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# Layer solutions for a class of semilinear elliptic equations involving fractional Laplacians

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**Abstract**

This paper is concerned with the nonlinear equation involving the fractional Laplacian: $(-\triangle)^s v(x) = b(x)f(v(x))$, $x \in \mathbb{R}$, where $s \in (0,1)$, $b : \mathbb{R} \to \mathbb{R}$ is a periodic, positive, even function and $-f$ is the derivative of a double-well potential $G$. That is, $G \in C^{2,\gamma}$ $(0 < \gamma < 1)$, $G(1) = G(-1) < G(\tau)$ $\forall \tau \in (-1,1)$, $G'(-1) = G'(1) = 0$. We show the existence of layer solutions of the equation for $s \geq \frac{1}{2}$ and for some odd nonlinearities by variational methods, which is a bounded solution having the limits $\pm 1$ at $\pm\infty$. Asymptotic estimates for layer solutions as $|x| \to +\infty$ and the asymptotic behavior of them as $s \uparrow 1$ are also obtained.

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**Keywords:** fractional Laplacian; layer solutions; existence; local minimizers

## 1 Introduction

In this paper we study the fractional Laplacian

$$(-\triangle)^s v(x) = b(x)f\bigl(v(x)\bigr), \quad x \in \mathbb{R}, \tag{1.1}$$

where $s \in (0,1)$, and $(-\triangle)^s$ is the fractional Laplacian defined by

$$(-\triangle)^s v = C_s \text{ P.V.} \int_{\mathbb{R}} \frac{v(x) - v(y)}{|x-y|^{1+2s}}\, dy.$$

Here P.V. stands for the Cauchy principle value and $C_s$ is a positive constant multiplier depending only on $s$.

The fractional Laplacian is a nonlocal operator which can be localized as

$$\begin{cases} -\operatorname{div}(y^a \nabla u) = 0 & \text{in } \mathbb{R}^2_+, \\ -\lim_{y \downarrow 0^+} y^a \frac{\partial u}{\partial y} = \frac{1}{d_s} b(x) f(u) & \text{on } \partial\mathbb{R}^2_+, \end{cases} \tag{1.2}$$

where $a = 1 - 2s \in (-1,1)$, $d_s = 2^{2s-1}\frac{\Gamma(s)}{\Gamma(1-s)}$ and $u(x,0) = v(x)$. Moreover $u(\cdot,\cdot)$ can be expressed by a Poisson kernel,

$$u(x,y) = P_s(\cdot,y) * v = p_s \int_{\mathbb{R}} \frac{y^{2s}}{(|z|^2 + y^2)^{\frac{1+2s}{2}}} v(x-z)\, dz \quad \text{for every } y > 0,$$

which is called the $s$-extension of $v$. $p_s$ is a positive constant depending only on $s$. For more details as regards the fractional Laplacian, readers can refer to [1–7] and the references therein.

In view of the celebrated De Giorgi conjecture (see [8–10]), Cabré and Sire [2, 3] considered layer solutions of the nonlocal equation

$$(-\triangle)^s v = f(v) \quad \text{in } \mathbb{R}. \tag{1.3}$$

The necessary and sufficient conditions for the existence of one-dimensional layer solutions were given as

$$G(1) = G(-1) < G(s) \quad \forall s \in (-1,1), \qquad G'(1) = G'(-1) = 0,$$

where $G' = -f$. All these were obtained by a Hamiltonian equality and a Modica-type estimate for layer solutions. By the sliding method, the layer solution of (1.3) was proved to be the unique local minimizer which increases in $x$ with values varying from –1 to 1. The regularity, Hopf principle, maximum principle as well as a Harnack inequality for (1.3) or for its extension equation (1.2) (in this case $b = 1$) were given. Some of them will be used in our paper.

If $b$ is not a constant and is periodic, the perturbed equation (1.1) becomes complicated. The aim of this paper is to study the layer solution of (1.1) with periodic perturbed nonlinearity.

**Definition 1.1** A function $v \in (L^\infty \cap C^\beta)(\mathbb{R})$ $(0 < \beta < 1)$ is said to be a layer solution of (1.1), if $v$ solves (1.1),

$$(-\triangle)^s v(x) = b(x) f\bigl(v(x)\bigr), \quad x \in \mathbb{R}$$

and

$$\lim_{x \to \pm\infty} v(x) = \pm 1.$$

**Definition 1.2** A function $u \in L^\infty(\mathbb{R}^2_+) \cap C^\beta(\overline{\mathbb{R}^2_+})$ is said to be a layer solution of (1.2), if $u$ solves (1.2),

$$\begin{cases} -\operatorname{div}(y^a \nabla u) = 0 & \text{in } \mathbb{R}^2_+, \\ -\lim_{y \downarrow 0^+} y^a \frac{\partial u}{\partial y} = \frac{1}{d_s} b(x) f(u) & \text{on } \partial\mathbb{R}^2_+ \end{cases}$$

and

$$\lim_{x \to \pm\infty} u(x,0) = \pm 1.$$

Namely, $u(x,0)$ is the corresponding layer solution of (1.1).

Different from the unperturbed case (1.3), the inhomogeneous term $b(x)f(u)$ depends explicitly on $x$ in (1.1) and (1.2); the sliding method cannot be used and layer solutions of them have no monotonicity in the direction of $x$. The method for obtaining layer solutions in [2] and [3] cannot be used in our case directly; some difficulties need to be solved.

In the paper, we consider the extension problem (1.2). Obviously, (1.2) has a variational structure.

Denote

$$\Omega \subset \mathbb{R}^2_+, \text{a bounded Lipschitz domain},$$
$$B_R(x,y) \subset \mathbb{R}^2, \text{a ball centered at } (x,y) \in \mathbb{R}^2 \text{ with radius } R,$$
$$B^+_\epsilon(x,0) = B_\epsilon(x,0) \cap \mathbb{R}^2_+,$$
$$\partial^0\Omega = \{(x,0) \in \partial\Omega \cap \partial\mathbb{R}^2_+ | \exists \epsilon > 0, B^+_\epsilon(x,0) \subset \Omega\},$$
$$\partial^+\Omega = \overline{\partial\Omega \cap \mathbb{R}^2_+}.$$

For $u \in H^1(y^a, \Omega)$, the norm is

$$\|u\|_{H^1(y^a,\Omega)} = \left(\int_\Omega y^a |\nabla u|^2 \,dx\,dy\right)^{\frac{1}{2}} + \left(\int_\Omega y^a |u|^2 \,dx\,dy\right)^{\frac{1}{2}}.$$

The energy functional of $u$ on $\Omega$ is given by

$$\mathcal{E}(u,\Omega) = d_s \int_\Omega \frac{y^a}{2} |\nabla u|^2 \,dx\,dy + \int_{\partial^0\Omega} b(x) G\big(u(x,0)\big)\,dx. \tag{1.4}$$

We state our main results in the following.

We show, via a Liouville result, the existence of layer solutions of (1.1) for $s \geq \frac{1}{2}$ and for some odd nonlinearities.

**Theorem 1.1** *Let $s \geq \frac{1}{2}$. Assume that $b, f \in C^{1,\gamma}(\mathbb{R})$ $(0 < \gamma < 1)$:*
(1) *$b : \mathbb{R} \to \mathbb{R}$ is 1-periodic, even, not constant and positive; denote $\overline{b} = \max_R b$ and $\underline{b} = \min_R b$;*
(2) *$f(-\tau) = -f(\tau)$ for any $\tau \in [-1,1]$, $f(-1) = f(1) = f(0) = 0$, $f > 0$ in $(0,1)$ and $f < 0$ in $(-1,0)$.*

*Obviously, if $G' = -f$,*

$$G(-1) = G(1) < G(\tau) \quad \text{for } \tau \in (-1,1), \qquad G'(1) = G'(-1) = 0.$$

*There exists a layer solution $v \in C^{2,\beta}(R)$ (for some $0 < \beta < 1$) of (1.1):*

$$\begin{cases} (-\partial_{xx})^s v(x) = b(x) f(v(x)) & \text{in } \mathbb{R}, \\ v \to \pm 1 & \text{as } x \to \pm\infty. \end{cases} \tag{1.5}$$

*In addition, $v$ is odd.*

Furthermore we obtain asymptotic estimates of the layer solutions of (1.1) by comparing with a layer solution of the unperturbed equation (1.3).

**Theorem 1.2** *Let $b \in C^{1,\gamma} \cap L^\infty$ is positive. Let $f \in C^{1,\gamma}(\mathbb{R})$ $(\gamma > \max(0, 1-2s))$ satisfy*
(i) *$G(-1) = G(1) < G(\tau)$ for $\tau \in (-1,1)$, $G'(1) = G'(-1) = 0$;*
(ii) *$G''(1) > 0$, $G''(-1) > 0$.*

*If $v$ is a layer solution of* (1.1), *then the following asymptotic estimates hold*:

$$cx^{-2s} \le |1-v| \le Cx^{-2s} \quad \text{for } x>1, \tag{1.6}$$

$$c|x|^{-2s} \le |1+v| \le C|x|^{-2s} \quad \text{for } x<-1 \tag{1.7}$$

*for some constants* $0<c<C$.

Finally we investigate the asymptotic behavior of $v^s$ as $s \uparrow 1$ and obtain a local elliptic equation, which is stated as follows.

**Theorem 1.3** *Let* $s \in [\frac{1}{2}, 1)$. *Let* $\{v^{s_k}\}$ *be a sequence of layer solutions of* (1.1) *in Theorem* 1.1. *Then, there exists a subsequence denoted again by* $\{v^{s_k}\}$ *converging locally uniformly to a function* $v^1 \in C^2(R)$ *as* $s_k \uparrow 1$, *which is also a layer solution of the local elliptic equation*

$$\begin{cases} -v^1_{xx}(x) = b(x)f(v^1(x)) & \text{in } \mathbb{R}, \\ \lim_{x\to\pm\infty} v^1(x) = \pm 1. \end{cases} \tag{1.8}$$

*In addition*,

$$\frac{1}{2}\left(v^1_x\right)^2 = b(x)\left\{G\left(v^1(x)\right) - G(1)\right\} + \int_x^{+\infty} b'(t)\left\{G\left(v^1(t)\right) - G(1)\right\}dt. \tag{1.9}$$

For convenience of the presentation we will use $C$ for a general positive constant; such a $C$ is usually different in different contexts.

## 2 Some preliminaries and properties

In this paper, we mainly study the extension equation (1.2). To make our problems clear, we present several properties of layer solutions.

**Lemma 2.1** *Let $u$ be a bounded solution of* (1.2),

$$\begin{cases} -\operatorname{div}(y^a \nabla u) = 0 & \text{in } \mathbb{R}^2_+, \\ -\lim_{y\downarrow 0^+} y^a \frac{\partial u}{\partial y} = \frac{1}{d_s} b(x) f(u) & \text{on } \partial\mathbb{R}^2_+ \end{cases}$$

*and*

$$\lim_{x\to\pm\infty} u(x,0) = L^{\pm} \tag{2.1}$$

*with two constants $L^{\pm}$. Then*,

(1)

$$f\left(L^+\right) = f\left(L^-\right) = 0; \tag{2.2}$$

(2)

$$\lim_{x\to\pm\infty} u(x,y) = L^{\pm} \tag{2.3}$$

*for every* $y \ge 0$;

(3)

$$\|u - L^{\pm}\|_{L^\infty(B_R^+(x,0))} \to 0 \quad \textit{as } x \to \pm\infty; \tag{2.4}$$

(4)

$$\|\nabla_x u\|_{L^\infty(B_R^+(x,0))} + \left\| y^a \frac{\partial u}{\partial y} \right\|_{L^\infty(B_R^+(x,0))} \to 0 \quad \textit{as } x \to \pm\infty. \tag{2.5}$$

*Proof* Our proof uses the invariance of the problem under periodic translations in $x$ and a compactness argument.

Denote $u^n(x,y) = u(x+n,y)$ for $n \in \mathbb{Z}$. Since $b$ is 1-periodic, $u^n$ still satisfies the equations

$$\begin{cases} -\operatorname{div}(y^a \nabla u^n) = 0 & \text{in } \mathbb{R}^2_+, \\ -\lim_{y\downarrow 0^+} y^a \frac{\partial u^n}{\partial y} = \frac{1}{d_s} b(x) f(u^n(x,0)) & \text{on } \partial\mathbb{R}^2_+. \end{cases} \tag{2.6}$$

By regularity results in [2] and [5], we see that up to a subsequence,

$$\begin{aligned} &u^n \to u^{\pm\infty} \quad \text{in } C^0_{\text{loc}}\big(\overline{\mathbb{R}^2_+}\big), \\ &\nabla_x u^n \to \nabla_x u^{\pm\infty} \quad \text{in } C^0_{\text{loc}}\big(\overline{\mathbb{R}^2_+}\big), \\ &y^a \frac{\partial u^n}{\partial y} \to y^a \frac{\partial u^\infty}{\partial y} \quad \text{in } C^0_{\text{loc}}\big(\overline{\mathbb{R}^2_+}\big) \end{aligned}$$

as $n \to \pm\infty$. Then $u^{\pm\infty}$ solves the equations

$$\begin{cases} -\operatorname{div}(y^a \nabla u^{\pm\infty}) = 0 & \text{in } \mathbb{R}^2_+, \\ -\lim_{y\downarrow 0^+} y^a \frac{\partial u^{\pm\infty}}{\partial y} = \frac{1}{d_s} b(x) f(u^{\pm\infty}) & \text{on } \partial\mathbb{R}^2_+, \end{cases} \tag{2.7}$$

and it follows that $u^{\pm\infty}(x,0) \equiv L^{\pm}$ for every $x \in \mathbb{R}$.

Consider the Dirichlet problem

$$\begin{cases} -\operatorname{div}(y^a \nabla u^{\pm\infty}) = 0 & \text{in } \mathbb{R}^2_+, \\ u^{\pm\infty}(x,0) \equiv L^{\pm} & \text{on } \partial\mathbb{R}^2_+. \end{cases} \tag{2.8}$$

$u^{\pm\infty} \equiv L^{\pm}$ is the unique solution of (2.8) by Corollary 3.5 in [2]. As a consequence, (2.2) and (2.5) are obvious. □

The following lemma is a necessary condition for a local minimizer of the energy functional $\mathcal{E}$.

**Lemma 2.2** *Let $u$ be a local minimizer of the energy functional $\mathcal{E}$ under perturbations in* $[-1,1]$. *That is, for any bounded Lipschitz domain $\Omega \subset \mathbb{R}^2_+$ and for any $\xi \in H^1(y^a,\Omega)$ having compact support in $\Omega \cup \partial^0\Omega$ such that $u + \xi \in [-1,1]$,*

$$\mathcal{E}(u,\Omega) \le \mathcal{E}(u+\xi,\Omega).$$

*Let*

$$\lim_{x\to\pm\infty} u(x,0) = \pm 1. \tag{2.9}$$

*Then*

$$G(1) = G(-1) \le G(\tau) \quad \textit{for all } \tau \in (-1,1). \tag{2.10}$$

*Proof* To show (2.10), it is sufficient to prove that $G(1) \le G(\tau)$ and $G(-1) \le G(\tau)$ for all $\tau \in [-1,1]$. Suppose $G(\tau_0) < G(1)$ for some point $\tau_0 \in [-1,1]$ by contradiction. For simplicity, assume that $G(\tau_0) = 0$ by adding a constant.

By (2.9),

$$\liminf_{l\to+\infty} \mathcal{E}\big(u, B_R^+(l,0)\big) \ge \liminf_{l\to+\infty} \int_{\partial^0 B_R^+(l,0)} b(x)G\big(u(x,0)\big) \ge 2\underline{b}\varepsilon R \tag{2.11}$$

for some $\varepsilon > 0$.

Let $\xi_R$ be a cut-off function with values in $[0,1]$,

$$\xi_R = \begin{cases} 1 & \text{in } B_{(1-\eta)R}, \\ 0 & \text{in } \mathbb{R}^{n+1} \setminus B_R, \end{cases}$$

where $\eta \in (0,1)$ will be specified later, and $|\nabla \xi_R| \le \frac{1}{\eta R}$.

Define $\xi_{R,l}(x,y) = \xi_R(x-l,y)$. Let $w = \tau_0\xi_{R,l} + (1-\xi_{R,l})u$, then $w = u$ on $\partial^+ B_R^+(l,0)$ and $w \equiv \tau_0$ in $B_{(1-\eta)R}^+(l,0)$. We have

$$\begin{aligned}
\limsup_{l\to+\infty} \mathcal{E}_{B_R^+(l,0)}(w) = {} & \limsup_{l\to+\infty} \left\{ d_s \int_{B_R^+(l,0)} \frac{y^a}{2} \big|(1-\xi_{R,l})\nabla u + (\tau_0 - u)\nabla \xi_{R,l}\big|^2 \right. \\
& \left. + \int_{\partial^0 B_R^+(l,0)} b(x_1)G(w) \right\} \\
\le {} & 2d_s \int_{B_R^+} y^a |\nabla \xi_{R,l}|^2 + 2\overline{b} \max_{[-1,1]} G \cdot \eta R \\
\le {} & \frac{Cd_s R^a}{\eta^2} + 2\overline{b} \max_{[-1,1]} G \cdot \eta R.
\end{aligned} \tag{2.12}$$

We use (2.5) in the first inequality above.

Having chosen $\eta = \frac{\underline{b}\varepsilon}{2\overline{b}\max_{[-1,1]} G}$, by (2.11) and (2.12),

$$\limsup_{l\to+\infty} \mathcal{E}\big(w, B_R^+(l,0)\big) < \liminf_{l\to+\infty} \mathcal{E}\big(u, B_R^+(l,0)\big)$$

for large $R > 1$. This contradiction leads to $G(1) \le G(\tau)$ for all $\tau \in [-1,1]$. By the same discussion, $G(-1) \le G(\tau)$ for all $\tau \in [-1,1]$. Thus we complete the proof. □

As in [2], we construct a Hamiltonian equality which will be used in the proof of Theorem 1.3. For this purpose a lemma is in order, for whose proof see Lemma 5.1 in [2].

**Lemma 2.3** *Let $u \in L^\infty(\mathbb{R}^2_+)$ be a solution of* (1.2). *Then for every $x \in \mathbb{R}$, $\int_0^\infty y^a |\nabla u|^2\,dy < \infty$. In addition, the integral can be differentiated with respect to $x \in \mathbb{R}$ under the integral sign. We have*

$$\lim_{M \to +\infty} \int_M^\infty y^a |\nabla u|^2\,dy = 0 \tag{2.13}$$

*uniformly in $x \in \mathbb{R}$. If $u$ is a layer solution of* (1.2),

$$\lim_{|x| \to +\infty} \int_0^\infty y^a |\nabla u|^2\,dy = 0. \tag{2.14}$$

**Proposition 2.1** (Hamiltonian equality) *Let $u$ be a layer solution of* (1.2) *for $a \in (-1, \frac{1}{2})$, i.e.,*

$$\begin{cases} -\operatorname{div}(y^a \nabla u) = 0 & \text{in } \mathbb{R}^2_+, \\ -\lim_{y \downarrow 0^+} y^a \frac{\partial u}{\partial y} = \frac{1}{d_s} b(x) f(u) & \text{on } \partial\mathbb{R}^2_+, \\ u(x,0) \to \pm 1 & \text{as } x \to \infty. \end{cases}$$

*For every $x \in \mathbb{R}$, the Hamiltonian equality holds*:

$$\begin{aligned} d_s \int_0^\infty \frac{y^a}{2} \{u_x^2 - u_y^2\}\,dy = b(x)\{G(u(x,0)) - G(1)\} \\ + \int_x^\infty b'(t)\{G(u(t,0)) - G(1)\}\,dt. \end{aligned} \tag{2.15}$$

*As a consequence,*

$$\int_{-\infty}^{+\infty} b'(x)\{G(u(x,0)) - G(1)\}\,dx = 0. \tag{2.16}$$

*Proof* We note that the integral in (2.16) is well defined since $s = \frac{1-a}{2} \in (\frac{1}{4}, 1)$ and

$$G(u(x,0)) - G(1) = \frac{G''(t)}{2}\bigl(u(x,0) - 1\bigr)^2 = O\bigl(|x|^{-4s}\bigr) \quad \text{as } |x| \to \infty,$$

where $t$ is some point between $u(x,0)$ and 1.

By Lemma 2.3, the left integral in (2.15) can be differentiated with respect to $x$,

$$\begin{aligned} \frac{d}{dx} d_s \int_0^\infty \frac{y^a}{2}\{u_x^2 - u_y^2\}\,dy &= d_s \int_0^\infty y^a \{u_x u_{xx} - u_y u_{yx}\}\,dy \\ &= d_s \int_0^\infty y^a \left\{u_{xx} + u_{yy} + \frac{a}{y} u_y\right\} u_x\,dy + d_s \lim_{y \to 0^+} y^a u_y u_x \\ &= -b(x) f\bigl(u(x,0)\bigr) u_x(x,0). \end{aligned}$$

In the second equality above we use the fact that $\lim_{y \to \infty} y^a u_y u_x = 0$ (see [2]). We have

$$\begin{aligned} &\frac{d}{dx}\left\{b(x)\bigl(G(u(x,0)) - G(1)\bigr) + \int_x^{+\infty} b'(t)\bigl(G(u(t,0)) - G(1)\bigr)\,dx\right\} \\ &\quad = -b(x) f\bigl(u(x,0)\bigr) u_x(x,0). \end{aligned}$$

Thus,

$$d_s \int_0^\infty \frac{y^a}{2}\{u_x^2 - u_y^2\}\,dy \equiv b(x)\{G(u(x,0)) - G(1)\} + \int_x^{+\infty} b'(t)\{G(u(t,0)) - G(1)\}\,dt + C. \tag{2.17}$$

Let $x \to +\infty$, the left of (2.17) converging to zero by (2.14); thus $C = 0$ and (2.15) is proved. Letting $x \to -\infty$, (2.16) is also obtained. □

To study asymptotic estimates of layer solutions of (1.1), we recall an explicit layer solution of the unperturbed problem (1.3).

**Lemma 2.4** ([3], Theorem 3.1) *Let $s \in (0,1)$. For every $t > 0$, the $C^\infty$ function*

$$v_s^t(x) = \operatorname{sign}(x)\frac{2}{\pi}\int_0^\infty \frac{\sin(z)}{z} e^{-t(\frac{z}{|x|})^{2s}}\,dz \tag{2.18}$$

*is the layer solution to the fractional equation*

$$(-\partial_{xx})^s v_s^t = f_s^t(v_s^t) \quad \text{in } \mathbb{R}, \tag{2.19}$$

*for a nonlinearity $f_s^t \in C^1([-1,1])$ which is odd and twice differentiable in $[-1,1]$ and which satisfies*

$$f_s^t(0) = f_s^t(1) = 0, \quad f_s^t > 0 \text{ in } (0,1), \qquad (f_s^t)'(\pm 1) = -\frac{1}{t}.$$

*In addition, the following limits exist:*

$$\lim_{|x|\to\infty} |x|^{1+2s}(\partial_x v_s^t)(x) = t\frac{4s}{\pi}\sin(\pi s)\Gamma(2s) > 0 \tag{2.20}$$

*and, as a consequence,*

$$\lim_{x\to\pm\infty} |x|^{2s}\left|(v_s^t)(x) \mp 1\right| = t\frac{2}{\pi}\sin(\pi s)\Gamma(2s) > 0. \tag{2.21}$$

## 3 Existence and asymptotic estimates

To prove the existence of layer solutions, we introduce a Liouville result where $a \le 0$ is required. This is the reason why we restrict ourselves to the case $s \ge \frac{1}{2}$ in Theorem 1.1.

**Proposition 3.1** *Let $a \le 0$. Suppose $u$ is a bounded nonnegative function which satisfies weakly the problem*

$$\begin{cases} -\operatorname{div}(y^a \nabla u) \le 0 & \text{in } \mathbb{R}_+^2, \\ -\lim_{y\downarrow 0^+} y^a \frac{\partial u}{\partial y} \le 0 & \text{on } \partial\mathbb{R}_+^2. \end{cases} \tag{3.1}$$

*Then $u \equiv C$ a.e. in $\mathbb{R}_+^2$.*

*Proof* Since $a \le 0$, $R^a \le 1$ for $R > 1$. Let $\xi$ be a smooth function with values in $[0,1]$, $\xi = 1$ in $B_R$ and $\xi = 0$ outside of $B_{2R}$, $|\nabla\xi| \le CR^{-1}$. Multiplying (3.1) with $u\xi^2$ and integrating by parts, we have that

$$\begin{aligned}\int_{B_R^+} y^a|\nabla u|^2 \le \int_{B_{2R}^+} y^a|\nabla u|^2\xi^2 \le 2\int_{\mathbb{R}^2_+} y^a\xi u|\nabla u||\nabla\xi| \\ \le 2\left\{\int_{B_{2R}^+\setminus B_R^+} y^a|\nabla u|^2\xi^2\right\}^{\frac{1}{2}}\left\{\int_{B_{2R}^+\setminus B_R^+} y^a|\nabla\xi|^2u^2\right\}^{\frac{1}{2}} \\ \le C\left\{\int_{B_{2R}^+\setminus B_R^+} y^a|\nabla u|^2\xi^2\right\}^{\frac{1}{2}}\left(RR^{1+a}R^{-2}\right)^{\frac{1}{2}} \\ \le C\left\{\int_{B_{2R}^+\setminus B_R^+} y^a|\nabla u|^2\xi^2\right\}^{\frac{1}{2}}.\end{aligned}$$

Thus $\int_{\mathbb{R}^2_+} y^a|\nabla u|^2 \le C$ for some constant $C$ independent of $R$. Let $R \to \infty$, $\int_{B_{2R}^+\setminus B_R^+} y^a|\nabla u|^2\xi^2 \to 0$. We deduce that $\int_{\mathbb{R}^2_+} y^a|\nabla u|^2 = 0$ and $u \equiv C$ a.e. in $\mathbb{R}^2_+$. □

Next we prove an existence result about the local minimizer of $\mathcal{E}$.

**Lemma 3.1** *Let $\Omega \subset \mathbb{R}^2_+$ be a bounded Lipschitz domain. Let $w_0 \in C^0(\overline{\Omega}) \cap H^1(y^a,\Omega)$ be a given function with $|w_0| \le 1$; $b$ is a bounded positive function.*

*Suppose that*

$$f(1) \le 0 \le f(-1),$$

*the energy functional $\mathcal{E}(u,\Omega)$ admits a minimizer $u \in \mathcal{C}_{w_0,a} = \{w \in H^1(y^a,\Omega), -1 \le w \le 1$ a.e. in $\Omega, w = w_0$ on $\partial^+\Omega$ in the weak sense$\}$, which solves weakly*

$$\begin{cases} -\operatorname{div}(y^a\nabla u) = 0 & \text{in } \Omega, \\ -\lim_{y\downarrow 0^+} y^a\frac{\partial u}{\partial y} = \frac{1}{d_s}b(x)f(u(x,0)) & \text{on } \partial^0\Omega, \\ u = w_0 & \text{on } \partial^+\Omega. \end{cases} \tag{3.2}$$

*Moreover, $u$ is a stable solution of* (3.2), *i.e.*,

$$d_s\int_\Omega y^a|\nabla\xi|^2\,dx\,dy - \int_{\partial^0\Omega} b(x)f'(u)\xi^2\,dx \ge 0, \tag{3.3}$$

*for every $\xi \in H^1(\Omega,y^a)$ such that $\xi \equiv 0$ on $\partial^+\Omega$ in the weak sense.*

*Proof* Consider the set $H_{w_0,a}(\Omega) = \{w \in H^1(y^a,\Omega), w \equiv w_0$ on $\partial^+\Omega$ in the weak sense$\} \supset \mathcal{C}_{w_0,a}$, $H_{w_0,a}(\Omega) \neq \emptyset$ since $w_0 \in H_{w_0,a}(\Omega)$. Denote

$$\widetilde{f} = \begin{cases} f(1) & \text{if } t \ge 1, \\ f & \text{if } -1 < t < 1, \\ f(-1) & \text{if } t \le -1, \end{cases}$$

and $\widetilde{G} = -\int_0^u \widetilde{f}$. Up to an additive constant, $\widetilde{G} = G$ in $[-1,1]$.

Consider the energy functional

$$\widetilde{\mathcal{E}}(u,\Omega) = d_s \int_\Omega \frac{y^a}{2}|\nabla u|^2\,dx\,dy + \int_{\partial^0\Omega} b(x)\widetilde{G}\big(u(x,0)\big)\,dx. \tag{3.4}$$

If $\widetilde{\mathcal{E}}$ has an absolute minimizer $u$ in $\mathcal{C}_{w_0,a}(\Omega)$, the statement of Lemma 3.1 is proved.

For every function $w \in H_{w_0,a}(\Omega)$, $w - w_0 \in H^1(y^a,\Omega)$ and vanishes on $\partial^+\Omega$ in the weak sense. We can extend $w - w_0$ in $\mathbb{R}^2_+$ by zeroes outside of $\Omega$ and $w - w_0 \in H^1(y^a,\mathbb{R}^2_+)$. By the trace theorem and the Sobolev imbedding theorem (see [7, 11, 12]),

$$H^1\big(y^a,\mathbb{R}^2_+\big) \hookrightarrow L^p(\mathbb{R})$$

for $p = \frac{2}{1-2s}$ if $s < \frac{1}{2}$ or for any $1 \le p < \infty$ if $s \ge \frac{1}{2}$. Moreover, $H^1(y^a,\mathbb{R}^2_+) \hookrightarrow\hookrightarrow L^2(\partial^0\Omega)$.

Since $\widetilde{G}$ has linear growth at infinity, $\widetilde{\mathcal{E}}$ is well defined, bounded below and coercive in $H_{w_0,a}$. There exists an absolute minimizer $u \in H_{w_0,a}$. By the first order variation, we have

$$\begin{cases} -\operatorname{div}(y^a\nabla u) = 0 & \text{in } \Omega, \\ -\lim_{y\downarrow 0^+} y^a\frac{\partial u}{\partial y} = \frac{1}{d_s}b(x_1)\widetilde{f}(u(x,0)) & \text{on } \partial^0\Omega. \end{cases} \tag{3.5}$$

Multiply $(u-1)^+$ with (3.5) and integrate in $\Omega$,

$$d_s\int_\Omega y^a\big|\nabla(u-1)^+\big|^2\,dx\,dy - \int_{\partial^0\Omega} b(x)f(1)(u-1)^+\,dx = 0.$$

Since $f(1) \le 0$, $\int_\Omega y^a|\nabla(u-1)^+|^2\,dx\,dy \le 0$. Thus $(u-1)^+ \equiv 0$ a.e. in $\Omega$, *i.e.*, $u \le 1$ a.e. in $\Omega$. Similarly we also get $u \ge -1$ a.e. in $\Omega$. Hence $u \in \mathcal{C}_{w_0,a}(\Omega)$. (3.2) follows from (3.5), and (3.3) comes from the second order variation of $\mathcal{E}$. □

**Remark 3.1** Suppose that $b$ is an even function, $f$ and $w_0$ are odd with respect to $x$, with a slight modification we can also show that there is an odd minimizer in the admissible set $\{w \in \mathcal{C}_{w_0,a} | w(-x,y) = -w(x,y) \text{ for every } y \ge 0\}$.

Now we start to show the existence of layer solutions of (1.2).

**Theorem 3.1** *Let $s \ge \frac{1}{2}$. Let $b \in (C^{1,\gamma} \cap L^\infty)(\mathbb{R})$ and $f \in C^{1,\gamma}(\mathbb{R})$ $(0 < \gamma < 1)$:*

(a) *$b : \mathbb{R} \to \mathbb{R}$ is an even positive function, $b(x+1) = b(x)$ $\forall x \in \mathbb{R}$,*

(b) *$f(-\tau) = -f(\tau)$ for any $\tau \in [-1,1]$, $f(-1) = f(1) = f(0) = 0$, $f > 0$ in $(0,1)$ and $f < 0$ in $(-1,0)$.*

*Then there exists a layer solution $u$ of (1.2) in $\mathbb{R}^2_+$:*

$$\begin{cases} -\operatorname{div}(y^a\nabla u) = 0 & \text{in } \mathbb{R}^2_+, \\ -\lim_{y\downarrow 0^+} y^a\frac{\partial u}{\partial y} = \frac{1}{d_s}b(x)f(u(x,0)) & \text{on } \partial\mathbb{R}^2_+, \end{cases} \tag{3.6}$$

*which is odd with respect to $x$, i.e., $u(-x,y) = -u(x,y)$, and, for every $y \ge 0$,*

$$\lim_{x\to\pm\infty} u(x,y) = \pm 1. \tag{3.7}$$

*Furthermore, $u$ is a local minimizer of the energy functional $\mathcal{E}$ under odd perturbations in $[-1,1]$, and it is stable in the sense that*

$$d_s \int_{\mathbb{R}^2_+} y^a |\nabla \xi|^2 \,dx\,dy - \int_{\mathbb{R}} b(x) f'\bigl(u(x,0)\bigr)\xi^2 \,dx \geq 0 \tag{3.8}$$

*for every function $\xi \in C^1(\overline{\mathbb{R}^2_+})$ with compact support in $\overline{\mathbb{R}^2_+}$, $\xi(-x,y) = -\xi(x,y)$ and $u + \xi \in [-1,1]$.*

*Proof* The proof is divided into three parts. For simplicity, we make $G(1) = G(-1) = 0$ by adding a constant.

Step 1. We show that there exists a solution with values in $[-1,1]$ of (3.6) which is odd with respect to the variable $x$ for every $y \geq 0$.

Let $Q_R = [-R,R] \times [0,R]$ and $w_0 = \frac{\arctan x}{\arctan R}$. Define the admissible set

$$\mathcal{C}_{w_0,a,o} = \bigl\{w \in C_{w_0,a}(Q_R), \forall y \geq 0, w(-x,y) = -w(x,y)\bigr\}.$$

By Remark 3.1, there is a minimizer $u_R$ in $\mathcal{C}_{w_0,a,o}$,

$$\begin{cases} -\operatorname{div}(y^a \nabla u_R) = 0 & \text{in } Q_R, \\ -\lim_{y\downarrow 0^+} y^a \frac{\partial u_R}{\partial y} = \frac{1}{d_s} b(x) f(u_R(x,0)) & \text{on } \partial^0 Q_R, \\ u_R = w_0 & \text{on } \partial^+ Q_R. \end{cases} \tag{3.9}$$

Define

$$u_R := \begin{cases} u_R(-x,y) & \text{if } u_R(x,y) < 0 \text{ and } x > 0, \\ u_R(x,y) & \text{if } u_R(x,y) \geq 0 \text{ and } x > 0 \end{cases}$$

and $u_R(x,y) := -u_R(-x,y)$ for $x \leq 0$. Thus $u_R \geq 0$ for $x > 0$ and $y \geq 0$. Obviously $u_R$ is still a minimizer of $\mathcal{E}(\cdot, Q_R)$.

By the regularity results in [2], $u_R, \nabla_x u_R, y^a \frac{\partial u_R}{\partial y} \in C^\beta(Q_R)$ for some $0 < \beta < 1$ and the continuous module is uniform bounded. Up to a subsequence, $u_R \to u$, $(u_R)_x \to u_x$ and $y^a \frac{\partial u_R}{\partial y} \to y^a \frac{\partial u}{\partial y}$ in $C^0(\overline{B^+_s})$ as $R \to \infty$ for all $R > s + 2$. By the canonical diagonal procedure, $u$ solves

$$\begin{cases} -\operatorname{div}(y^a \nabla u) = 0 & \text{in } \mathbb{R}^2_+, \\ -\lim_{y\downarrow 0^+} y^a \frac{\partial u}{\partial y} = \frac{1}{d_s} b(x) f(u(x,0)) & \text{on } \partial \mathbb{R}^2_+, \\ u(-x,y) = -u(x,y) & \text{in } \overline{\mathbb{R}^2_+}, \end{cases} \tag{3.10}$$

and by the Hopf maximum principle $-1 < u < 1$.

Step 2. We show that there exists at least a subsequence $x_n \to \infty$ such that $u(x_n, 0) \to 1$.

First we claim that $u$ is a local minimizer under odd perturbations in $[-1,1]$. That is,

$$\mathcal{E}(u, \Omega) \leq \mathcal{E}(w, \Omega)$$

for any $\Omega \subset \mathbb{R}^2_+$ and for any odd function $w \in H^1(y^a, \Omega)$ with $|w| \leq 1$ and $w = u$ on $\partial^+\Omega$ in the weak sense.

Let $\xi \in C_c^1(B_s^+ \cup \partial^0 B_s^+)$ is odd with respect to $x$ for every $y \geq 0$ and $u_R + \xi \in [-1,1]$. Since $-1 < u_R < 1$, $u_R + (1-\epsilon)\xi \in (-1,1)$ for $\epsilon \in (0,1)$. We have

$$\mathcal{E}(u_R, B_s^+) \leq \mathcal{E}(u_R + (1-\epsilon)\xi, B_s^+) \quad \text{for } R > s+2.$$

Let $R \to \infty$, and

$$\mathcal{E}(u, B_s^+) \leq \mathcal{E}(u + (1-\epsilon)\xi, B_s^+)$$

for every $s > 0$ and $u + (1-\epsilon)\xi \in [-1,1]$. Our claim is proved.

If $w(-x,y) = -w(x,y)$,

$$\mathcal{E}(w, B_s^+) = 2\mathcal{E}(w, B_s^{++}) = 2\left\{d_s \int_{B_s^{++}} \frac{y^a}{2}|\nabla w|^2\,dx\,dy + \int_{\partial^0 B_s^{++}} b(x)G(w)\,dx\right\},$$

where $B_s^{++} = \{(x,y) \in B_s^+, x > 0, y \geq 0\}$. Therefore $u$ is also a local minimizer of $\mathcal{E}$ in $\mathbb{R}_{++}^{n+1} = \{(x,y) \in \mathbb{R}_+^2, x > 0, y \geq 0\}$ with perturbations in $[-1,1]$, *i.e.*,

$$\mathcal{E}(u,\Omega) \leq \mathcal{E}(w,\Omega)$$

for any $\Omega \subset \mathbb{R}_{++}^2$ and for any $w \in H^1(y^a,\Omega)$ with $|w| \leq 1$ and $w = u$ on $\partial^+\Omega$ in weak sense.

Suppose $u(x_n,0) \nrightarrow 1$ for any sequence $x_n \to \infty$ by contradiction. $|u(x,0)| < 1-\epsilon$ for some $0 < \epsilon < 1$ and $x \in \mathbb{R}$. Hence $0 \leq u(x,y) < 1-\epsilon$ for all $x > 0$ and $y \geq 0$ by the fact that $u(\cdot,y) = P_s(\cdot,y) * u(\cdot,0)$.

Let $R > 1$. Let $\varphi_R$ be a cut-off function with values 1 in $B_{(1-\eta)R}^+$ and zeroes outside of $B_R^+$, $|\nabla\varphi_R| \leq \frac{C}{\eta R}$ for some $0 < \eta < 1$ determined later.

Denote $\varphi_R = \varphi_R(|(x-l,y)|)$. Let $w = 1\cdot\varphi_R + (1-\varphi_R)u \in H^1(y^a, B_R^+(l,0))$, $w \equiv u$ on $\partial^+ B_R(l,0)$. For $l > R$,

$$\begin{aligned}
\mathcal{E}(w, B_R^+(l,0)) &= d_s \int_{B_R^+(l,0)} \frac{y^a}{2}\left|(1-\varphi_R)\nabla u + (1-u)\nabla\varphi_R\right|^2 dx\,dy + \int_{\partial^0 B_R^+(l,0)} b(x)G(w)\,dx \\
&\leq d_s \int_{B_R^+(l,0)} \frac{y^a}{2}|\nabla u|^2\,dx\,dy + d_s \int_{B_R^+(l,0)} \frac{y^a}{2}|\nabla\varphi_R|^2\,dx\,dy \\
&\quad + d_s\left\{\int_{B_R^+(l,0)} y^a|\nabla u|^2\,dx\,dy\right\}^{\frac{1}{2}}\left\{\int_{B_R^+} y^a|\nabla\varphi_R|^2\,dx\,dy\right\}^{\frac{1}{2}} \\
&\quad + \int_{\partial^0(B_R^+\setminus B_{(1-\eta)R}^+)} b(x)G(w)\,dx \\
&\leq d_s \int_{B_R^+(l,0)} \frac{y^a}{2}|\nabla u|^2\,dx\,dy + \left(C\eta^{-2}R^{-2}RR^{1+a}\right) \\
&\quad + \left\{CR\left[\int_1^R y^a y^{-2}\,dy + \int_0^1 \left(y^a + y^{-a}\right)dy\right]\right\}^{\frac{1}{2}}\left(C\eta^{-2}R^a\right)^{\frac{1}{2}} \\
&\quad + 2\overline{b}\max_{[0,1]} G\cdot\eta R \\
&\leq d_s \int_{B_R^+(l,0)} \frac{y^a}{2}|\nabla u|^2\,dx\,dy + C\eta^{-2}R^a + C\eta^{-1}R^{\frac{1+a}{2}} + 2\overline{b}\max_{[0,1]} G\cdot\eta R.
\end{aligned}$$

Here the constant $C$ does not depend on $R$, we use the gradient estimates (see [2]) in the second line from the bottom.

On the other hand,

$$\mathcal{E}\big(u, B_R^+(l,0)\big) \geq d_s \int_{B_R^+(l,0)} \frac{y^a}{2} |\nabla u|^2 \, dx \, dy + 2\underline{b} \min_{[0,1-\epsilon]} G \cdot R.$$

Choose $\eta = \frac{\underline{b} \min_{[0,1-\epsilon]} G}{2\overline{b} \max_{[0,1]} G}$, $\mathcal{E}(u, B_R^+(l,0)) > \mathcal{E}(w, B_R^+(l,0))$ for large $R$. This contradiction leads to the result that there exists at least a sequence $x_n \to \infty$ such that $u(x_n, 0) \to 1$.

Step 3. We show that $u$ is the layer solution, *i.e.*, $\lim_{x \to \pm\infty} u(x,0) = \pm 1$.

Let $u^n(x,y) = u(x+n, y)$ and $n \in \mathbb{Z}^+$. By the regularity results [2], up to a subsequence,

$$\begin{aligned}
&u^n \to u^\infty \quad \text{in } C^0_{\mathrm{loc}}\big(\overline{\mathbb{R}^2_+}\big), \\
&u^n_x \to u^\infty_x \quad \text{in } C^0_{\mathrm{loc}}\big(\overline{\mathbb{R}^2_+}\big), \\
&y^a \frac{\partial u^n}{\partial y} \to y^a \frac{\partial u^\infty}{\partial y} \quad \text{in } C^0_{\mathrm{loc}}\big(\overline{\mathbb{R}^2_+}\big)
\end{aligned}$$

as $n \to \infty$.

$$\begin{cases} -\operatorname{div}(y^a \nabla u^\infty) = 0 & \text{in } \mathbb{R}^2_+, \\ -\lim_{y \downarrow 0^+} y^a \frac{\partial u^\infty}{\partial y} = \frac{1}{d_s} b(x) f(u^\infty(x,0)) & \text{on } \partial\mathbb{R}^2_+, \\ 0 \leq u^\infty \leq 1 & \text{in } \mathbb{R}^2_+. \end{cases} \tag{3.11}$$

Define $\widetilde{u} = 1 - u^\infty$, we have

$$\begin{cases} -\operatorname{div}(y^a \nabla \widetilde{u}) = 0 & \text{in } \mathbb{R}^2_+, \\ -\lim_{y \downarrow 0^+} y^a \frac{\partial \widetilde{u}}{\partial y} = -\frac{1}{d_s} b(x) f(u^\infty(x,0)) \leq 0 & \text{on } \partial\mathbb{R}^2_+, \\ 0 \leq \widetilde{u} \leq 1 & \text{in } \mathbb{R}^2_+. \end{cases} \tag{3.12}$$

$\widetilde{u} \equiv C$ by Proposition 3.1, $f(u^\infty(x,0)) = f(C) \equiv 0$ and $u^\infty \equiv 0$ or 1. Thus $u^\infty \equiv 1$ by step 2. That is, $u \to 1$ as $x \to \infty$. $u \to -1$ as $x \to -\infty$ is achieved by odd symmetry.

$u$ is the desired layer solution. □

*Proof of Theorem* 1.1 It follows from Theorem 3.1; for the regularity of $v$ see [2]. □

Lastly we give asymptotic estimates for layer solutions of (1.1) as $|x| \to \infty$.

*Proof of Theorem* 1.2 Let $v$ be a layer solution of (1.1),

$$\begin{cases} (-\partial_{xx})^s v(x) = b(x) f(v(x)) & \text{in } \mathbb{R}, \\ \lim_{x \to \pm\infty} v = \pm 1. \end{cases} \tag{3.13}$$

Then

$$(-\partial_{xx})^s (1-v) - b(x) f'(\xi_1)(1-v) = 0 \quad \text{in } \mathbb{R}, \tag{3.14}$$

where $\xi_1$ is some point between $v(x)$ and 1.

Consider the layer solution $v_s^t$ of the unperturbed problem in Lemma 2.4,

$$(-\partial_{xx})^s(1-v_s^t) - (f_s^t)'(\xi_2)(1-v_s^t) = 0 \quad \text{in } \mathbb{R} \tag{3.15}$$

with $\xi_2$ is some point between $v_s^t(x)$ and 1.

Since $-(f_s^t)'(1) = \frac{1}{t}$, choose $t$ large enough such that $\frac{2}{t} < -\underline{b}f'(1)$ and choose $x_0 \in \mathbb{R}$ such that $-(f_s^t)'(\xi_2) < \frac{2}{t} < -\underline{b}f'(\xi_1)$ for all $x > x_0$.

Choose $C > 0$ such that $C(1-v_s^t) > 1-v$ in $(-\infty, x_0]$, which can be done since $v_s^t, v \to -1$ as $x \to -\infty$.

Define

$$d(x) = \begin{cases} \frac{2}{t} & \text{in } (x_0, +\infty), \\ \frac{Cf_s^t(v_s^t)-b(x)f(v)}{C(1-v_s^t)-(1-v)} & \text{in } (-\infty, x_0], \end{cases}$$

$d(x) \in L^\infty$. We have

$$\begin{cases} (-\partial_{xx})^s\{C(1-v_s^t)-(1-v)\} + d(x)\{C(1-v_s^t)-(1-v)\} \geq 0 & \text{in } \mathbb{R}, \\ C(1-v_s^t)-(1-v) > 0 & \text{in } (-\infty, x_0]. \end{cases} \tag{3.16}$$

Obviously, if $\inf_{\mathbb{R}}\{C(1-v_s^t)-(1-v)\} < 0$, it is achieved at some point $\underline{x} \in (x_0, +\infty)$. Since $d > 0$ in $(x_0, +\infty)$, $(-\partial_{xx})^s\{C(1-v_s^t)-(1-v)\}(\underline{x}) \geq 0$ from the first inequality of (3.16), which contradicts with the fact that

$$\begin{aligned} &(-\partial_{xx})^s\{C(1-v_s^t)-(1-v)\}(\underline{x}) \\ &\quad = \int_R \frac{\{C(1-v_s^t)-(1-v)\}(\underline{x}) - \{C(1-v_s^t)-(1-v)\}(y)}{|\underline{x}-y|^{1+2s}}\,dy < 0. \end{aligned}$$

Therefore $(1-v) \leq C(1-v_s^t)$ for $C > 0$ given from above.

On the other hand, choose small $t > 0$ such that $-\overline{b}f'(1) < \frac{1}{2t}$ and choose $x^0 \in \mathbb{R}$ such that $-\overline{b}f'(\xi_1) < \frac{1}{2t} < -(f_s^t)'(\xi_2)$ for all $x > x^0$. Choose $c > 0$ such that $c(1-v_s^t) < 1-v$ in $(-\infty, x^0]$.

Define

$$\widetilde{d}(x) = \begin{cases} \frac{1}{2t} & \text{in } (x^0, +\infty), \\ \frac{b(x)f(v)-cf_s^t(v_s^t)}{(1-v)-c(1-v_s^t)} & \text{in } (-\infty, x^0] \end{cases}$$

and obviously $\widetilde{d}(x) \in L^\infty$.

Then,

$$\begin{cases} (-\partial_{xx})^s\{(1-v)-c(1-v_s^t)\} + \widetilde{d}(x)\{(1-v)-c(1-v_s^t)\} \geq 0 & \text{in } \mathbb{R}, \\ (1-v)-c(1-v_s^t) > 0 & \text{in } (-\infty, x^0]. \end{cases} \tag{3.17}$$

If $\inf_R\{(1-v)-c(1-v_s^t)\} < 0$, it is only achieved at some point $\overline{x} \in (x^0, +\infty)$. Since $\widetilde{d} > 0$ in $(x^0, +\infty)$, $(-\partial_{xx})^s\{(1-v)-c(1-v_s^t)\}(\overline{x}) \geq 0$ from the first inequality of (3.17), which contradicts the fact that $(-\partial_{xx})^s\{(1-v)-c(1-v_s^t)\}(\overline{x}) < 0$. Thus $c(1-v_s^t) \leq (1-v)$ for some $0 < c < C$ given from above.

Therefore,

$$cx^{-2s} \le |1-v| \le Cx^{-2s} \quad \text{for } x > 1$$

by Lemma 2.4. Similarly,

$$c|x|^{-2s} \le |1+v| \le C|x|^{-2s} \quad \text{for } x < -1.$$

Here $c$ and $C$ maybe different from above. □

## 4 Asymptotic as $s \uparrow 1$

In this section we prove Theorem 1.3, which consists of two lemmas.

**Lemma 4.1** *Let* $\{v^{s_k}\}$ *be a sequence of layer solutions of* (1.1) *in Theorem* 1.1. *Then there exists a subsequence denoted again by* $\{v^{s_k}\}$, *converging locally uniformly to* $v^1$ *which solves the local elliptic equation*

$$-v^1_{xx}(x) = b(x)f\left(v^1\right) \quad \text{in } \mathbb{R}. \tag{4.1}$$

*Proof* Consider $u_{a_k}$, the $s$-extension of $v^{s_k}$, which solves

$$\begin{cases} -\operatorname{div}(y^{a_k}\nabla u_{a_k}) = 0 & \text{in } \mathbb{R}^2_+, \\ -(1+a_k)\lim_{y\downarrow 0^+} y^{a_k}\partial_y u_{a_k} = C_{a_k} b(x)f(u_{a_k}(x,0)) & \text{on } \partial\mathbb{R}^2_+, \end{cases} \tag{4.2}$$

where $a_k = 1-2s_k$ and $C_{a_k} = \frac{1+a_k}{d_{s_k}} = \frac{2(1-s_k)}{d_{s_k}}$. Obviously $a_k \downarrow -1$ as $s_k \uparrow 1$.

Let $\xi \in C^1_c(\overline{\mathbb{R}^2_+})$. Multiplying (4.2) with $\xi$ and integrating in $\mathbb{R}^2_+$,

$$(1+a_k)\int_{\mathbb{R}^2_+} y^{a_k}\nabla u_{a_k}\nabla\xi\,dx\,dy - C_{a_k}\int_{\mathbb{R}} b(x)f\left(u_{a_k}(x,0)\right)\xi\,dx = 0. \tag{4.3}$$

Choose $\xi(x,y) = \xi_1(x)\xi_2(y)$, $\xi_1 \in C^1_c(\mathbb{R})$ and $\xi_2$ is a cut-off function which equals 1 in $[0,1]$ and 0 in $[2,\infty)$, $|\xi_2'| \le C$ for some constant $C > 0$. Thus (4.3) can be rewritten as

$$\begin{aligned} &(1+a_k)\int_{\mathbb{R}^2_+} y^{a_k}\left\{\xi_1'(x)\xi_2(y)\partial_x u_{a_k} + \xi_1(x)\xi_2'(y)\partial_y u_{a_k}\right\}dx\,dy \\ &\quad = C_{a_k}\int_{\mathbb{R}} b(x)f\left(u_{a_k}(x,0)\right)\xi_1(x)\,dx. \end{aligned} \tag{4.4}$$

By the regularity results in [2], the continuous module does not depend on $s$ for $s > s_0 > \frac{1}{2}$. Up to a subsequence,

$$\begin{aligned} &u_{a_k} \to u_{-1} \quad \text{in } C^0_{\text{loc}}\left(\overline{\mathbb{R}^2_+}\right), \\ &(u_{a_k})_x \to (u_{-1})_x \quad \text{in } C^0_{\text{loc}}\left(\overline{\mathbb{R}^2_+}\right) \end{aligned}$$

and

$$C_{a_k} = \frac{2(1-s_k)}{d_{s_k}} = \frac{2(1-s_k)}{2^{2s_k-1}\frac{\Gamma(s_k)}{\Gamma(1-s_k)}} \to 1$$

as $s_k \uparrow 1$ (or equivalently $a_k \downarrow -1$). Then

$$C_{a_k} \int_{\mathbb{R}} b(x) f\big(u_{a_k}(x,0)\big)\xi_1 \, dx \to \int_{\mathbb{R}} b(x) f\big(u_{-1}(x,0)\big)\xi_1 \, dx \quad \text{as } a_k \downarrow -1. \tag{4.5}$$

For the first integral in (4.4), we consider

$$\begin{aligned} &(1+a_k)\int_0^\infty y^{a_k}\xi_2(y)\partial_x u_{a_k}\,dy \\ &\quad = (1+a_k)\int_0^\delta y^{a_k}\xi_2(y)\big\{\partial_x u_{a_k} - u'_{-1}(x)\big\}\,dy \\ &\qquad + (1+a_k)\int_0^\delta y^{a_k}\xi_2(y)u'_{-1}(x)\,dy + (1+a_k)\int_\delta^\infty y^{a_k}\xi_2(y)\partial_x u_{a_k}\,dy \\ &\quad = I_1 + I_2 + I_3, \end{aligned} \tag{4.6}$$

$$|I_1| \le (1+a_k)\int_0^\delta y^{a_k}\xi_2(y)\big|\partial_x u_{a_k} - u'_{-1}(x)\big|\,dy \le \epsilon\delta^{1+a_k} \tag{4.7}$$

for $0<\delta<1$ and small $\epsilon>0$. Here we use the fact that $\partial_x u_{a_k} \to u'_{-1}(x)$ locally uniformly in $\overline{\mathbb{R}^2_+}$. We have

$$I_2 = u'_{-1}(x)(1+a_k)\int_0^\delta y^{a_k}\,dy = \delta^{1+a_k}u'_{-1} \to u'_{-1} \quad \text{as } a_k \downarrow -1. \tag{4.8}$$

Since $|\nabla u_{a_k}| \le \frac{C}{y}$ for $y>0$ and $C$ independent of $a_k$ (see [2]),

$$|I_3| \le C(1+a_k)\int_\delta^\infty y^{a_k-1}\,dy = C\frac{1+a_k}{a_k}\delta^{a_k} \to 0 \quad \text{as } a_k \downarrow -1. \tag{4.9}$$

By (4.6)-(4.9),

$$(1+a_k)\int_0^\infty y^{a_k}\xi_2(y)\partial_x u_{a_k}\,dy \to u'_{-1}$$

and

$$(1+a_k)\int_{\mathbb{R}^2_+} y^{a_k}\xi_1'(x)\xi_2(y)\partial_x u_{a_k}\,dx\,dy \to \int_{\mathbb{R}} \xi_1'(x)u'_{-1}\,dx, \tag{4.10}$$

$$\begin{aligned} &\left|(1+a_k)\int_{\mathbb{R}^2_+} y^{a_k}\xi_1(x)\xi_2'(y)\partial_y u_{a_k}\,dx\,dy\right| \\ &\quad \le \int_{\mathbb{R}} \big|\xi_1(x)\big|\,dx(1+a_k)\int_1^2 y^{a_k}\big|\xi_2'(y)\big||\partial_y u_{a_k}|\,dy \\ &\quad \le C(1+a_k)\int_1^2 y^{a_k-1}\,dy = C\frac{1+a_k}{a_k}\big(2^{a_k}-1\big) \to 0 \end{aligned} \tag{4.11}$$

as $a_k \downarrow -1$.

Therefore, by (4.4), (4.5), (4.10), and (4.11),

$$\int_{\mathbb{R}} u'_{-1}(x)\xi_1'(x)\,dx = \int_{\mathbb{R}} b(x) f\big(u_{-1}(x)\big)\xi_1(x)\,dx. \tag{4.12}$$

That is,

$$-v^1_{xx} = b(x)f(v^1) \tag{4.13}$$

in the weak sense ($u_{-1} = v^1$). By the regularity theory of elliptic equations, $v^1$ is also a classical solution of (4.13). □

**Lemma 4.2** *$v^1$ is also a layer solution of* (4.1), *i.e.*, $v^1 \to \pm 1$ *as* $x \pm \infty$.

*Proof* Claim 1. $v^1$ is a local minimizer in $(0,\infty)$ under perturbations in $[-1,1]$. That is,

$$\mathcal{F}(v^1, I) \le \mathcal{F}(v^1 + \xi_1, I) \tag{4.14}$$

for any bounded open interval $I \subset (0,\infty)$ and for any $\xi_1 \in C^1_0(I)$ such that $|v^1 + \xi_1| \le 1$, where

$$\mathcal{F}(w, I) := \int_I \left\{ \frac{|w_x|^2}{2} + b(x)G(w) \right\} dx \quad \text{for every } w \in H^1(I).$$

Indeed, for the test function $\xi$ in Lemma 4.1 with the additional property that $|u_{a_k} + \xi| \le 1$, we have

$$\begin{aligned}
0 &\le E\big(u_{a_k} + (1-\epsilon)\xi, I \times [0,R]\big) - E\big(u_{a_k}, I \times [0,R]\big) \\
&= \frac{1+a_k}{2} \int_{I\times[0,R]} y^{a_k} \big|\nabla\big(u_{a_k} + (1-\epsilon)\xi\big)\big|^2 \, dx\, dy + C_{a_k} \int_I b(x)G\big(u_{a_k} + (1-\epsilon)\xi\big)\, dx \\
&\quad - \frac{1+a_k}{2} \int_{I\times[0,R]} y^{a_k} |\nabla u_{a_k}|^2 \, dx\, dy - C_{a_k} \int_I b(x)G(u_{a_k})\, dx \\
&= \frac{1+a_k}{2} \int_{I\times[0,R]} y^{a_k} \big|\partial_x u_{a_k} + (1-\epsilon)\xi_1'(x)\xi_2(y)\big|^2 \, dx\, dy \\
&\quad - \frac{1+a_k}{2} \int_{I\times[0,R]} y^{a_k} |\partial_x u_{a_k}|^2 \, dx\, dy \\
&\quad + (1+a_k) \int_{I\times[0,R]} y^{a_k} \partial_y u_{a_k} (1-\epsilon)\xi_1(x)\xi_2'(y)\, dx\, dy \\
&\quad + \frac{1+a_k}{2} \int_{I\times[0,R]} y^{a_k} \big((1-\epsilon)\xi_1(x)\xi_2'(y)\big)^2 \, dx\, dy \\
&\quad + C_{a_k} \int_I b(x)G\big(u_{a_k} + (1-\epsilon)\xi_1(x)\big)\, dx - C_{a_k} \int_I b(x)G(u_{a_k})\, dx.
\end{aligned} \tag{4.15}$$

As in the discussions in Lemma 4.1, let $a_k \downarrow -1$, and we have

$$\frac{1+a_k}{2} \int_{I\times[0,R]} y^{a_k} (\partial_x u_{a_k})^2 \, dx\, dy \to \int_I \frac{(u'_{-1})^2}{2}\, dx, \tag{4.16}$$

$$(1+a_k) \int_{I\times[0,R]} y^{a_k} \partial_x u_{a_k} (1-\epsilon)\xi_1'(x)\xi_2(y)\, dx\, dy \to \int_I u'_{-1}(x)(1-\epsilon)\xi_1'(x)\, dx, \tag{4.17}$$

$$\begin{aligned}
&\frac{1+a_k}{2}\int_{I\times[0,R]} y^{a_k}\big((1-\epsilon)\xi_1'(x)\xi_2(y)\big)^2\,dx\,dy\\
&\quad=\frac{1}{2}\int_I (1-\epsilon)^2\big(\xi_1'(x)\big)^2\,dx\bigg\{\int_0^1 (1+a_k)y^{a_k}\,dy+\int_1^R (1+a_k)y^{a_k}\big(\xi_2(y)\big)^2\,dy\bigg\}\\
&\quad\to\frac{1}{2}\int_I\big((1-\epsilon)\xi_1'(x)\big)^2\,dx. \qquad (4.18)
\end{aligned}$$

By (4.16)-(4.18),

$$\begin{aligned}
&\frac{1+a_k}{2}\int_{I\times[0,R]} y^{a_k}\big(\partial_x u_{a_k}+(1-\epsilon)\xi_1'(x)\xi_2(y)\big)^2\,dx\,dy\\
&\qquad-\frac{1+a_k}{2}\int_{I\times[0,R]} y^{a_k}(\partial_x u_{a_k})^2\,dx\,dy\\
&\quad\to\frac{1}{2}\int_I\big(u_{-1}'(x)+(1-\epsilon)\xi_1'(x)\big)^2\,dx-\frac{1}{2}\int_I\big(u_{-1}'(x)\big)^2\,dx, \qquad (4.19)
\end{aligned}$$

$$\begin{aligned}
&(1+a_k)\int_{I\times[0,R]} y^{a_k}\partial_y u_{a_k}\xi_1(x)\xi_2'(y)\,dx\,dy\\
&\quad=(1+a_k)\int_{I\times[1,2]} y^{a_k}\partial_y u_{a_k}\xi_1(x)\xi_2'(y)\,dx\,dy\\
&\quad\le C(1+a_k)\int_1^2 y^{a_k-1}\,dy\\
&\quad=\frac{C(1+a_k)}{a_k}\big\{2^{a_k}-1\big\}\to 0\quad\text{as } a_k\downarrow -1, \qquad (4.20)
\end{aligned}$$

$$\begin{aligned}
\frac{(1+a_k)}{2}\int_I\int_0^R y^{a_k}\big(\xi_1(x)\xi_2'(y)\big)^2\,dx\,dy&=\frac{(1+a_k)}{2}\int_I\int_1^2 y^{a_k}\big(\xi_1(x)\xi_2'(y)\big)^2\,dx\,dy\\
&\le C(1+a_k)\int_1^2 y^{a_k}\,dy\\
&=C\big(2^{a_k+1}-1\big)\to 0\quad\text{as } a_k\downarrow -1, \qquad (4.21)
\end{aligned}$$

$$\begin{aligned}
&C_{a_k}\int_I b(x)G\big(u_{a_k}+(1-\epsilon)\xi_1(x)\big)\,dx-C_{a_k}\int_I b(x)G(u_{a_k})\,dx\\
&\quad\to\int_I b(x)G\big(u_{-1}+(1-\epsilon)\xi_1(x)\big)\,dx-\int_I b(x)G(u_{-1})\,dx. \qquad (4.22)
\end{aligned}$$

By (4.15), (4.19)-(4.22), our claim is proved.

Claim 2. $v^1\to 1$ as $x\to\infty$.

Define $v^{1,n}(x)=v^1(x+n)$ for $n\in\mathbb{Z}^+$, up to a subsequence, $v^{1,n}\to v^{1,\infty}$ in $C^2_{\text{loc}}$ as $n\to\infty$,

$$\begin{cases}-v^{1,\infty}_{xx}(x)=b(x)f(v^{1,\infty}(x)), & x\in\mathbb{R},\\ 0\le v^{1,\infty}\le 1.\end{cases} \qquad (4.23)$$

Since $f\ge 0$ and $b>0$, $-v^{1,\infty}_{xx}\ge 0$ in $\mathbb{R}$ and $v^{1,\infty}\equiv 0$ or 1.

We show that $v^1\to 0$ or 1 as $x\to\infty$. Indeed, if there are two sequences $\{x_n\}$ and $\{y_n\}$ such that $v^1(x_n)\to 0$ and $v^1(y_n)\to 1$ as $n\to\infty$, there must exist $z_n\in(x_n,y_n)$ such that $v^1(z_n)=\frac{1}{2}$.

Denote $\widetilde{v}^1_n(x)=v^1(x+[z_n])$ where $[z_n]$ is the integer part of $z_n$. $\widetilde{v}^1_n(z_n-[z_n])=v^1(z_n)=\frac{1}{2}$ and up to a subsequence $\widetilde{v}^1_n\to\widetilde{v}^1_\infty$ in $C^2_{\text{loc}}$, $\widetilde{v}^1_\infty$ solves equation (4.23). Therefore $\widetilde{v}^1_\infty\equiv 0$ or 1.

For the above subsequence, there is a subsubsequence such that $z_n - [z_n] \to z^* \in [0,1]$ as $n \to \infty$ and $\widetilde{v}^1_\infty(z^*) = \frac{1}{2}$. This contradiction verifies $v^1 \to 0$ or 1 as $x \to \infty$.

To check $v^1 \to 1$ as $x \to \infty$, suppose that $v^1 \to 0$ as $x \to \infty$ by contradiction. Then,

$$\liminf_{l\to+\infty} \mathcal{F}\big(v^1, (l-R, l+R)\big) = \liminf_{l\to+\infty} \int_{l-R}^{l+R} \left\{ \frac{|v^1|^2}{2} + b(x)G\big(v^1\big) \right\} dx \geq 2\underline{b}R\epsilon$$

for some $\epsilon > 0$.

Let $\xi \in C_0^1(l-R, l+R)$, $\xi = 1$ if $|x-l| < (1-\eta)R$ and $\xi = 0$ if $|x-l| > R$ where $\eta$ will be determined later, $|\xi'| \leq \frac{1}{\eta R}$. Define $w = 1 \cdot \xi + (1-\xi)v^1$, then $w(l \pm R) = v^1(l \pm R)$. We have

$$\begin{aligned}
&\limsup_{l\to+\infty} \mathcal{F}\big(w, (l-R, l+R)\big) \\
&\quad = \limsup_{l\to+\infty} \int_{l-R}^{l+R} \left( \frac{1}{2}\big|(1-\xi)v^1_x + \big(1-v^1\big)\xi_x\big|^2 + b(x)G\big(1\cdot\xi + (1-\xi)v^1\big) \right) dx \\
&\quad \leq \int_{l-R}^{l+R} \xi_x^2 \, dx + \overline{b} \max_{[-1,1]} G \cdot 2\eta R \\
&\quad \leq \frac{1}{\eta^2 R} + \overline{b} \max_{[-1,1]} G \cdot 2\eta R.
\end{aligned}$$

Choose $\eta = \frac{\epsilon \underline{b}}{2\overline{b}\max_{[-1,1]} G}$,

$$\limsup_{l\to+\infty} \mathcal{F}\big(w, (l-R, l+R)\big) < \liminf_{l\to+\infty} \mathcal{F}\big(v^1, (l-R, l+R)\big)$$

for $R > 1$ large enough. Therefore $v^1 \to 1$ as $x \to \infty$, by odd symmetry, $v^1 \to -1$ as $x \to -\infty$, *i.e.*, $v^1$ is a layer solution of the local elliptic equation (4.13).

By the Hamiltonian equality (2.15),

$$\begin{aligned}
&b(x)\big\{G\big(v^1(x)\big) - G(1)\big\} + \int_x^{+\infty} b'(t)\big\{G\big(v^1(t)\big) - G(1)\big\}\, dt \\
&\quad = \frac{1}{2}\big(v^1_x\big)^2 = \lim_{a_k \downarrow -1} (1+a_k) \int_0^\infty \frac{y^{a_k}}{2} (\partial_x u_{a_k})^2 \\
&\quad = \lim_{a_k \downarrow -1} (1+a_k) \int_0^\infty \frac{y^{a_k}}{2} (\partial_y u_{a_k})^2 \\
&\qquad + \lim_{a_k \downarrow -1} C_{a_k} b(x)\big\{G\big(u_{a_k}(x,0)\big) - G(1)\big\} \\
&\qquad + \lim_{a_k \downarrow -1} C_{a_k} \int_x^\infty b'(t)\big\{G\big(u_{a_k}(t,0)\big) - G(1)\big\}\, dt.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\lim_{a\downarrow -1} (1+a) \int_0^\infty \frac{y^a}{2} (\partial_y u_a)^2 = {}& \int_x^{+\infty} b'(t)\big\{G\big(v^1(t)\big) - G(1)\big\}\, dt \\
& - \lim_{a_k \downarrow -1} C_{a_k} \int_x^\infty b'(t)\big\{G\big(u_{a_k}(t,0)\big) - G(1)\big\}\, dt. \qquad \square
\end{aligned}$$

*Proof of Theorem* 1.3 It follows from Lemmas 4.1 and 4.2. $\square$

**Competing interests**

The author declares that she has no competing interests.

**Author's contributions**

The author read and approved the final manuscript.

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**References**

1. Cabré, X, Solà-Morales, J: Layer solutions in a half-space for boundary reactions. Commun. Pure Appl. Math. **58**(12), 1678-1732 (2005)
2. Cabré, X, Sire, Y: Nonlinear equations for fractional Laplacians I: regularity, maximum principles, and Hamiltonian estimates. Ann. Inst. Henri Poincaré, Anal. Non Linéaire. (2013). doi:10.1016/j.anihpc.2013.02.001
3. Cabré, X, Sire, Y: Nonlinear equations for fractional Laplacian II: existence, uniqueness, and qualitative properties of solutions. arXiv:1111.0796 (2011)
4. Caffarelli, L, Silvestre, L: An extension problem related to the fractional Laplacian. Commun. Partial Differ. Equ. **32**(8), 1245-1260 (2007)
5. Fabes, EB, Kenig, CE, Serapioni, RP: The local regularity of solutions of degenerate elliptic equations. Commun. Stat., Theory Methods **7**(1), 77-116 (1982)
6. Palatucci, G, Savin, O, Valdinoci, E: Local and global minimizers for a variational energy involving a fractional norm. Ann. Mat. Pura Appl. **192**(4), 673-718 (2013)
7. Nekvinda, A: Characterization of traces of the weighted Sobolev space $W^{1,p}(\Omega, d_M^\epsilon)$ on $M$. Czechoslov. Math. J. **43**(4), 695-711 (1993)
8. Alama, S, Bronsard, L, Gui, C: Stationary layered solutions in $\mathbb{R}^2$ for an Allen-Cahn systems with multiple well potential. Calc. Var. Partial Differ. Equ. **5**(4), 359-390 (1997)
9. Ghoussoub, N, Gui, C: On a conjecture of De Giorgi and some related problems. Math. Ann. **311**, 481-491 (1998)
10. Ghoussoub, N, Gui, C: On De Giorgi's conjecture in dimensions 4 and 5. Ann. Math. **157**(1), 313-334 (2003)
11. Adams, RA: Sobolev Spaces. Pure and Applied Mathematics, vol. 65. Academic Press, New York (1975)
12. Gilbarg, D, Trudinger, NS: Elliptic Partial Differential Equations of Second Order. Springer, Berlin (1997)

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