Boichuk and Pokutnyi *Boundary Value Problems* 2014, 2014:4
http://www.boundaryvalueproblems.com/content/2014/1/4

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Solutions of the Schrödinger equation in a Hilbert space

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Dedicated to the academician Kiguradze I.T.

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Abstract

Necessary and sufficient conditions for the existence of a solution of a boundary-value problem for the Schrödinger equation are obtained in the linear and nonlinear cases. Analytic solutions are represented using the generalized Green operator.

Keywords: normally resolvable operator; generalized Green operator; Schrödinger equation

Introduction

The Schrödinger equation is the subject of numerous publications, and it is impossible to analyze all of them in detail. For this reason, we only briefly describe the methods and ideas that underlie the approach proposed in this paper for the investigation of the linear and a weakly nonlinear Schrödinger equation with different boundary conditions.

In this work, we develop constructive methods of analysis of linear and weakly nonlinear boundary-value problems, which occupy a central place in the qualitative theory of differential equations. The specific feature of these problems is that the operator of the linear part of the equation does not have an inverse. This does not allow one to use the traditional methods based on the principles of contracting mappings and a fixed point. These problems include the most complicated and inadequately studied problems known as critical (or resonance) problems [1–4]. Therefore, for the investigation of periodic problems for the Schrödinger equation, we develop the technique of generalized inverse operators [5–8] for the original linear operator in Banach and Hilbert spaces.

On the other hand, we use the notion of a strong generalized solution of an operator equation developed in [9]. The origins of this approach go back to the works of Weil and Sobolev. Using the process of completion, one can introduce the concept of a strong pseudoinverse operator for an arbitrary linear bounded operator and thus relax the requirement that the range of its values be closed. In this way, one can prove the existence of solutions of different types for the linear Schrödinger equation with arbitrary inhomogeneities. Thus, one may say that, in a certain sense, the Schrödinger equation is always solvable. There are three possible types of solutions: classical generalized solutions, strong generalized solution, and strong pseudosolutions [10].

For the analysis of a weakly nonlinear Schrödinger equation, we develop the ideas of the Lyapunov-Schmidt method and efficient methods of perturbation theory, namely the

Vishik-Lyusternik method [11]. The combination of different approaches allows us to take a different look at the Schrödinger equation with a constant unbounded operator in the linear part and obtain all its solutions by using the generalized Green operator of this problem constructed in this work. Possible generalizations are discussed in the final part of the paper. By an example of the abstract van der Pol equation, we illustrate the results that can be obtained by using the proposed method.

Auxiliary result (linear case)

Statement of the problem

Consider the following boundary-value problem for the Schrödinger equation in a Hilbert space H_T:

$$\frac{d\varphi(t)}{dt} = -iH_0\varphi(t) + f(t), \quad t \in [0; w], \tag{1}$$

$$\varphi(0) - \varphi(w) = \alpha \in D, \tag{2}$$

where $H_T = H \oplus H$, H is Hilbert space and vector-function $f(t)$ is integrable; for simplicity, the unbounded operator H_0 has the following form [12] for each $t \in [0; w]$:

$$H_0 = i\begin{pmatrix} 0 & T \\ -T & 0 \end{pmatrix} = \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} = i\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}.$$

In a more general case, the operator H_0 has the form

$$H_0 = iJ\begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} = i\begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}J, \quad J = J^* = J^{-1},$$

where T is a strongly positive self-adjoint operator in the Hilbert space H. Since the operator T is closed, the domain $D(T)$ of the operator T is a Hilbert space with scalar product (Tu, Tu). The operator H_0 is self-adjoint in the domain $D = D(T) \oplus D(T)$ with product

$$\bigl(\langle u, v\rangle, \langle u, v\rangle\bigr)_{H^T} = (Tu, Tu)_H + (Tv, Tv)_H,$$

the infinitesimal generator of a strongly continuous evolution semigroup has the form

$$U(t) := U(t, 0) = \begin{pmatrix} \cos tT & \sin tT \\ -\sin tT & \cos tT \end{pmatrix},$$

$$U^n(t) = \begin{pmatrix} \cos ntT & \sin ntT \\ -\sin ntT & \cos ntT \end{pmatrix} = U(nt),$$

$\|U^n(t)\| = 1$, $n \in N$ (nonexpanding group), $\varphi(t) = (\varphi_1(t), \varphi_2(t))^T$, $\alpha = (\alpha_1, \alpha_2)^T$, and $f(t) = (f_1(t), f_2(t))^T$. The mild solutions of equation (1) can be represented in the form

$$\varphi(t) = U(t)c + \int_0^t U(t)U^{-1}(\tau)f(\tau)\,d\tau,$$

for any element $c \in H_T$. Substituting this in condition (2), we conclude that the solvability of the boundary-value problem (1), (2) is equivalent to the solvability of the following operator equation:

$$\bigl(I - U(w)\bigr)c = g, \tag{3}$$

where $g = \alpha + U(w)\int_0^w U^{-1}(\tau)f(\tau)\,d\tau$. Consider the case where the set of values of $I - U(w)$ is closed $R(I - U(w)) = \overline{R(I - U(w))}$. Since $\|U^n(w)\| = \|U(wn)\| = 1$ for all $n \in N$, we can conclude [13] that the operator system (3) is solvable if and only if

$$U_0(w)g = 0, \tag{4}$$

where

$$U_0(w) = \lim_{n\to\infty} \frac{\sum_{k=0}^n U^k(w)}{n} = \lim_{n\to\infty} \frac{\sum_{k=0}^n U(kw)}{n},$$

is the orthoprojector that projects the space H_T onto the subspace $1 \in \sigma(U(w))$. Under this condition, the solutions of (3) have the form

$$c = U_0(w)\overline{c} + \left(\sum_{k=0}^{\infty}(\mu - 1)^k \left\{\sum_{l=0}^{\infty} \mu^{-l-1}\bigl(U(w) - U_0(w)\bigr)^l\right\}^{k+1} - U_0(w)\right)g,$$

for $0 < \mu - 1 < \frac{1}{\|R_\mu(U(w))\|}$, and any $\overline{c} \in H_T$. Then we can formulate the first result as a lemma.

Lemma 1 *Suppose that the operator $I - U(w)$ has a closed image $R(I - U(w)) = \overline{R(I - U(w))}$.*

1. *Solutions of the boundary-value problem* (1), (2) *exist if and only if*

$$U_0(w)\left(\alpha + \int_0^w U^{-1}(\tau)f(\tau)\,d\tau\right) = 0. \tag{5}$$

2. *Under condition* (5), *solutions of* (1), (2) *have the form*

$$\varphi(t, \overline{c}) = U(t)U_0(w)\overline{c} + \bigl(G[f, \alpha]\bigr)(t), \tag{6}$$

where

$$\begin{aligned}\bigl(G[f, \alpha]\bigr)(t) = U(t)&\sum_{k=0}^{\infty}(\mu - 1)^k \left\{\sum_{l=0}^{\infty} \mu^{-l-1}\bigl(U(w) - U_0(w)\bigr)^l\right\}^{k+1} \\ &\times \left(\alpha + \int_0^w U(w)U^{-1}(\tau)f(\tau)\,d\tau\right) \\ &- U(t)U_0(w)\left(\alpha + \int_0^w U(w)U^{-1}(\tau)f(\tau)\,d\tau\right) \\ &+ \int_0^t U(t)U^{-1}(\tau)f(\tau)\,d\tau,\end{aligned}$$

is the generalized Green operator of the boundary-value problem (1), (2) *for* $0 < \mu - 1 < 1/\|R_\mu(U(w))\|$.

We now show that the condition $R(I - U(w)) = \overline{R(I - U(w))}$ of Lemma 1 can be omitted and, in different senses, the boundary-value problem (1), (2) is always solvable.

(1) Classical generalized solutions.

Consider the case where the set of values of $I - U(w)$ is closed ($R(I - U(w)) = \overline{R(I - U(w))}$). Then [5] $g \in R(I - U(w))$ if and only if $\mathcal{P}_{N((I-U(w))^*)}g = 0$, and the set of solutions of (3) has the form [5] $c = G[g] + U_0(w)\overline{c}$, $\forall \overline{c} \in H_T$, where [5, 13]

$$G[g] = \bigl(I - U(w)\bigr)^+ g = \bigl(I - \bigl(U(w) - U_0(w)\bigr)^{-1} - U_0(w)\bigr)g$$

is the generalized Green operator (or it has the form of a convergent series).

(2) Strong generalized solutions. Consider the case where $R(I - U(w)) \neq \overline{R(I - U(w))}$ and $g \in \overline{R(I - U(w))}$. We show that the operator $I - U(w)$ can be extended to $\overline{I - U(w)}$ in such a way that $R(\overline{I - U(w)})$ is closed.

Since the operator $I - U(w)$ is bounded, the following representation of H_T in the form of a direct sum is true:

$$H_T = N\bigl(I - U(w)\bigr) \oplus X, \qquad H_T = \overline{R\bigl(I - U(w)\bigr)} \oplus Y,$$

where $X = N(I - U(w))^{\perp} = \overline{R(I - U(w))}$ and $Y = \overline{R(I - U(w))}^{\perp} = N(I - U(w))$. Let $E = H_T/N(I - U(w))$ be the quotient space of H_T and let $\mathcal{P}_{\overline{R(I-U(w))}}$ and $\mathcal{P}_{N(I-U(w))}$ be the orthoprojectors onto $\overline{R(I - U(w))}$ and $N(I - U(w))$, respectively. Then the operator

$$\mathcal{I} - \mathcal{U}(w) = \mathcal{P}_{\overline{R(I-U(w))}}\bigl(I - U(w)\bigr)j^{-1}p : X \to R\bigl(I - U(w)\bigr) \subset \overline{R\bigl(I - U(w)\bigr)}$$

is linear, continuous, and injective. Here,

$$p : X \to E = H_T/N\bigl(I - U(w)\bigr) \quad \text{and} \quad j : H_T \to E,$$

are a continuous bijection and a projection, respectively. The triple (H_T, E, j) is a locally trivial bundle with typical fiber $H_1 = \mathcal{P}_{N(I-U(w))}H_T$ [14]. In this case [9, p.26,29], we can define a strong generalized solution of the equation

$$\bigl(\mathcal{I} - \mathcal{U}(w)\bigr)x = g, \quad x \in X. \tag{7}$$

We complete the space X with the norm $\|x\|_{\overline{X}} = \|(\mathcal{I} - \mathcal{U}(w))x\|_F$, where $F = \overline{R(I - U(w))}$ [9]. Then the extended operator

$$\overline{\mathcal{I} - \mathcal{U}(w)} : \overline{X} \to \overline{R\bigl(I - U(w)\bigr)}, \quad X \subset \overline{X}$$

is a homeomorphism of $\overline{X}$ and $\overline{R(I - U(w))}$. By the construction of a strong generalized solution [9], the equation

$$\bigl(\overline{\mathcal{I} - \mathcal{U}(w)}\bigr)\overline{\xi} = g,$$

has a unique solution $(\overline{\mathcal{I} - \mathcal{U}(w)})^{-1}g$, which is called the generalized solution of equation (7).

Remark 1 It should be noted that there exist the following extensions of spaces and the corresponding operators:

$$\overline{p}: \overline{X} \to \overline{E}, \qquad \overline{j}: \overline{H}_T \to \overline{E}, \qquad \overline{\mathcal{P}}_X = \mathcal{P}_{\overline{X}}: \overline{H}_T \to \overline{X}, \qquad \overline{G}: \overline{R(I - U(w))} \to \overline{X},$$

where

$$\overline{H}_T = N(I - U(w)) \oplus \overline{X}; \qquad \overline{p}(x) = p(x), \quad x \in X; \qquad \overline{j}(x) = j(x), \quad x \in H_T,$$
$$\overline{\mathcal{P}}_X(x) = \mathcal{P}_X(x), \quad x \in H_T \ (\mathcal{P}_X = \mathcal{P}_X^2 = \mathcal{P}_X^*); \qquad \overline{G}[g] = G[g], \quad g \in R(I - U(w)).$$

Then the operator $\overline{I - U(w)} = (\overline{\mathcal{I} - \mathcal{U}(w)})\mathcal{P}_{\overline{X}}: \overline{H}_T \to H_T$ is an extension of $I - U(w)$, and $\overline{(I - U(w))}c = (I - U(w))c$ for all $c \in H_T$.

(3) Strong pseudosolutions.

Consider an element $g \notin \overline{R(I - U(w))}$. This condition is equivalent to $\mathcal{P}_{N((I-U(w))^*)}g \neq 0$. In this case, there are elements of $\overline{H}_T$ that minimize the norm $\|\overline{(I - U(w))}\xi - g\|_{H_T}$:

$$\xi = \overline{(\mathcal{I} - \mathcal{U}(w))}^{-1} g + \mathcal{P}_{N(I-U(w))}\overline{c}, \quad \forall \overline{c} \in H_T.$$

These elements are called *strong pseudosolutions* by analogy with [5].

We now formulate the full theorem on solvability.

Theorem 1 *The boundary-value problem* (1), (2) *is always solvable.*

(1) (a) *Classical or strong generalized solutions of* (1), (2) *exist if and only if*

$$U_0(w)\left(\alpha + \int_0^w U^{-1}(\tau)f(\tau)\,d\tau\right) = 0. \tag{8}$$

If $(\alpha + \int_0^w U^{-1}(\tau)f(\tau)\,d\tau) \in R(I - U(w))$, *then solutions of* (1), (2) *are classical.*

(b) *Under assumption* (8), *the solutions of* (1), (2) *have the form*

$$\varphi(t, \overline{c}) = U(t)U_0(w)\overline{c} + (\overline{G[f, \alpha]})(t),$$

where $(\overline{G[f, \alpha]})(t)$ *is an extension of the operator* $(G[f, \alpha])(t)$.

(2) (a) *Strong pseudosolutions exist if and only if*

$$U_0(w)\left(\alpha + \int_0^w U^{-1}(\tau)f(\tau)\,d\tau\right) \neq 0. \tag{9}$$

(b) *Under assumption* (9), *the strong pseudosolutions of* (1), (2) *have the form*

$$\varphi(t, \overline{c}) = U(t)U_0(w)\overline{c} + (\overline{G[f, \alpha]})(t),$$

where

$$\begin{aligned}(\overline{G[f, \alpha]})(t) &= U(t)\overline{G}[g] + \int_0^t U(t)U^{-1}(\tau)f(\tau)\,d\tau \\ &= U(t)\overline{(\mathcal{I} - \mathcal{U}(w))}^{-1} g + \int_0^t U(t)U^{-1}(\tau)f(\tau)\,d\tau.\end{aligned}$$

1 Main result (nonlinear case)

1.1 Modification of the Lyapunov-Schmidt method

In the Hilbert space H_T defined above, we consider the boundary-value problem

$$\frac{d\varphi(t)}{dt} = -iH_0\varphi(t) + \varepsilon Z\big(\varphi(t), t, \varepsilon\big) + f(t), \tag{10}$$

$$\varphi(0,\varepsilon) - \varphi(w,\varepsilon) = \alpha. \tag{11}$$

We seek a generalized solution $\varphi(t,\varepsilon)$ of the boundary-value problem (10), (11) that becomes one of the solutions of the generating equation (1), (2) $\varphi_0(t,\overline{c})$ in the form (6) for $\varepsilon = 0$.

To find a necessary condition for the operator function $Z(\varphi, t, \varepsilon)$, we impose the joint constraints

$$Z(\cdot,\cdot,\cdot) \in C\big([0;w], H_T\big) \times C[0,\varepsilon_0] \times C\big[\|\varphi - \varphi_0\| \leq q\big],$$

where q is a positive constant.

The main idea of the next results was used in [15] for the investigation of bounded solutions.

Let us show that this problem can be solved with the use of the following operator equation for generating amplitudes:

$$F(\overline{c}) = U_0(w) \int_0^w U^{-1}(\tau) Z\big(\varphi_0(\tau,\overline{c}), \tau, 0\big)\, d\tau = 0. \tag{12}$$

Theorem 2 (Necessary condition) *Suppose that the nonlinear boundary-value problem* (10), (11) *has a generalized solution* $\varphi(\cdot,\varepsilon)$ *that becomes one of the solutions* $\varphi_0(t,\overline{c})$ *of the generating equation* (1), (2) *with constant* $\overline{c} = c^0$ *and* $\varphi(t,0) = \varphi_0(t,c^0)$ *for* $\varepsilon = 0$. *Then this constant must satisfy the equation for generating amplitudes* (12).

Proof If the boundary-value problem (10), (11) has classical generalized solutions, then, by Lemma 1, the following solvability condition must be satisfied:

$$U_0(w)\left(\alpha + \int_0^w U^{-1}(\tau)\big\{f(\tau) + \varepsilon Z\big(\varphi(\tau,\varepsilon), \tau, \varepsilon\big)\big\}\, d\tau\right) = 0. \tag{13}$$

By using condition (5), we establish that condition (13) is equivalent to the following:

$$U_0(w) \int_0^w U^{-1}(\tau) Z\big(\varphi(\tau,\varepsilon), \tau, \varepsilon\big)\, d\tau = 0.$$

Since $\varphi(t,\varepsilon) \to \varphi_0(t,c^0)$ as $\varepsilon \to 0$, we finally obtain [by using the continuity of the operator function $Z(\varphi,t,\varepsilon)$] the required assertion.

To find a sufficient condition for the existence of solutions of the boundary-value problem (10), (11), we additionally assume that the operator function $Z(\varphi,t,\varepsilon)$ is strongly differentiable in a neighborhood of the generating solution ($Z(\cdot,t,\varepsilon) \in C^1[\|\varphi - \varphi_0\| \leq q]$).

This problem can be solved with the use of the operator

$$B_0 = \frac{dF(\overline{c})}{d\overline{c}}\bigg|_{\overline{c}=c_0} = U_0(w) \int_0^w U^{-1}(t) A_1(t)\, dt : H \to H,$$

where $A_1(t) = Z^1(\nu,t,\varepsilon)|_{\nu=\varphi_0,\varepsilon=0}$ (Fréchet derivative). □

Theorem 3 (Sufficient condition) *Suppose that the operator B_0 satisfies the following conditions*:

(1) *The operator B_0 is Moore-Penrose pseudoinvertible*;

(2) $\mathcal{P}_{N(B_0^*)}U_0(w) = 0$.

Then, for an arbitrary element $c = c^0 \in H_T$ satisfying the equation for generating amplitudes (12), *there exists at least one solution of* (10), (11).

This solution can be found by using the following iterative process:

$$\overline{v}_{k+1}(t,\varepsilon) = \varepsilon G\bigl[Z\bigl(\varphi_0\bigl(\tau,c^0\bigr) + v_k,\tau,\varepsilon\bigr),\alpha\bigr](t),$$

$$c_k = -B_0^+ U_0(w)\int_0^w U^{-1}(\tau)\bigl\{A_1(\tau)\overline{v}_k(\tau,\varepsilon) + \mathcal{R}\bigl(v_k(\tau,\varepsilon),\tau,\varepsilon\bigr)\bigr\}\,d\tau,$$

$$\mathcal{R}\bigl(v_k(t,\varepsilon),t,\varepsilon\bigr) = Z\bigl(\varphi_0\bigl(t,c^0\bigr) + v_k(t,\varepsilon),t,\varepsilon\bigr) - Z\bigl(\varphi_0\bigl(t,c^0\bigr),t,0\bigr) - A_1(t)v_k(t,\varepsilon),$$

$$\mathcal{R}(0,t,0) = 0, \qquad \mathcal{R}_x^1(0,t,0) = 0,$$

$$v_{k+1}(t,\varepsilon) = U(t)U_0(w)c_k + \overline{v}_{k+1}(t,\varepsilon),$$

$$\varphi_k(t,\varepsilon) = \varphi_0\bigl(t,c^0\bigr) + v_k(t,\varepsilon), \quad k = 0,1,2,\ldots, \qquad v_0(t,\varepsilon) = 0, \varphi(t,\varepsilon) = \lim_{k\to\infty}\varphi_k(t,\varepsilon).$$

1.2 Relationship between necessary and sufficient conditions

First, we formulate the following assertion:

Corollary *Suppose that a functional $F(\overline{c})$ has the Fréchet derivative $F^{(1)}(\overline{c})$ for each element c^0 of the Hilbert space H satisfying the equation for generating constants* (12). *If $F^1(\overline{c})$ has a bounded inverse, then the boundary-value problem* (10), (11) *has a unique solution for each c^0*.

Remark 2 If the assumptions of the corollary are satisfied, then it follows from its proof that the operators B_0 and $F^{(1)}(c^0)$ are equal. Since the operator $F^{(1)}(\overline{c})$ is invertible, it follows that assumptions 1 and 2 of Theorem 3 are necessarily satisfied for the operator B_0. In this case, the boundary-value problem (10), (11) has a unique bounded solution for each $c^0 \in H_T$ satisfying (12). Therefore, the invertibility condition for the operator $F^1(\overline{c})$ expresses the relationship between the necessary and sufficient conditions. In the finite-dimensional case, the condition of invertibility of the operator $F^{(1)}(\overline{c})$ is equivalent to the condition of simplicity of the root c^0 of the equation for generating amplitudes [5].

In this way, we modify the well-known Lyapunov-Schmidt method. It should be emphasized that Theorems 2 and 3 give us a condition for the chaotic behavior of (10) and (11) [16].

1.3 Example

We now illustrate the obtained assertion. Consider the following differential equation in a separable Hilbert space H:

$$\ddot{y}(t) + Ty(t) = \varepsilon\bigl(1 - \bigl\|y(t)\bigr\|^2\bigr)\dot{y}(t), \tag{14}$$

$$y(0) = y(w), \qquad \dot{y}(0) = \dot{y}(w), \tag{15}$$

where T is an unbounded operator with compact T^{-1}. Then there exists an orthonormal basis $e_i \in H$ such that $y(t) = \sum_{i=1}^{\infty} c_i(t)e_i$ and $Ty(t) = \sum_{i=1}^{\infty} \lambda_i c_i(t)e_i$, $\lambda_i \to \infty$. In this case, the operator system (10), (11) for the boundary-value problem (14), (15) is equivalent to the following countable system of ordinary differential equations ($c_k(t) = x_k(t)$):

$$\begin{aligned} &\dot{x}_k(t) = \sqrt{\lambda_k} y_k(t), \quad k = 1, 2, \dots, \\ &\dot{y}_k(t) = -\sqrt{\lambda_k} x_k(t) + \varepsilon\sqrt{\lambda_k}\left(1 - \sum_{j=1}^{\infty} x_j^2(t)\right) y_k(t), \end{aligned} \tag{16}$$

$$x_k(0) = x_k(w), \qquad y_k(0) = y_k(w). \tag{17}$$

We find the solutions of these equations in the space $W_2^1([0;w])$ that, for $\varepsilon = 0$, turn into one of the solutions of the generating equation. Consider the critical case $\lambda_i = 4\pi^2 i^2/w^2$, $i \in N$. Let $w = 2\pi$. In this case, the set of all periodic solutions of (16), (17) has the form

$$\begin{aligned} &x_k(t) = \cos(kt)c_1^k + \sin(kt)c_2^k, \\ &y_k(t) = -\sin(kt)c_1^k + \cos(kt)c_2^k \end{aligned}$$

for all pairs of constants $c_1^k, c_2^k \in R$, $k \in N$. The equation for generating amplitudes (12) is equivalent in this case to the following countable systems of algebraic nonlinear equations:

$$\begin{aligned} &\left(c_1^k\right)^3 + 2\sum_{j=1, j\neq k}\left(c_1^k\left(c_1^j\right)^2 + c_1^k\left(c_2^j\right)^2\right) + c_1^k\left(c_2^k\right)^2 - 4c_1^k = 0, \\ &\left(c_2^k\right)^3 + 2\sum_{j=1, j\neq k}\left(c_2^k\left(c_1^j\right)^2 + c_2^k\left(c_2^j\right)^2\right) + \left(c_1^k\right)^2 c_2^k - 4c_2^k = 0, \quad k \in N. \end{aligned}$$

Then we can obtain the next result.

Theorem 4 (Necessary condition for the van der Pol equation) *Suppose that the boundary-value problem* (16), (17) *has a bounded solution* $\varphi(\cdot, \varepsilon)$ *that becomes one of the solutions of the generating equations with pairs of constants* (c_1^k, c_2^k), $k \in N$. *Then only a finite number of these pairs are not equal to zero. Moreover, if* $(c_1^{k_i}, c_2^{k_i}) \neq (0, 0)$, $i = \overline{1, N}$, *then these constants lie on an* N*-dimensional torus in the infinite-dimensional space of constants*:

$$\left(c_1^{k_i}\right)^2 + \left(c_2^{k_i}\right)^2 = \left(\frac{2}{\sqrt{2N-1}}\right)^2, \quad i = \overline{1, N}.$$

Remark Similarly, we can study the Schrödinger equation with a variable operator and more general boundary conditions (as noted in the introduction).

Consider the differential Schrödinger equation

$$\frac{d\varphi(t)}{dt} = -iH(t)\varphi(t) + f(t), \quad t \in J \tag{18}$$

in a Hilbert space H with the boundary condition

$$Q\varphi(\cdot) = \alpha, \tag{19}$$

where, for each $t \in J \subset R$, the unbounded operator $H(t)$ has the form $H(t) = H_0 + V(t)$, $H_0 = H_0^*$ is an unbounded self-adjoint operator with domain $D = D(H_0) \subset H$, and the mapping $t \to V(t)$ is strongly continuous. The operator Q is linear and bounded and acts from the Hilbert space H to H_1. As in [12], we define the operator-valued function

$$\tilde{V}(t) = e^{itH_0} V(t) e^{-itH_0}.$$

In this case, $\tilde{V}(t)$ admits the Dyson representation [12, p.311]; denote its propagator by $\tilde{U}(t,s)$. If $U(t,s) = e^{-itH_0}\tilde{U}(t,s)e^{isH_0}$, then $\psi_s(t) = U(t,s)\psi$ is a weak solution of (14) with the condition $\varphi_s(s) = \psi$ in the sense that, for any $\eta \in D(H_0)$, the function $(\eta, \psi_s(t))$ is differentiable and

$$\frac{d}{dt}\big(\eta, \psi_s(t)\big) = -i\big(H_0\eta, \psi_s(t)\big) - i\big(V(t)\eta, \psi_s(t)\big) + \big(f(t), \psi_s(t)\big), \quad t \in J.$$

A detailed study of the boundary-value problem (18), (19) will be given in a separate paper.

Competing interests
The authors did not provide this information.

Authors' contributions
The authors did not provide this information.

Received: 1 October 2013 Accepted: 11 December 2013 Published: 06 Jan 2014

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10.1186/1687-2770-2014-4
Cite this article as: Boichuk and Pokutnyi: **Solutions of the Schrödinger equation in a Hilbert space.** *Boundary Value Problems* 2014, 2014:4