivative of order q with the lower limit zero, $A : D(A) \subseteq X \to X$ is the generator of a C_0-semigroup $\{S(t), t \geq 0\}$ on a Banach space X, $f : J \to X$ is continuous, $x(0)$ and y_k are the elements of X, $0 = t_0 < t_1 < t_2 < \cdots < t_\delta < t_{\delta+1} = T$, and $x(t_k^+) = \lim_{h \to 0^+} = x(t_k + h)$ and $x(t_k^-) = x(t_k)$ represent respectively the right and left limits of $x(t)$ at $t = t_k$.

Secondly, we design parameter drift for the above periodic motion. We study the following impulsive periodic BVPs with parameter perturbations:

$$\begin{cases} {}^{c}D_{0,t}^{q}x(t) = Ax(t) + f(t) + p(t, x(t), \xi), & t \in J, t \neq t_k, \\ x(0) = x(T), \\ x(t_k^+) = x(t_k^-) + y_k, & k = 1, 2, \dots, \delta, \end{cases} \tag{2}$$

where p is a given function and $\xi \in \Lambda = (-\tilde{\xi}, \tilde{\xi})$ is a small parameter perturbation that may be caused by some adaptive control algorithms or parameter drift.

Finally, we consider semilinear impulsive periodic problems:

$$\begin{cases} {}^{c}D_{0,t}^{q}x(t) = Ax(t) + f(t, x(t)), & t \in J, t \neq t_k, \\ x(0) = x(T), \\ x(t_k^+) = x(t_k^-) + I_k(x(t_k^-)), & k = 1, 2, \dots, \delta, \end{cases} \tag{3}$$

where $f : J \times X \to X$ is continuous and $I_k : X \to X$ is continuous.

The rest of this paper is organized as follows. In Section 2, the existence and boundedness of the operator $B = [I - \mathcal{T}(T)]^{-1}$ are given. In Section 3, the existence and boundedness of PC-mild solutions and the design parameter drift for such a periodic motion are presented. In Section 4, existence results of PC-mild solutions for impulsive periodic problems are showed. Finally, an example is presented to illustrate the theory.

2 Existence and boundedness of operator $B = [I - \mathcal{T}(T)]^{-1}$

Suppose $T > 0$, let $J = [0, T]$. Let $M = \sup_{t \geq 0} \|S(t)\| < \infty$. We denote by $C(J, X)$ the Banach space of all continuous functions from J into X with the norm $\|x\|_C = \sup\{\|x(t)\| : t \in J\}$. We also introduce the set of functions $PC(J, X) = \{x : J \to X : x \text{ is continuous at } t \in J \setminus \{t_1, t_2, \dots, t_\delta\}$, and x is continuous from the left and has right-hand limits at $t \in \{t_1, t_2, \dots, t_\delta\}\}$. Endowed with the norm

$$\|x\|_{PC} = \max\left\{ \sup_{t \in J} \|x(t+0)\|, \sup_{t \in J} \|x(t-0)\| \right\},$$

it is easy to see $(PC(J, X), \|\cdot\|_{PC})$ is a Banach space.

For measurable functions $l : J \to R$, define the norm $\|l\|_{L^p(J,R)} = (\int_J |l(t)|^p \, dt)^{\frac{1}{p}}, 1 \leq p < \infty$. We denote by $L^p(J, R)$ the Banach space of all Lebesgue measurable functions l with $\|l\|_{L^p(J,R)} < \infty$.

Definition 2.1 ([14]) The fractional integral of order γ with the lower limit zero for a function f is defined as

$$I_{0,t}^{\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t \frac{f(s)}{(t-s)^{1-\gamma}} \, ds, \quad t > 0, \gamma > 0,$$

provided the right side is point-wise defined on $[0, \infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2 ([14]) The Riemann-Liouville derivative of order γ with the lower limit zero for a function $f : [0, \infty) \to R$ can be written as

$$^{L}D_{0,t}^{\gamma} f(t) = \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\gamma+1-n}} \, ds, \quad t > 0, n-1 < \gamma < n.$$

Definition 2.3 ([14]) The Caputo derivative of order γ for a function $f : [0, \infty) \to R$ can be written as

$$^{c}D_{0,t}^{\gamma}f(t) = {}^{L}D_{0,t}^{\gamma}\left(f(t) - \sum_{k=0}^{n-1}\frac{t^{k}}{k!}f^{(k)}(0)\right), \quad t > 0, n-1 < \gamma < n.$$

Remark 2.4 If f is an abstract function with values in X, then the integrals which appear in Definitions 2.1 and 2.2 are taken in Bochner's sense.

As in our previous work [13], by a *PC*-mild solution of (1) we mean the function $x \in PC(J, X)$ satisfying

$$x(t) = \mathcal{T}(t)x(0) + \sum_{i=1}^{k}\mathcal{T}(t - t_{i})y_{i}$$

$$+ \int_{0}^{t}(t - s)^{q-1}\mathcal{S}(t - s)f(s)\,ds, \quad t \in (t_{i}, t_{i+1}], k = 1, 2, \ldots, \delta,$$

and $x(0) = x(T)$, where

$$\mathcal{T}(t) = \int_{0}^{\infty}\xi_{q}(\theta)S(t^{q}\theta)\,d\theta, \qquad \mathcal{S}(t) = q\int_{0}^{\infty}\theta\xi_{q}(\theta)S(t^{q}\theta)\,d\theta$$

and

$$\xi_{q}(\theta) = \frac{1}{q}\theta^{-1-\frac{1}{q}}\vartheta_{q}\left(\theta^{-\frac{1}{q}}\right) \geq 0,$$

$$\vartheta_{q}(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-qn-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q), \quad \theta \in (0, \infty),$$

here ξ_{q} is a probability density function defined on $(0, \infty)$, that is,

$$\xi_{q}(\theta) \geq 0, \quad \theta \in (0, \infty) \quad \text{and} \quad \int_{0}^{\infty}\xi_{q}(\theta)\,d\theta = 1.$$

Lemma 2.5 (see Lemma 2.9 [15]) *The operator $\mathcal{T}$ has the following properties*:

   (i) *For any fixed $t \geq 0$, $\mathcal{T}(t)$ and $\mathcal{S}(t)$ are linear and bounded operators, i.e., for any $x \in X$, $\|\mathcal{T}(t)x\| \leq M\|x\|$ and $\|\mathcal{S}(t)x\| \leq \frac{M}{\Gamma(q)}\|x\|$.*
  (ii) *Both $\{\mathcal{S}(t), t \geq 0\}$ and $\{\mathcal{T}(t), t \geq 0\}$ are strongly continuous.*
 (iii) *For every $t > 0$, $\mathcal{T}(t)$ and $\mathcal{S}(t)$ are compact operators if $S(t)$ is compact.*

Suppose that here the bounded operator $B : X \to X$ exists given by

$$B = \left[I - \mathcal{T}(T)\right]^{-1}. \tag{4}$$

We present sufficient conditions for the existence and boundedness of the operator B.

Lemma 2.6 (see Theorem 3.3 and Remark 3.4 [16]) *The operator B defined in* (4) *exists and is bounded, if one of the following three conditions holds*:

(i) $S(t)$ is compact for each $t > 0$ and the homogeneous linear nonlocal problem

$$\begin{cases} {}^cD_{0,t}^q x(t) = Ax(t), & t \in J, \\ x(0) = x(T), \end{cases}$$

has no non-trivial PC-mild solutions.

(ii) If $\|\mathcal{T}(T)\| < 1$, then the operator $I - \mathcal{T}(T)$ is invertible and $[I - \mathcal{T}(T)]^{-1} \in L_b(X)$.

(iii) If $\|S(t)\| < 1$ for $t \in (0, T]$, then $\mathcal{T}(nT) \to 0$ as $n \to \infty$ and the operator $I - \mathcal{T}(T)$ is invertible and $[I - \mathcal{T}(T)]^{-1} \in L_b(X)$.

3 Linear impulsive periodic problems and robustness

In this section, we consider the existence of *PC*-mild solutions of (1) and of design parameter drift for (2).

We list the following assumptions.

(HA): $A : D(A) \to X$ is the infinitesimal generator of a C_0-semigroup $\{S(t), t \geq 0\}$.

(HF): $f : J \to X$ is strongly measurable for $t \in J$ and there exist a constant $q_1 \in (0, q)$ and a real-valued function $h(\cdot) \in L^{\frac{1}{q_1}}(J, R)$ such that $\|f(t)\| \leq h(t)$, for each $t \in J$.

(HB): The operator B defined in (4) exists and is bounded.

We first give an existence theorem of *PC*-mild solutions of (1).

Theorem 3.1 *Assume that* (HA), (HF), *and* (HB) *are satisfied. Then* (1) *has a PC-mild solution given by*

$$x_T(t) = \mathcal{T}(t)Bz + \sum_{i=1}^{k} \mathcal{T}(t - t_i)y_i$$

$$+ \int_0^t (t - s)^{q-1} \mathcal{S}(t - s)f(s)\, ds, \quad t \in (t_i, t_{i+1}], k = 1, 2, \ldots, \delta, \tag{5}$$

where

$$z = \sum_{i=1}^{\delta} \mathcal{T}(t - t_i)y_i + \int_0^T (T - s)^{q-1} \mathcal{S}(T - s)f(s)\, ds. \tag{6}$$

Further, we have

$$\|x_T(t)\| \leq M(ML_1 + 1)\left[\sum_{i=1}^{\delta} \|y_i\| + \frac{T^{q-q_1}H}{\Gamma(q)(\frac{q-q_1}{1-q_1})^{1-q_1}}\right], \tag{7}$$

where $L_1 = \|B\|$ and $H = \|h\|_{L^{\frac{1}{q_1}}(J,R)}$.

Proof We consider an impulsive Cauchy problem,

$$\begin{cases} {}^cD_{0,t}^q x(t) = Ax(t) + f(t), & t \in J, t \neq t_k, \\ x(0) = Bz := \bar{x}, & z \text{ defined in (6)}, \\ x(t_k^+) = x(t_k^-) + y_k, & k = 1, 2, \ldots, \delta. \end{cases} \tag{8}$$

It follows from the expression of the initial value $x(0) = Bz := \bar{x}$ that the mild solution $x_T(\cdot)$ of (8) corresponding to the initial value $\bar{x}$ must be the *PC*-mild solution of (1).

For the estimation of $x_T(\cdot)$, by Lemma 2.5,

$$\left\| x_T(t) \right\| \leq \left\| \mathcal{T}(t)Bz \right\| + \sum_{i=1}^{k} \left\| \mathcal{T}(t - t_i) \right\| \|y_i\| + \int_0^t (t - s)^{q-1} \left\| \mathcal{S}(t - s)f(s) \right\| ds$$

$$\leq M\|B\|\|z\| + M\sum_{i=1}^{k} \|y_i\| + \frac{M}{\Gamma(q)} \int_0^t (t - s)^{q-1} h(s)\, ds$$

$$\leq M\|B\|\left(\sum_{i=1}^{\delta} \left\| \mathcal{T}(t - t_i) \right\| \|y_i\| + \int_0^T (T - s)^{q-1} \left\| \mathcal{S}(T - s)f(s) \right\| ds \right)$$

$$+ M\sum_{i=1}^{\delta} \|y_i\| + \frac{M}{\Gamma(q)} \int_0^t (t - s)^{q-1} h(s)\, ds$$

$$\leq M^2\|B\|\left(\sum_{i=1}^{\delta} \|y_i\| + \frac{1}{\Gamma(q)} \int_0^T (T - s)^{q-1} h(s)\, ds \right)$$

$$+ M\sum_{i=1}^{\delta} \|y_i\| + \frac{M}{\Gamma(q)} \int_0^t (t - s)^{q-1} h(s)\, ds$$

$$\leq M^2\|B\|\left[\sum_{i=1}^{\delta} \|y_i\| + \frac{1}{\Gamma(q)}\left(\int_0^T (T - s)^{\frac{q-1}{1-q_1}} ds \right)^{1-q_1} \left(\int_0^T h^{\frac{1}{q_1}}(s)\, ds \right)^{q_1} \right]$$

$$+ \left[M\sum_{i=1}^{\delta} \|y_i\| + \frac{M}{\Gamma(q)}\left(\int_0^t (t - s)^{\frac{q-1}{1-q_1}} ds \right)^{1-q_1} \left(\int_0^t h^{\frac{1}{q_1}}(s)\, ds \right)^{q_1} \right]$$

$$\leq M(ML_1 + 1)\left[\sum_{i=1}^{\delta} \|y_i\| + \frac{T^{q-q_1}H}{\Gamma(q)(\frac{q-q_1}{1-q_1})^{1-q_1}} \right],$$

where $L_1 = \|B\|$ and $H = \|h\|_{L^{\frac{1}{q_1}}(J,R)}$. The desired results are obtained. $\square$

Remark 3.2 In Theorem 3.1, we replace (HB) by $\{S(t), t \geq 0\}$; it is a compact semigroup and

$$\begin{cases} {}^c D_{0,t}^q x(t) = Ax(t), & t \in J, \\ x(0) = x(T), \end{cases}$$

has no non-trivial mild solutions. Then one can use the Fredholm alternative theorem to derive that the operator equation $[I - \mathcal{T}(T)]x(0) = z$ has a unique solution $x(0) = [I - \mathcal{T}(T)]^{-1}z := \bar{x}$. Thus, the PC-mild solution of (1) is unique.

Define $PC_T(J, X) = \{x \in PC(J, X) : x(0) = x(T)\}$ with $\|x\|_{PC_T} = \sup\{\|x(t)\| : t \in J\}$ for $x \in PC_T(J, X)$. It can be seen that endowed with the norm $\|\cdot\|_{PC_T}$, $PC_T(J, X)$ is a Banach space. Denote

$$\mathcal{S}_\rho = \left\{ x \in PC_T(J, X) : \|x\|_{PC_T} \leq \rho \right\},$$

$$\mathcal{B}(x, \rho_1) = \left\{ x \in PC_T(J; X) : \|x - x_T\|_{PC_T} \leq \rho_1 \right\},$$

where

$$\rho = M(ML_1 + 1)\left[\sum_{i=1}^{\delta}\|y_i\| + \frac{T^{q-q_1}H}{\Gamma(q)(\frac{q-q_1}{1-q_1})^{1-q_1}} + \frac{T^q}{\Gamma(q+1)}\sup_{|\xi|\leq\tilde{\xi}}\chi(\xi)\right],$$

$$\rho_1 = \frac{M(ML_1 + 1)T^q}{\Gamma(1+q)}\sup_{|\xi|\leq\tilde{\xi}}\chi(\xi),$$

and χ is a nonnegative function.

We introduce the assumption (HP):

(HP1): $p : J \times \mathcal{S}_\rho \times \Lambda \to X$ is measurable in t.

(HP2): There exists a nonnegative function ϖ such that $\lim_{\xi\to0}\varpi(\xi) = \varpi(0) = 0$ and for any $t \in J$, $x, y \in \mathcal{S}_\rho$, and $\xi \in \Lambda$, and we have

$$\|p(t,x,\xi) - p(t,y,\xi)\| \leq \varpi(\xi)\|x - y\|.$$

(HP3): There exists a nonnegative function χ such that $\lim_{\xi\to0}\chi(\xi) = \chi(0) = 0$, and for any $t \in J$, $x \in \mathcal{S}_\rho$, and $\xi \in \Lambda$, we have

$$\|p(t,x,\xi)\| \leq \chi(\xi).$$

By a *PC*-mild solution of (2), we mean the function $x \in PC(J, X)$ satisfying

$$x(t) = \mathcal{T}(t)x(0) + \sum_{i=1}^{k}\mathcal{T}(t - t_i)y_i$$
$$+ \int_0^t (t-s)^{q-1}\mathcal{S}(t-s)\big[f(s) + p\big(s, x(s), \xi\big)\big]\,ds, \quad k = 1, 2, \ldots, \delta,$$

and $x(0) = x(T)$.

The following result shows that given a periodic motion we can design periodic motion controllers that are robust with respect to parameter drift.

Theorem 3.3 *Let* (HA), (HF), (HB), *and* (HP) *hold. Then there is a* $\xi_0 \in (0, \tilde{\xi})$ *such that, for* $|\xi| \leq \xi_0$, (2) *has a PC-mild solution* x_T^ξ *satisfying*

$$\big\|x_T^\xi - x_T\big\|_{PC_T} \leq \rho_1$$

and $\lim_{\xi\to0} x_T^\xi(t) = x_T(t)$ *uniformly on* $t \in J$ *where* x_T *is the mild solution of* (1).

Proof By (HB), one can choose

$$x_0 = B\left[z + \int_0^T (T-s)^{q-1}\mathcal{S}(T-s)p\big(s, x(s), \xi\big)\,ds\right] \in X$$

to be fixed. Consider the map $\mathcal{O}$ on $\mathcal{B}(x_T, \rho_1)$ given by

$$(\mathcal{O}x)(t) = \mathcal{T}(t)x_0 + \sum_{i=1}^{k}\mathcal{T}(t - t_i)y_i + \int_0^t (t-s)^{q-1}\mathcal{S}(t-s)\big[f(s) + p\big(s, x(s), \xi\big)\big]\,ds.$$

Obviously, $\mathcal{O}x \in PC_T(J, X)$.

By the assumption (HP), we can choose a $\xi_0 \in (0, \tilde{\xi})$ such that

$$\frac{M(ML_1 + 1)T^q}{\Gamma(1 + q)} \sup_{|\xi| \le \xi_0} \chi(\xi) \le \rho_1$$

and

$$\eta = \frac{M(ML_1 + 1)T^q}{\Gamma(1 + q)} \sup_{|\xi| \le \xi_0} \varpi(\xi) < 1.$$

For $\xi \in (-\xi_0, \xi_0)$ and $x, y \in \mathcal{B}(x_T, \rho_1)$, one can verify that

$$\|\mathcal{O}x - x_T\|_{PC_T} \le \frac{M(ML_1 + 1)T^q}{\Gamma(1 + q)} \sup_{|\xi| \le \xi_0} \chi(\xi) \le \rho_1 \tag{9}$$

and

$$\|\mathcal{O}x - \mathcal{O}y\|_{PC_T} \le \eta\|x - y\|_{PC_T}.$$

This implies that $\mathcal{O}$ is a contraction mapping on $\mathcal{B}(x_T, \rho_1)$. Then, $\mathcal{O}$ has a unique fixed point $x_T^\xi \in \mathcal{B}(x_T, \rho_1)$ given by

$$x_T^\xi(t) = \mathcal{T}(t)x_0 + \int_0^t (t - s)^{q-1} \mathcal{S}(t - s)\big[f(s) + p\big(s, x_T^\xi(s), \xi\big)\big]\,ds, \tag{10}$$

which is just the *PC*-mild solution of (2).

From the expressions (9) and (10), one can get $\|x_T^\xi - x_T\|_{PC_T} \le \rho_1$. It is easy to see that $\lim_{\xi \to 0} x_T^\xi(t) = x_T(t)$ uniformly on $t \in J$. $\qquad\square$

4 Semilinear impulsive periodic problems

We impose the following assumptions.

(HF1): $f : J \times X \to X$ is continuous and there exist a constant $q_1 \in (0, q)$ and a
real-valued function $L_f(\cdot) \in L^{\frac{1}{q_1}}(J, R^+)$ such that $\|f(t, x) - f(t, y)\| \le L_f(t)\|x - y\|$
for all $x, y \in X$.

(HF2): $f : J \times X \to X$ is continuous and maps a bounded set into a bounded set.

(HF3): For each $x_0 \in X$, there exists a constant $r > 0$ such that

$$M(ML_1 + 1)\left(\delta M_I + \frac{T^q}{\Gamma(1 + q)} \sup_{s \in J, \phi \in Y_\Gamma} \big\|f\big(s, \phi(s)\big)\big\|\right) \le r,$$

where

$$Y_\Gamma = \big\{\phi \in PC(J, X) : \|\phi\| \le r \text{ for } t \in J\big\}.$$

(HI1): $I_k : X \to X$ is continuous and there exists a constant $L_I > 0$ such that
$\|I_k(x) - I_k(y)\| \le L_I\|x - y\|$ for all $x, y \in X$, $k = 1, 2, \ldots, \delta$.

(HI2): $I_k : X \to X$ is continuous and there exists a constant $M_I > 0$ such that
$\|I_k(x)\| \le M_I$, for all $x \in X$, $k = 1, 2, \ldots, \delta$.

Theorem 4.1 *Let* (HA), (HB), (HI1), *and* (HF1) *be satisfied. Then* (3) *has a unique PC-mild solution on J provided that*

$$M(ML_1 + 1)\left(\delta L_I + \frac{T^*}{\Gamma(q)}\right) < 1, \quad T^* = \left[\left(\frac{1-q_1}{q-q_1}\right)T^{\frac{q-q_1}{1-q_1}}\right]^{1-q_1} \|L_f\|_{L^{\frac{1}{q_1}}(J,R^+)}. \tag{11}$$

Proof Consider

$$\begin{cases} {}^cD^q_{0,t}x(t) = Ax(t) + f(t, x(t)), & t \in J, t \neq t_k, \\ x(0) = Bz_1, \\ x(t_k^+) = x(t_k^-) + I_k(x(t_k^-)), & k = 1, 2, \ldots, \delta, \end{cases} \tag{12}$$

where

$$z_1 = \sum_{i=1}^{\delta} \mathcal{T}(t - t_i)I_i\left(x(t_i^-)\right) + \int_0^T (T-s)^{q-1}\mathcal{S}(T-s)f\left(s, x(s)\right)ds, \tag{13}$$

and we define an operator Q on $PC(J, X)$:

$$(Qx)(t) = \mathcal{T}(t)Bz_1 + \sum_{i=1}^{k} \mathcal{T}(t - t_i)I_i\left(x(t_i^-)\right)$$

$$+ \int_0^t (t-s)^{q-1}\mathcal{S}(t-s)f\left(s, x(s)\right)ds, \quad t \in (t_i, t_{i+1}], k = 1, 2, \ldots, \delta. \tag{14}$$

Clearly, Q is well defined on $PC(J, X)$ due to our assumptions.

Then, we only need to show that Q is a contraction on $PC(J, X)$.

In general, for each $t \in (t_k, t_{k+1}]$, $k = 1, 2, \ldots, \delta$ we have

$$\left\|(Qx)(t) - (Qy)(t)\right\|$$

$$\leq M^2 L_1 L_I \sum_{i=1}^{\delta} \left\|x(t_i^-) - y(t_i^-)\right\| + ML_I \sum_{i=1}^{k} \left\|x(t_i^-) - y(t_i^-)\right\|$$

$$+ \frac{M}{\Gamma(q)}\left(ML_1 \int_0^T (T-s)^{q-1}L_f(s)\left\|x(s) - y(s)\right\|ds\right.$$

$$+ \int_0^t (t-s)^{q-1}L_f(s)\left\|x(s) - y(s)\right\|ds\right)$$

$$\leq ML_I(ML_1 + 1)\delta\|x - y\|_{PC}$$

$$+ \frac{M\|x - y\|_{PC}}{\Gamma(q)}\left(ML_1 \int_0^T (T-s)^{q-1}L_f(s)ds + \int_0^t (t-s)^{q-1}L_f(s)ds\right)$$

$$\leq ML_I(ML_1 + 1)\delta\|x - y\|_{PC}$$

$$+ \frac{M\|x - y\|_{PC}}{\Gamma(q)}\left[ML_1\left(\int_0^T (T-s)^{\frac{q-1}{1-q_1}}ds\right)^{1-q_1}\|L_f\|_{L^{\frac{1}{q_1}}(J,R^+)}\right.$$

$$+ \left(\int_0^t (t-s)^{\frac{q-1}{1-q_1}}ds\right)^{1-q_1}\|L_f\|_{L^{\frac{1}{q_1}}([0,t_{k+1}],R^+)}\right]$$

$$\leq M(ML_1 + 1)\left(\delta L_I + \frac{T^*}{\Gamma(q)}\right)\|x - y\|_{PC}.$$

Hence, the condition (11) allows us to conclude, in view of the Banach contraction mapping principle, that Q has a unique fixed point $x \in PC(J,X)$, which is just the unique PC-mild solution of (3). □

Theorem 4.2 *Suppose that* (HA), (HB), (HI2), *and* (HF2) *and* (HF3) *are satisfied. Then for every* $x_0 \in X$, (3) *has at least a PC-mild solution on* J.

Proof Consider the mapping

$$Q : PC(J,X) \to PC(J,X)$$

by

$$(Qv)(t) = (Q_1 v)(t) + (Q_2 v)(t),$$

where

$$(Q_1 v)(t) = \mathcal{T}(t)Bz_1 + \int_0^t (t-s)^{q-1} \mathcal{S}(t-s) f\big(s, v(s)\big)\, ds, \quad t \in J \setminus \{t_1, t_2, \ldots, t_\delta\}; \tag{15}$$

z_1 is defined in (13) and

$$(Q_2 v)(t) = \begin{cases} 0, & t \in [0, t_1], \\ \sum_{i=1}^k \mathcal{T}(t-t_i) I_i(v(t_i^-)), & t \in (t_k, t_{k+1}], k = 1, \ldots, \delta. \end{cases} \tag{16}$$

For each $t \in (t_k, t_{k+1}]$, $v \in Y_\Gamma$,

$$\big\| (Qv)(t) \big\| \le \big\| (Q_1 v)(t) \big\| + \big\| (Q_2 v)(t) \big\|$$

$$\le M(ML_1 + 1)\left(\delta M_I + \frac{T^q}{\Gamma(1+q)} \sup_{s \in J, \phi \in Y_\Gamma} \big\| f\big(s, \phi(s)\big) \big\| \right).$$

Thus, we see that $Q : Y_\Gamma \to Y_\Gamma$.

Just like the proof in our previous work [13], one can prove that Q is a continuous mapping from Y_Γ to Y_Γ and it is a compact operator. Now, Schauder's fixed point theorem implies that Q has a fixed point, which gives rise to a PC-mild solution. □

5 Example

We consider impulsive fractional differential equations with periodic boundary conditions,

$$\begin{cases} {}^c D_{0,t}^{\frac{1}{2}} x(t,y) = \frac{\partial^2}{\partial y^2} x(t,y) + \frac{|x(t,y)|}{\gamma(1+9e^t)(1+|x(t,y)|)}, & y \in (0, \pi), t \in [0, \frac{1}{2}) \cup (\frac{1}{2}, 1], \\ x(t,0) = x(t,\pi) = 0, \\ x(0,y) = x(1,y), & y \in (0, \pi), \\ x(\frac{1}{2}^+) = x(\frac{1}{2}^-) + \frac{|x(\frac{1}{2}^-, y)|}{\gamma(9+t)(1+|x(\frac{1}{2}^-, y)|)}, & y \in (0, \pi), \end{cases} \tag{17}$$

in $X = L^2(0, \pi)$ where $\gamma > 0$ will be chosen later.

Define $Ax = -\frac{\partial^2}{\partial y^2}x$ for $x \in D(A)$ where $D(A) = \{x \in X : \frac{\partial x}{\partial y}, \frac{\partial^2 x}{\partial y^2} \in X$ and $x(0) = x(\pi) = 0\}$. Then A is the infinitesimal generator of a C_0-semigroup $\{T(t), t \geq 0\}$ in $L^2(0, \pi)$. Moreover, $T(\cdot)$ is also compact and $\|T(t)\| \leq e^{-t} \leq 1$, $t \geq 0$. By the Fredholm alternative theorem, $[I - \mathcal{T}(1)]^{-1}$ exists and is bounded where $\mathcal{T}(\cdot)$ is defined in Section 2.

Define $f(t, x(t))(y) = \frac{|x(t,y)|}{\gamma(1+9e^t)(1+|x(t,y)|)}$, $t \in [0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$, $x \in X$, $y \in (0, \pi)$ and $f : [0, 1] \times X \to X$ is a continuous function, $\|f(t, x_1) - f(t, x_2)\| \leq L_f \|x_1 - x_2\|$, with $L_f = \frac{1}{10\gamma} \in L^{\frac{1}{4}}([0, 1], R^+)$. Define $I_1(x(\frac{1}{2}^-))(y) = \frac{|x(\frac{1}{2}^-, y)|}{\gamma(9+t)(1+|x(\frac{1}{2}^-, y)|)}$, $x \in X$, $y \in (0, \pi)$. $\|I_1(x_1) - I_1(x_2)\| \leq L_I \|x_1 - x_2\|$ for all $x_1, x_2 \in X$ with $L_I = \frac{1}{10\gamma}$.

Moreover,

$$M(ML_1 + 1)\left(\delta L_I + \frac{T^*}{\Gamma(q)}\right) = \left\|[I - \mathcal{T}(1)]^{-1}\right\|\left(\frac{1}{10\gamma} + \frac{\sqrt[4]{27}}{10\gamma\,\Gamma(\frac{1}{2})}\right) < \frac{2}{5\gamma}\left\|[I - \mathcal{T}(1)]^{-1}\right\|.$$

Thus, one can choose $\gamma > \frac{2}{5}\|[I - \mathcal{T}(1)]^{-1}\|$ such that (11) holds. Therefore, (17) has a unique PC-mild solution on $[0, 1]$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

This work was carried out in collaboration between all authors. JRW raised these interesting problems in this research. JRW and XLY proved the theorems, interpreted the results and wrote the article. All authors defined the research theme, read and approved the manuscript.

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References

1. Bainov, DD, Simeonov, PS: Impulsive Differential Equations: Periodic Solutions and Applications. Longman, New York (1993)
2. Samoilenko, AM, Perestyuk, NA: Impulsive Differential Equations. World Scientific Series on Nonlinear Science. Series A: Monographs and Treatises, vol. 14. World Scientific, Singapore (1995)
3. Benchohra, M, Henderson, J, Ntouyas, SK: Impulsive Differential Equations and Inclusions, vol. 2. Hindawi Publishing Corporation, New York (2006)
4. Lakshmikantham, V, Bainov, DD, Simeonov, PS: Theory of Impulsive Differential Equations. World Scientific, Singapore (1989)
5. Abada, N, Benchohra, M, Hammouche, H: Existence and controllability results for nondensely defined impulsive semilinear functional differential inclusions. J. Differ. Equ. **246**, 3834-3863 (2009)
6. Ahmed, NU: Existence of optimal controls for a general class of impulsive systems on Banach space. SIAM J. Control Optim. **42**, 669-685 (2003)
7. Akhmet, MU: On the smoothness of solutions of impulsive autonomous systems. Nonlinear Anal. TMA **60**, 311-324 (2005)
8. Fan, Z, Li, G: Existence results for semilinear differential equations with nonlocal and impulsive conditions. J. Funct. Anal. **258**, 1709-1727 (2010)
9. Liang, J, Liu, JH, Xiao, T-J: Nonlocal impulsive problems for nonlinear differential equations in Banach spaces. Math. Comput. Model. **49**, 798-804 (2009)
10. Liu, J: Nonlinear impulsive evolution equations. Dyn. Contin. Discrete Impuls. Syst. **6**, 77-85 (1999)
11. Battelli, F, Fečkan, M: Chaos in singular impulsive O.D.E. Nonlinear Anal. TMA **28**, 655-671 (1997)
12. Wang, J, Fečkan, M, Zhou, Y: Ulam's type stability of impulsive ordinary differential equations. J. Math. Anal. Appl. **395**, 258-264 (2012)
13. Wang, J, Fečkan, M, Zhou, Y: On the new concept of solutions and existence results for impulsive fractional evolution equations. Dyn. Partial Differ. Equ. **8**, 345-361 (2011)

14. Kilbas, AA, Srivastava, HM, Trujillo, JJ: Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, vol. 204. Elsevier, Amsterdam (2006)
15. Wang, J, Zhou, Y: A class of fractional evolution equations and optimal controls. Nonlinear Anal., Real World Appl. **12**, 262-272 (2011)
16. Wang, J, Zhou, Y, Fečkan, M: Alternative results and robustness for fractional evolution equations with periodic boundary conditions. Electron. J. Qual. Theory Differ. Equ. **2011**, 97 (2011)