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Variational approach to a class of impulsive differential equations

Dajun Guo*

*Correspondence:
guodj@sdu.edu.cn
Department of Mathematics, Shandong University, Jinan, Shandong 250100, People's Republic of China

Abstract
In this article, the author discusses the existence of solutions for a class of impulsive differential equations by means of a variational approach different from earlier approaches.
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1 Introduction

The theory of impulsive differential equations has been emerging as an important area of investigation in recent years [1–3]. There is a vast literature on the existence of solutions by using topological methods, including fixed point theorems, Leray-Schauder degree theory, and fixed point index theory [4–15]. But it is quite difficult to apply the variational approach to an impulsive differential equation; therefore, there was no result in this area for a long time. Only in the recent five years, there appeared a few articles which dealt with some impulsive differential equations by using variational methods [16–20]. Motivated by [17], in this article we shall use a different variational approach to discuss the existence of solutions for a class of impulsive differential equations and we only deal with classical solutions.

Consider the boundary value problem (BVP) for the second-order nonlinear impulsive differential equation:

$$\begin{cases} -u''(t) = f(t, u(t)), \quad \forall t \in J', \\ \Delta u|_{t=t_k} = c_k \quad (k = 1, 2, 3, \dots, m), \\ \Delta u'|_{t=t_k} = d_k \quad (k = 1, 2, 3, \dots, m), \\ u(0) = u(1) = 0, \end{cases} \tag{1}$$

where $J = [0,1]$, $0 < t_1 < \cdots < t_k < \cdots < t_m < 1$, $J' = J \backslash \{t_1, \dots, t_k, \dots, t_m\}$, c_k and d_k ($k = 1, 2, \dots, m$) are any real numbers, $f(t,u)$ is a real function defined on $J \times R$, where R denotes the set of all real numbers, and $f(t,u)$ is continuous on $J' \times R$, left continuous at $t = t_k$, *i.e.*

$$\lim_{t \to t_k - 0, w \to u} f(t, w) = f(t_k, u)$$

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for any $u \in R$ ($k = 1, 2, \dots, m$), and the right limit at $t = t_k$ exists, *i.e.*

$$\lim_{t \to t_k+0, w \to u} f(t, w)$$

(denoted by $f(t_k^+, u)$) exists for any $u \in R$ ($k = 1, 2, \dots, m$). $\Delta u|_{t=t_k}$ denotes the jump of $u(t)$ at $t = t_k$, *i.e.*

$$\Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-),$$

where $u(t_k^+)$ and $u(t_k^-)$ represent the right and left limits of $u(t)$ at $t = t_k$, respectively. Similarly,

$$\Delta u'|_{t=t_k} = u'(t_k^+) - u'(t_k^-),$$

where $u'(t_k^+)$ and $u'(t_k^-)$ represent the right and left limits of $u'(t)$ at $t = t_k$, respectively. Let $PC[J, R] = \{u : u$ is a real function on J such that $u(t)$ is continuous at $t \neq t_k$, left continuous at $t = t_k$, and $u(t_k^+)$ exists, $k = 1, 2, \dots, m\}$ and $PC^1[J, R] = \{u \in PC[J, R] : u'(t)$ is continuous at $t \neq t_k$ and $u'(t_k^+), u'(t_k^-)$ exist, $k = 1, 2, \dots, m\}$. A function $u \in PC^1[J, R] \cap C^2[J', R]$ is called a solution of BVP (1) if $u(t)$ satisfies (1).

Let us list some conditions.

(H_1) There exist $p > 2$, $a > 0$ and $b > 0$ such that

$$\left|f(t, u)\right| \leq a + b|u|^{p-1}, \quad \forall t \in J, u \in R.$$

(H_2) There exist $0 < c < \frac{\pi^2}{4}$ and $d > 0$ such that

$$\int_0^u f(t, v)\, dv \leq cu^2 + d, \quad \forall t \in J, u \in R.$$

Lemma 1 *$u \in PC^1[J, R] \cap C^2[J', R]$ is a solution of BVP* (1) *if and only if $v \in C[J, R]$ is a solution of the integral equation*

$$v(t) = \int_0^1 G(t, s) g\bigl(s, v(s)\bigr)\, ds, \quad \forall t \in J, \tag{2}$$

where

$$G(t, s) = \begin{cases} s(1-t), & \forall 0 \leq s \leq t \leq 1; \\ t(1-s), & \forall 0 \leq t < s \leq 1, \end{cases} \tag{3}$$

$$g(t, v) = f\bigl(t, v + a(t) - a(1)t\bigr), \quad \forall t \in J, v \in R \tag{4}$$

and

$$v(t) = u(t) - a(t) + a(1)t, \qquad a(t) = \sum_{0<t_k<t} \bigl[c_k + (t - t_k) d_k\bigr], \quad \forall t \in J. \tag{5}$$

Proof For $u \in PC^1[J,R] \cap C^2[J',R]$, we have the formula (see [21], Lemma 1(b))

$$u(t) = u(0) + tu'(0) + \int_0^t (t-s)u''(s)\,ds + \sum_{0<t_k<t} \bigl\{\bigl[u\bigl(t_k^+\bigr) - u(t_k)\bigr] + (t-t_k)\bigl[u'\bigl(t_k^+\bigr) - u'\bigl(t_k^-\bigr)\bigr]\bigr\}, \quad \forall t \in J. \tag{6}$$

So, if $u \in PC^1[J,R] \cap C^2[J',R]$ is a solution of BVP (1), then, by (1) and (6), we have

$$\begin{aligned} u(t) &= tu'(0) - \int_0^t (t-s)f\bigl(s,u(s)\bigr)\,ds + \sum_{0<t_k<t} \bigl[c_k + (t-t_k)d_k\bigr] \\ &= tu'(0) - \int_0^t (t-s)f\bigl(s,u(s)\bigr)\,ds + a(t), \quad \forall t \in J. \end{aligned} \tag{7}$$

It is clear, by (5), that

$$a(t) = 0, \quad \forall 0 \le t \le t_1; \qquad a(1) = \sum_{k=1}^m \bigl[c_k + (1-t_k)d_k\bigr], \tag{8}$$

so

$$v'(0) = u'(0) + a(1). \tag{9}$$

Substituting (9) into (7), we get

$$\begin{aligned} v(t) &= tv'(0) - \int_0^t (t-s)f\bigl(s,u(s)\bigr)\,ds \\ &= tv'(0) - \int_0^t (t-s)f\bigl(s,v(s) + a(s) - a(1)s\bigr)\,ds \\ &= tv'(0) - \int_0^t (t-s)g\bigl(s,v(s)\bigr)\,ds, \quad \forall t \in J. \end{aligned} \tag{10}$$

By virtue of (5), we see that $v \in C[J,R]$ (in fact, $v \in C^1[J,R]$) and

$$v(1) = u(1) - a(1) + a(1) = u(1) = 0,$$

so, letting $t = 1$ in (10), we find

$$v'(0) = \int_0^1 (1-s)g\bigl(s,v(s)\bigr)\,ds. \tag{11}$$

Substituting (11) into (10), we get

$$\begin{aligned} v(t) &= \int_t^1 t(1-s)g\bigl(s,v(s)\bigr)\,ds + \int_0^t s(1-t)g\bigl(s,v(s)\bigr)\,ds \\ &= \int_0^1 G(t,s)g\bigl(s,v(s)\bigr)\,ds, \quad \forall t \in J, \end{aligned}$$

so $v(t)$ is a solution of the integral equation (2).

Conversely, suppose that $v \in C[J,R]$ is a solution of (2), *i.e.*

$$v(t) = (1-t)\int_0^t sg\big(s,v(s)\big)\,ds + t\int_t^1 (1-s)g\big(s,v(s)\big)\,ds, \quad \forall t \in J. \tag{12}$$

By (4), it is clear that $g(t,v(t))$ is continuous on J', so differentiation of (12) gives

$$\begin{aligned} v'(t) &= -\int_0^t sg\big(s,v(s)\big)\,ds + (1-t)tg\big(t,v(t)\big) \\ &\quad + \int_t^1 (1-s)g\big(s,v(s)\big)\,ds - t(1-t)g\big(t,v(t)\big) \\ &= -\int_0^t sg\big(s,v(s)\big)\,ds + \int_t^1 (1-s)g\big(s,v(s)\big)\,ds, \quad \forall t \in J'. \end{aligned} \tag{13}$$

Differentiating again, we get

$$v''(t) = -tg\big(t,v(t)\big) - (1-t)g\big(t,v(t)\big) = -g\big(t,v(t)\big), \quad \forall t \in J'. \tag{14}$$

From (13) we see that $v'(t_k^+)$ and $v'(t_k^-)$ $(k = 1,2,\ldots,m)$ exist and

$$v'\big(t_k^+\big) = v'\big(t_k^-\big) = -\int_0^{t_k} sg\big(s,v(s)\big)\,ds + \int_{t_k}^1 (1-s)g\big(s,v(s)\big)\,ds. \tag{15}$$

It follows from (4), (5), (12), (14), and (15) that $u \in PC^1[J,R] \cap C^2[J',R]$ and $u(t)$ satisfies (1). □

Lemma 2 *Let condition* (H_1) *be satisfied. If* $v \in L^p[J,R]$ *is a solution of the integral equation* (2), *then* $v \in C[J,R]$.

Proof It is clear, for function $a(t)$ defined by (5),

$$\big|a(t)\big| \le a_0, \quad \forall t \in J; \qquad a_0 = \sum_{k=1}^m \big(|c_k| + (1-t_k)|d_k|\big). \tag{16}$$

By (4), (5), (16), and condition (H_1), we have

$$\begin{aligned} \big|g(t,v)\big| &\le a + b\big|v + a(t) - a(1)t\big|^{p-1} \le a + b\big(|v| + 2a_0\big)^{p-1} \\ &\le a + b\big(2\max\big\{|v|, 2a_0\big\}\big)^{p-1} \le a + b2^{p-1}\big(|v|^{p-1} + (2a_0)^{p-1}\big), \quad \forall t \in J, v \in R, \end{aligned}$$

so,

$$\big|g(t,v)\big| \le a_1 + b_1|v|^{p-1}, \quad \forall t \in J, v \in R, \tag{17}$$

where

$$a_1 = a + b2^{2(p-1)}a_0^{p-1}, \qquad b_1 = b2^{p-1}.$$

It is clear that $g(t,v)$ satisfies the Caratheodory condition, *i.e.* $g(t,v)$ is measurable with respect to t on J for every $v \in R$ and is continuous with respect to v on R for almost $t \in J$ (in fact, $g(t,v)$ is discontinuous only at $t = t_k$ ($k = 1, 2, \ldots, m$)), so (17) implies [22, 23] that the operator g defined by

$$(gv)(t) = g\bigl(t, v(t)\bigr), \quad \forall t \in J \tag{18}$$

is bounded and continuous from $L^p[J,R]$ into $L^q[J,R]$, where $\frac{1}{p} + \frac{1}{q} = 1$ ($q > 1$).

Let $v \in L^p[J,R]$ be a solution of the integral equation (2). Then by the Hölder inequality,

$$\bigl|v(t_1) - v(t_2)\bigr| \le \left(\int_0^1 \bigl|G(t_1,s) - G(t_2,s)\bigr|^p\,ds\right)^{\frac{1}{p}} \left(\int_0^1 \bigl|g\bigl(s, v(s)\bigr)\bigr|^q\,ds\right)^{\frac{1}{q}}, \quad \forall t_1, t_2 \in J,$$

which implies by virtue of the uniform continuity of $G(t,s)$ on $J \times J$ that $v \in C[J,R]$. □

2 Variational approach

Theorem 1 *If conditions* (H_1) *and* (H_2) *are satisfied, then BVP* (1) *has at least one solution* $u \in PC^1[J,R] \cap C^2[J',R]$.

Proof By Lemma 1 and Lemma 2, we need only to show that the integral equation (2) has a solution $v \in L^p[J,R]$. The integral equation (2) can be written in the form

$$v = Ggv, \tag{19}$$

where G is the linear integral operator defined by

$$(Gv)(t) = \int_0^1 G(t,s)v(s)\,ds, \quad \forall t \in J, \tag{20}$$

and the nonlinear operator g is defined by (18), which is bounded and continuous from $L^p[J,R]$ into $L^q[J,R]$ ($\frac{1}{p} + \frac{1}{q} = 1$). It is well known that $G(t,s)$ is a L^2 positive-definite kernel with eigenvalues $\{\frac{1}{n^2\pi^2}\}$ ($n = 1, 2, 3, \ldots$) and, by the continuity of $G(t,s)$, we have

$$\int_0^1 \int_0^1 \bigl[G(t,s)\bigr]^p\,ds\,dt < \infty, \tag{21}$$

so [22, 23] the linear operator G defined by (20) is completely continuous from $L^2[J,R]$ into $L^2[J,R]$ and also from $L^q[J,R]$ into $L^p[J,R]$, and $G = HH^*$, where $H = G^{\frac{1}{2}}$ (the positive square-root operator of G) is completely continuous from $L^2[J,R]$ into $L^p[J,R]$ and H^* denotes the adjoint operator of H, which is completely continuous from $L^q[J,R]$ into $L^2[J,R]$. We now show that (19) has a solution $v \in L^p[J,R]$ is equivalent to the equation

$$u = H^*gHu \tag{22}$$

has a solution $u \in L^2[J,R]$. In fact, if $v \in L^p[J,R]$ is a solution of (19), *i.e.* $v = HH^*gv$, then $H^*gv = H^*gHH^*gv$, so, $u = H^*gv \in L^2[J,R]$ and u is a solution of (22). Conversely, if $u \in L^2[J,R]$ is a solution of (22), then $Hu = HH^*gHu = GgHu$, so, $v = Hu \in L^p[J,R]$ and v is a solution of (19). Consequently, we need only to show that (22) has a solution $u \in L^2[J,R]$. It is well known [22, 23] that the functional Φ defined by

$$\Phi(u) = \frac{1}{2}(u,u) - \int_0^1 dt \int_0^{(Hu)(t)} g(t,v)\,dv, \quad \forall u \in L^2[J,R] \tag{23}$$

is a C^1 functional on $L^2[J,R]$ and its Fréchet derivative is

$$\Phi'(u) = u - H^* gHu, \quad \forall u \in L^2[J,R]. \tag{24}$$

Hence we need only to show that there exists a $u \in L^2[J,R]$ such that $\Phi'(u) = \theta$ (θ denotes the zero element of $L^2[J,R]$), *i.e.* u is a critical point of functional Φ.

By (4), (5), (16), and condition (H_1), we have

$$\int_0^u g(t,v)\,dv = \int_0^{u+a(t)-a(1)t} f(t,w)\,dw - \int_0^{a(t)-a(1)t} f(t,w)\,dw, \quad \forall t \in J, u \in R \tag{25}$$

and

$$\begin{aligned} \left|\int_0^{a(t)-a(1)t} f(t,w)\,dw\right| &\le \left|a(t)-a(1)t\right|\left(a + b\left|a(t)-a(1)t\right|^{p-1}\right) \\ &\le 2a_0\left(a + b2^{p-1}a_0^{p-1}\right) = a_2, \quad \forall t \in J. \end{aligned} \tag{26}$$

So, (25), (26), and condition (H_2) imply

$$\begin{aligned} \int_0^{(Hu)(t)} g(t,v)\,dv &\le \int_0^{(Hu)(t)+a(t)-a(1)t} f(t,w)\,dw + a_2 \\ &\le c\left\{(Hu)(t) + a(t) - a(1)t\right\}^2 + d + a_2 \\ &\le 2c\left\{\left[(Hu)(t)\right]^2 + \left[a(t)-a(1)t\right]^2\right\} + d + a_2 \\ &\le 2c\left[(Hu)(t)\right]^2 + 8ca_0^2 + d + a_2, \quad \forall u \in L^2[J,R], t \in J. \end{aligned} \tag{27}$$

It is well known [24],

$$\|G\| = \lambda_1 = \frac{1}{\pi^2}, \tag{28}$$

where G is defined by (20) and is regarded as a positive-definite operator from $L^2[J,R]$ into $L^2[J,R]$, and λ_1 denotes the largest eigenvalue of G. It follows from (23), (27), and (28) that

$$\begin{aligned} \Phi(u) &\ge \frac{1}{2}(u,u) - 2c(Hu,Hu) - 8ca_0^2 - d - a_2 \\ &= \frac{1}{2}(u,u) - 2c(Gu,u) - 8ca_0^2 - d - a_2 \ge \frac{1}{2}(u,u) - \frac{2c}{\pi^2}(u,u) - 8ca_0^2 - d - a_2 \\ &= \left(\frac{1}{2} - \frac{2c}{\pi^2}\right)\|u\|^2 - 8ca_0^2 - d - a_2, \quad \forall u \in L^2[J,R], \end{aligned} \tag{29}$$

which implies by virtue of $0 < c < \frac{\pi^2}{4}$ (see condition (H_2)) that

$$\lim_{\|u\|\to\infty} \Phi(u) = \infty. \tag{30}$$

So, there exists a $r > 0$ such that

$$\Phi(u) > \Phi(\theta) = 0, \quad \forall u \in L^2[J,R], \|u\| > r. \tag{31}$$

It is well known [22, 23] that the ball $T(\theta,r) = \{u \in L^2[J,R] : \|u\| \le r\}$ is weakly closed and weakly compact and the functional $\Phi(u)$ is weakly lower semicontinuous, so, there exists $u^* \in T(\theta,r)$ such that

$$\Phi(u^*) = \inf_{u\in T(\theta,r)} \Phi(u) \le \Phi(\theta). \tag{32}$$

It follows from (31) and (32) that

$$\Phi(u^*) = \inf_{u\in L^2[J,R]} \Phi(u).$$

Hence $\Phi'(u^*) = \theta$ and the theorem is proved. □

Example 1 Consider the BVP

$$\begin{cases} -u''(t) = \frac{9}{2}u(t)\sin(t-u(t)) - t^3, & \forall t \in J', \\ \Delta u|_{t=t_k} = c_k \quad (k=1,2,\dots,m), \\ \Delta u'|_{t=t_k} = d_k \quad (k=1,2,\dots,m), \\ u(0) = u(1) = 0, \end{cases} \tag{33}$$

where $J = [0,1]$, $0 < t_1 < \cdots < t_k < \cdots < t_m < 1$, $J' = J\backslash\{t_1,\dots,t_k,\dots,t_m\}$, c_k and d_k ($k = 1,2,\dots,m$) are any real numbers.

Conclusion BVP (33) has at least one solution $u \in PC^1[J,R] \cap C^2[J',R]$.

Proof Evidently, (33) is a BVP of the form (1) with

$$f(t,u) = \frac{9}{2}u\sin(t-u) - t^3. \tag{34}$$

It is clear that $f \in C[J\times R,R]$. By (34), we have

$$|f(t,u)| \le \frac{9}{2}|u| + 1, \quad \forall t \in J, u \in R. \tag{35}$$

Moreover, it is well known that

$$|u| \le \frac{1}{2}(1+u^2), \quad \forall u \in R. \tag{36}$$

So, (35) and (36) imply that

$$|f(t,u)| \le \frac{9}{4}u^2 + \frac{13}{4}, \quad \forall t \in J, u \in R,$$

and consequently, condition (H_1) is satisfied for $p = 3$, $a = \frac{13}{4}$ and $b = \frac{9}{4}$. On the other hand, choose ϵ_0 such that

$$0 < \epsilon_0 < \frac{1}{4}(\pi^2 - 9). \tag{37}$$

For $|u| \geq \frac{1}{\epsilon_0}$, we have $|u| \leq \epsilon_0 u^2$, so,

$$|u| \leq \epsilon_0 u^2 + \frac{1}{\epsilon_0}, \quad \forall u \in R. \tag{38}$$

By (35), we have

$$\int_0^u f(t,v)\,dv \leq \frac{9}{4}u^2 + |u|, \quad \forall t \in J, u \in R. \tag{39}$$

It follows from (38) and (39) that

$$\int_0^u f(t,v)\,dv \leq \left(\frac{9}{4} + \epsilon_0\right)u^2 + \frac{1}{\epsilon_0}, \quad \forall t \in J, u \in R. \tag{40}$$

Since, by virtue of (37),

$$0 < \frac{9}{4} + \epsilon_0 < \frac{\pi^2}{4},$$

we see that (40) implies that condition (H_2) is satisfied for $c = \frac{9}{4} + \epsilon_0$ and $d = \frac{1}{\epsilon_0}$. Hence, our conclusion follows from Theorem 1. □

By using the Mountain Pass Lemma and the Minimax Principle established by Ambrosetti and Rabinowitz [25, 26], we have obtained in [23] the existence of a nontrivial solution and the existence of infinitely many nontrivial solutions for a class of nonlinear integral equations. Since (2) is a special case of such nonlinear integral equations, we get the following result for (2).

Lemma 3 (Special case of Theorem 1 and Theorem 2 in [23]) *Suppose the following.*

(a) *There exist* $p > 2$ *and* $a > 0$, $b > 0$ *such that*

$$|g(t,v)| \leq a + b|v|^{p-1}, \quad \forall t \in J, v \in R.$$

(b) *There exist* $0 \leq \tau < \frac{1}{2}$ *and* $M > 0$ *such that*

$$\int_0^v g(t,w)\,dw \leq \tau v g(t,v), \quad \forall t \in J, |v| \geq M.$$

(c) $\frac{g(t,v)}{v} \to 0$ *as* $v \to 0$ *uniformly for* $t \in J$ *and* $\frac{g(t,v)}{v} \to \infty$ *as* $|v| \to \infty$ *uniformly for* $t \in J$.

Then the integral equation (2) *has at least one nontrivial solution in* $L^p[J,R]$. *If, in addition,*

(d) $g(t,-v) = -g(t,v)$, $\forall t \in J$, $v \in R$.

Then the integral equation (2) *has infinite many nontrivial solutions in* $L^p[J,R]$.

Let us list more conditions for the function $f(t,u)$.

(H_3) There exist $0 \leq \tau < \frac{1}{2}$ and $M > 0$ such that

$$\int_0^u f\bigl(t, v + a(t) - a(1)t\bigr)\,dv \leq \tau u f\bigl(t, u + a(t) - a(1)t\bigr), \quad \forall t \in J, |u| \geq M.$$

(H_4) $\frac{f(t,u+a(t)-a(1)t)}{u} \to 0$ as $u \to 0$ uniformly for $t \in J$, and $\frac{f(t,u+a(t)-a(1)t)}{u} \to \infty$ as $|u| \to \infty$ uniformly for $t \in J$.

(H_5) $f(t,-u+a(t)-a(1)t) = -f(t,u+a(t)-a(1)t)$, $\forall t \in J$, $u \in R$.

Theorem 2 *Suppose that conditions* (H_1), (H_3), *and* (H_4) *are satisfied. Then BVP* (1) *has at least one solution* $u \in PC^1[J,R] \cap C^2[J',R]$. *If, in addition, condition* (H_5) *is satisfied, then BVP* (1) *has infinitely many solutions* $u_n \in PC^1[J,R] \cap C^2[J',R]$ ($n = 1,2,3,\ldots$).

Proof In the proof of Lemma 2, we see that condition (H_1) implies condition (a) of Lemma 3 (see (17)). On the other hand, it is clear that conditions (H_3), (H_4), (H_5) are the same as conditions (b), (c), (d) in Lemma 3, respectively. Hence the conclusion of Theorem 2 follows from Lemma 3, Lemma 2, and Lemma 1. □

Example 2 Consider the BVP

$$\begin{cases} -u''(t) = \begin{cases} [u(t)-t]^3, & \forall 0 \le t < \frac{1}{2}; \\ [u(t)+3t-3]^3, & \forall \frac{1}{2} < t \le 1, \end{cases} \\ \Delta u|_{t=\frac{1}{2}} = 1, \\ \Delta u'|_{t=\frac{1}{2}} = -4, \\ u(0) = u(1) = 0. \end{cases} \tag{41}$$

Conclusion BVP (41) has infinite many solutions $u_n \in PC^1[J,R] \cap C^2[J',R]$ ($n = 1,2,3,\ldots$).

Proof Obviously, (41) is a BVP of form (1). In this situation, $J = [0,1]$, $m = 1$, $t_1 = \frac{1}{2}$, $J' = [0,1] \backslash \{\frac{1}{2}\}$, $c_1 = 1$, $d_1 = -4$, and

$$f(t,u) = \begin{cases} (u-t)^3, & \forall 0 \le t \le \frac{1}{2}; \\ (u+3t-3)^3, & \forall \frac{1}{2} < t \le 1. \end{cases} \tag{42}$$

It is clear that $f(t,u)$ is continuous on $J' \times R$, left continuous at $t = t_1$, and the right limit $f(t_1^+,u)$ exists. By (42), we have

$$\begin{aligned} |f(t,u)| &\le \left(|u| + \frac{3}{2}\right)^3 \le \left(2\max\left\{|u|, \frac{3}{2}\right\}\right)^3 \\ &\le 2^3\left(|u|^3 + \left(\frac{3}{2}\right)^3\right) = 8|u|^3 + 27, \quad \forall t \in J, u \in R, \end{aligned}$$

so, condition (H_1) is satisfied for $p = 4$, $a = 27$ and $b = 8$. By (5), we have

$$a(t) = \begin{cases} 0, & \forall 0 \le t \le \frac{1}{2}; \\ 3-4t, & \forall \frac{1}{2} < t \le 1, \end{cases} \tag{43}$$

so, $a(1) = -1$ and (42) and (43) imply

$$f\big(t, u + a(t) - a(1)t\big) = u^3, \quad \forall t \in J, u \in R, \tag{44}$$

and, consequently, (H_3) is satisfied for $\tau = \frac{1}{4}$ and any $M > 0$. On the other hand, from (44) we see that conditions (H_4) and (H_5) are all satisfied. Hence, our conclusion follows from Theorem 2. □

Competing interests
The author declares that they have no competing interests.

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