Li and Li *Boundary Value Problems* 2014, 2014:49
http://www.boundaryvalueproblems.com/content/2014/1/49

Boundary Value Problems
a SpringerOpen Journal

RESEARCH **Open Access**

Mountain pass lemma and new periodic solutions of the singular second order Hamiltonian system

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Abstract
We generalize the classical Ambrosetti-Rabinowitz mountain pass lemma with the Palais-Smale condition for C^1 functional to some singular case with the Cerami-Palais-Smale condition and then we study the existence of new periodic solutions with a fixed period for the singular second-order Hamiltonian systems with a strong force potential.

MSC: 34C15; 34C25; 58F

Keywords: Ambrosetti-Rabinowitz's mountain pass lemma; singular second-order Hamiltonian systems; periodic solutions; Cerami-Palais-Smale condition

1 Introduction

Many authors [1–19] studied the existence of periodic solutions $t \mapsto x(t) \in \Omega$, with a prescribed period, of the following second-order differential equations:

$$\ddot{x} = -V'(t,x), \tag{1.1}$$

where $\Omega = \mathbb{R}^N - \{0\}$ ($N \in \mathbb{N}, N \geq 2$) and $V \in C^1(\mathbb{R} \times \Omega, \mathbb{R})$; $V'(t,\cdot)$ denotes the gradient of the function $V(t,\cdot)$ defined on Ω.

In 1975, Gordon [10] firstly used variational methods to study periodic solutions of planar 2-body type problems, he assumed the condition nowadays called Gordon's strong force condition.

Condition (V_1): There exists a neighborhood $\mathcal{N}$ of 0 and a function $U \in C^1(\Omega, \mathbb{R})$ such that:

(i) $\lim_{x\to 0} U(x) = -\infty$;

(ii) $-V(t,x) \geq |U'(x)|^2$ for every $x \in \mathcal{N} - \{0\}$ and $t \in [0,T]$.

Moreover,

(iii) $\lim_{x\to 0} V(t,x) = -\infty$.

In the 1980s and 1990s, Ambsosetti-Coti Zelati, Bahri-Rabinowitz, Greco *etc.* [1–9, 11–19] further studied 2-body type problems in R^N ($N \geq 2$).

Suppose that $V(t,x)$ is T-periodic in t; as regards the behavior of $V(t,x)$ at infinity, they suppose that one of the following conditions holds.

Condition (V_2): $\lim_{|x|\to\infty} V(t,x) = 0$, $\lim_{|x|\to\infty} V'(t,x) = 0$ (uniformly for t) and $V(t,x) < 0$ for every $t \in [0,T]$, $x \in \Omega$.

Condition (V_3): There exist $c_1, M_1, R_1, \nu > 0$ such that, for every $t \in [0,T]$ and $x \in \mathbb{R}^N$ with $|x| \geq R_1$:

(i) $|V'(t,x)| \leq M_1$;

(ii) $V(t,x) \geq c_1|x|^\nu$.

Condition (V_4): There exist $c_1, R_1 > 0, \theta > \frac{1}{2}, \nu > 1$ such that, for every $t \in [0,T]$, $|x| \geq R_1$:

(i) $\theta V'(t,x)x \leq V(t,x)$;

(ii) $V(t,x) \geq c_1|x|^\nu$.

Setting $K = \{x \in \Omega | V'(t,x) = 0 \text{ for every } t \in [0,T]\}$, they got the following results.

Theorem 1.1 (Greco [11]) *If* (V_1) *and one of* (V_2)-(V_4) *hold, and moreover* $K = \emptyset$, *then there is at least one non-constant* T*-periodic* C^2 *solution.*

Theorem 1.2 (Bahri-Rabinowitz [3], Greco [11]) *Suppose that* $\partial V/\partial t \equiv 0$, *so* $V(t,x) \equiv V(x)$; *moreover suppose we have the following condition.*

Condition (V_5): *K is compact* (*or empty*).

Then, if (V_1) *and one of* (V_2)-(V_4) *hold, there exist infinitely many non-constant* T*-periodic* C^2 *solutions.*

In this paper, we prove the following new theorem.

Theorem 1.3 *Suppose* $V \in C^1(\mathbb{R} \times \Omega, \mathbb{R})$ *satisfies the conditions*:

(V1) *For the given* $T > 0$, $V(t+T,x) = V(t,x)$.

(V2) $\forall (t,x) \in \mathbb{R} \times \Omega$, $V(t+\frac{T}{2}, -x) = V(t,x)$.

(V3) *There is* $a > 0$, $\alpha \geq 2$ *such that for any given* $\epsilon > 0$ *and*

$$\forall t \in [0,T], \quad |x| \leq \left(\frac{T}{12}\right)^{\frac{1}{2}} \left[(b\alpha)^{\frac{1}{(\alpha+2)}} + \epsilon\right],$$

we have

$$-V(t,x) \geq \frac{a}{|x|^\alpha},$$

where

$$b = a(2\pi)^\alpha T^{1-\frac{\alpha}{2}}.$$

(V4) *There exists* $M > 0$ *such that* $\forall (t,x) \in \mathbb{R} \times \Omega$,

$$3V(t,x) - V'(t,x)x \leq M.$$

(V5) $V(t,x) \to +\infty$ *as* $|x| \to +\infty$ *uniformly for* $0 \leq t \leq T$.

Then the system (1.1) *has at least a non-constant* T*-periodic solution.*

Corollary 1.1 *Suppose* $\alpha \geq 2$, $\beta \geq 3$, $a > 0$, $a' > 0$, $V \in C^1(\Omega, \mathbb{R})$ *and*

$$V(x) = -a|x|^{-\alpha}, \quad 0 < |x| \leq r_1 = \left(\frac{T}{12}\right)^{\frac{1}{2}} \left[(b\alpha)^{\frac{1}{(\alpha+2)}} + \epsilon\right];$$

$$V(x) = a'|x|^\beta, \quad |x| \geq r_2 > r_1;$$

then $\forall T > 0$, (1.1) *has at least a* T*-periodic solution.*

2 A few lemmas

Lemma 2.1 (Sobolev-Rellich-Kondrachov [20]) *We have*

$$H^1 = W^{1,2}(R/TZ, R^N) \subset C(R/TZ, R^N)$$

and the embedding is compact.

Lemma 2.2 (Eberlein-Shmulyan [20]) *A Banach space X is reflexive if and only if any bounded sequence in X has a weakly convergent subsequence.*

Lemma 2.3 ([21]) (i) *Let $q \in W^{1,2}(R/ZT, R^N)$ and $\int_0^T q(t)\,dt = 0$, then we have Wirtinger's inequality*:

$$\int_0^T |\dot{q}(t)|^2\,dt \geq \left(\frac{2\pi}{T}\right)^2 \int_0^T |q(t)|^2\,dt.$$

(ii) *Let $q \in W^{1,2}(R/ZT, R^N)$ and $\int_0^T q(t)\,dt = 0$, then we have Sobolev's inequality*:

$$\|q\|_\infty^2 \leq \frac{T}{12} \int_0^T |\dot{q}(t)|^2\,dt.$$

(iii) *Let ϕ be a convex function on the real line; $f : [a,b] \to R$ is a non-negative real-valued function which is Lebesgue-integrable, then*

$$\phi\left(\int_a^b f(x)\,dx\right) \leq \frac{1}{b-a} \int_a^b \phi\big((b-a)f(x)\big)\,dx.$$

Lemma 2.4 (Ekeland [8]) *Let X be a Banach space; suppose that Φ defined on X is Gateaux-differentiable and lower semi-continuous and bounded from below. Then there is a sequence $\{x_n\}$ such that*

$$\Phi(x_n) \to \inf \Phi,$$
$$\big(1 + \|x_n\|\big)\big\|\Phi'(x_n)\big\| \to 0.$$

Definition 2.1 (Palais and Smale [22]) Let X be a Banach space; $f \in C^1(X, R)$, if $\{x_n\} \subset X$ s.t.

$$f(x_n) \to c, \qquad f'(x_n) \to 0,$$

and $\{x_n\}$ has a strongly convergent subsequence; then we say that f satisfies the $(PS)_c$ condition.

Cerami [23] presented a weaker compact condition than the above classical $(PS)_c$ condition.

Definition 2.2 ([8]) Let X be a Banach space, $\Lambda \subset X$, and suppose that Φ is defined on Λ is Gateaux-differentiable, if the sequence $\{x_n\}$ is such that

$$\Phi(x_n) \to c,$$
$$\big(1 + \|x_n\|\big)\big\|\Phi'(x_n)\big\| \to 0,$$

then $\{x_n\}$ has a strongly convergent subsequence in Λ.

Then we say that f satisfies the $(CPS)_c$ condition.

We can give a weaker condition than the $(CPS)_c$ condition.

Definition 2.3 Let X be a Banach space, $\Lambda \subset X$, and suppose that Φ defined on Λ is Gateaux-differentiable; if the sequence $\{x_n\}$ is such that

$$\Phi(x_n) \to c,$$
$$\bigl(1 + \|x_n\|\bigr)\bigl\|\Phi'(x_n)\bigr\| \to 0,$$

and $\{x_n\}$ has a weakly convergent subsequence in Λ, then we say that f satisfies the $(WCPS)_c$ condition.

Lemma 2.5 (Ambrosetti-Rabinowitz [24], mountain pass lemma) *Let X be a Banach space, $\Lambda \subset X, f \in C^1(\Lambda, R)$. We have*

$$B_\rho = \bigl\{x \in \Lambda \,|\, \|x\| \le \rho\bigr\},$$
$$S_\rho = \partial B_\rho \cap X, \quad \rho > 0.$$

If there are two points $e_1 \in B_\rho - S_\rho$, $e_2 \in \Lambda - B_\rho$ such that

$$f|_{S_\rho} \ge \alpha > 0$$

and

$$f(e_1), f(e_2) \le 0,$$

then $C = \inf_{\phi \in \Gamma} \sup_{t \in [0,1]} f(\phi(t)) \ge \alpha$, where $\Gamma = \{h(t) \in C^1([0,1], \Lambda), h(0) = e_1, h(1) = e_2\}$. If f satisfies the $(CPS)_C$ condition on $\Lambda \subset X$, furthermore, if $f(x_n) \to +\infty$ as $x_n \to \partial\Lambda$, then C is a critical value for f.

3 The proof of Theorem 1.3

Let

$$H^1 = \bigl\{q : R \to R^n \,|\, q \in L^2, \dot{q} \in L^2, q(t+T) = q(t)\bigr\},$$
$$\Lambda = \left\{q \in H^1, q\left(t + \frac{T}{2}\right) = -q(t), q(t) \ne 0, \forall t\right\}.$$

Lemma 3.1 ([2, 25]) *If $V \in C^1(\mathbb{R} \times \Omega, \mathbb{R})$ satisfies the conditions* (V1)-(V2), *let*

$$f(q) = \frac{1}{2}\int_0^T |\dot{q}|^2\,dt - \int_0^T V(t,q)\,dt, \quad q \in \Lambda,$$

then the critical point of $f(q)$ on Λ is a T-periodic solution of (1.1).

Lemma 3.2 *If V satisfies* (V3), (V4) *in Theorem* 1.1, *then f satisfies the Cerami-Palais-Smale condition for any $c > 0$, that is, for any $\{x_n\} \subset \Lambda$:*

$$f(x_n) \to c, \qquad \bigl(1 + \|x_n\|\bigr)f'(x_n) \to 0, \tag{3.1}$$

$\{x_n\}$ has a strongly convergent subsequence and the limit is in Λ.

Proof By the condition (V3), it is well known [10] that $f(x_n) \to +\infty$ as $x_n \to \partial\Lambda$. Since $f(x_n) \to c$, we know that for any given $\epsilon > 0$, there exists N such that when $n > N$, we have

$$\frac{1}{2}\int_0^T |\dot{x}_n|^2\,dt - \int_0^T V(x_n)\,dt \le c + \epsilon. \tag{3.2}$$

By $(1 + \|x_n\|)f'(x_n) \to 0$, we have

$$f'(x_n)x_n \to 0, \tag{3.3}$$

$$f'(x_n)x_n = 2f(x_n) + \int_0^T \bigl[2V(t,x_n) - V'(t,x_n)x_n\bigr]\,dt \to 0. \tag{3.4}$$

So by (V4) and (3.2) and (3.4), we have $d > 0$ such that when n large enough, we have

$$\int_0^T |\dot{x}_n|^2\,dt \le d. \tag{3.5}$$

So $\int_0^T |\dot{x}_n|^2\,dt$ is bounded. Then $\{x_n\}$ has a weakly convergence subsequence, and it is standard to further prove that this subsequence is strongly convergent in Λ.

Now we can prove our theorem.

In order to apply for Ambrosetti-Rabinowitz's mountain pass lemma, we notice that

$$\forall x \in \Lambda, \qquad \int_0^T x(t)\,dt = 0,$$

so by (V3) and Wirtinger's inequality we have

$$\begin{aligned} f(x) &= \frac{1}{2}\int_0^T |\dot{x}|^2\,dt - \int_0^T V(t,x)\,dt & (3.6)\\ &\ge \frac{1}{2}\int_0^T |\dot{x}|^2\,dt + a\int_0^T |x|^{-\alpha}\,dt & (3.7)\\ &\ge \frac{1}{2}\int_0^T |\dot{x}|^2 + aT^{1+\frac{\alpha}{2}}\left(\int_0^T |x|^2\,dt\right)^{\frac{-\alpha}{2}} & (3.8)\\ &\ge \frac{1}{2}\int_0^T |\dot{x}|^2 + aT^{1+\frac{\alpha}{2}}\left(\frac{T}{2\pi}\right)^{-\alpha}\left(\int_0^T |\dot{x}|^2\,dt\right)^{\frac{-\alpha}{2}} & (3.9)\\ &= \frac{1}{2}s^2 + bs^{-\alpha} = \phi(s), & (3.10) \end{aligned}$$

where

$$s = \left(\int_0^T |\dot{x}|^2\,dt\right)^{1/2}, \qquad b = aT^{1+\frac{\alpha}{2}}\left(\frac{T}{2\pi}\right)^{-\alpha} = a(2\pi)^{\alpha}T^{1-\frac{\alpha}{2}}. \tag{3.11}$$

It is easy to see that if $s_0 = (b\alpha)^{\frac{1}{\alpha+2}}$, ϕ attains its infimum which is a positive number. $\forall\epsilon > 0$, we can take $\rho = s_0 + \epsilon$, take $e_1(t) \neq 0$, $\|\dot{e_1}\| = s_0 < \rho$. By Sobolev's inequality, we know that $(\int_0^T |\dot{x}|^2\,dt) \ge \frac{12}{T}\|x\|_\infty^2$, so if $\|x(t)\|_\infty \le (\frac{T}{12})^{\frac{1}{2}}[(b\alpha)^{\frac{1}{(\alpha+2)}} + \epsilon]$, then the above proofs hold.

Let us choose e_2 = constant value vector in R^n, $\dot{e}_2 = 0$. Then by (V1) and (V5), we have

$$f(e_2) = -\int_0^T V(t, e_2) \leq -T \min_{0\leq t\leq T} \left| V(t, e_2) \right| \to -\infty \quad \text{as } |e_2| = R \to +\infty. \tag{3.12}$$

So if $|e_2| = R$ is large enough, we have

$$f|_{e_2} \leq 0.$$

By Lemmas 2.5 and 3.2, f has a critical value $C > 0$, and the corresponding critical point is a T-periodic solution of the system (1.1). Furthermore, we claim that the critical point is non-constant; in fact, if otherwise, by the anti-$T/2$ periodic property, we know that the critical point must be constant zero, which is impossible since $f(0) = +\infty$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Received: 2 February 2014 Accepted: 18 February 2014 Published: 27 Feb 2014

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10.1186/1687-2770-2014-49
Cite this article as: Li and Li: Mountain pass lemma and new periodic solutions of the singular second order Hamiltonian system. *Boundary Value Problems* 2014, 2014:49