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Approximation of eigenvalues of boundary value problems

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Abstract

In the present paper we apply a sinc-Gaussian technique to compute approximate values of the eigenvalues of discontinuous Dirac systems, which contain an eigenvalue parameter in one boundary condition, with transmission conditions at the point of discontinuity. The error of this method decays exponentially in terms of the number of involved samples. Therefore the accuracy of the new technique is higher than the classical sinc-method. Numerical worked examples with tables and illustrative figures are given at the end of the paper showing that this method gives us better results.

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Keywords: sinc-Gaussian; sinc-method; Dirac systems; transmission conditions; discontinuous boundary value problems; truncation and amplitude errors

1 Introduction

Consider the discontinuous Dirac system which consists of the system of differential equations

$$\begin{pmatrix} y_2'(x) - r_1(x)y_1(x) \\ y_1'(x) + r_2(x)y_2(x) \end{pmatrix} = \begin{pmatrix} \lambda y_1(x) \\ -\lambda y_2(x) \end{pmatrix}, \quad x \in [-1,0) \cup (0,1], \tag{1.1}$$

with boundary conditions

$$U_1(\mathbf{y}) := \sin\alpha y_1(-1) - \cos\alpha y_2(-1) = 0, \tag{1.2}$$

$$U_2(\mathbf{y}) := (a_1 + \lambda\sin\beta)y_1(1) - (a_2 + \lambda\cos\beta)y_2(1) = 0 \tag{1.3}$$

and transmission conditions

$$U_3(\mathbf{y}) := y_1(0^-) - \delta y_1(0^+) = 0, \tag{1.4}$$

$$U_4(\mathbf{y}) := y_2(0^-) - \delta y_2(0^+) = 0, \tag{1.5}$$

where $\lambda \in \mathbb{C}$; $\mathbf{y} = \binom{y_1}{y_2}$; the real-valued functions $r_1(\cdot)$ and $r_2(\cdot)$ are continuous in $[-1,0)$ and $(0,1]$, and have finite limits $r_1(0^\pm) := \lim_{x\to 0^\pm} r_1(x)$, $r_2(0^\pm) := \lim_{x\to 0^\pm} r_2(x)$; $a_1, a_2, \delta \in \mathbb{R}$, $\alpha, \beta \in [0,\pi)$; $\delta \neq 0$ and $\rho := a_1\cos\beta - a_2\sin\beta > 0$.The aim of the present work is to compute the eigenvalues of (1.1)-(1.5) numerically by the sinc-Gaussian technique with errors analysis, truncation error and amplitude error.

Sampling theory is one of the most important mathematical tools used in communication engineering since it enables engineers to reconstruct signals from some of their sampled data. A fundamental result in information theory is the Whittaker-Kotel'nikov-Shannon (WKS) sampling theorem [1–3]. It states that any $f \in \mathcal{B}^2_\sigma$, $\sigma > 0$,

$$\mathcal{B}^2_\sigma := \left\{ f : f \text{ entire}, \left|f(\mu)\right| \leq Ce^{\sigma|\Im\mu|}, \int_{\mathbb{R}} \left|f(\mu)\right|^2 d\mu < \infty \right\},$$

can be reconstructed from its sampled values $\{f(n\pi/\sigma) : n \in \mathbb{Z}\}$ by the formula

$$f(\lambda) = \sum_{n\in\mathbb{Z}} f(n\pi/\sigma)\operatorname{sinc}(\sigma\lambda - n\pi), \quad \lambda \in \mathbb{C}, \tag{1.6}$$

where

$$\operatorname{sinc}(\lambda) := \begin{cases} \frac{\sin(\lambda)}{\lambda}, & \lambda \neq 0, \\ 1, & \lambda = 0. \end{cases} \tag{1.7}$$

Series (1.6) converges absolutely and uniformly on compact subsets of $\mathbb{C}$, and uniformly on $\mathbb{R}$, *cf.* [4]. Expansion (1.6) is used in several approximation problems which are known as sinc-methods; see, *e.g.*, [5–8]. In particular the sinc-method is used to approximate eigenvalues of boundary value problems; see, for example, [9–12]. The sinc-method has a slow rate of decay at infinity, which is as slow as $O(|\lambda^{-1}|)$. There have been several attempts to improve the rate of decay. One of the interesting ways is to multiply the sinc-function in (1.6) by a kernel function; see, *e.g.*, [13–15]. Let $h \in (0, \pi/\sigma]$ and $\gamma \in (0, \pi - h\sigma)$. Assume that $\Phi \in \mathcal{B}^2_\gamma$ such that $\Phi(0) = 1$, then for $\mathfrak{g} \in \mathcal{B}^2_\sigma$ we have the expansion [16]

$$\mathfrak{g}(\lambda) = \sum_{n=-\infty}^{\infty} \mathfrak{g}(nh)\operatorname{sinc}\left(h^{-1}\pi\lambda - n\pi\right)\Phi\left(h^{-1}\lambda - n\right). \tag{1.8}$$

The speed of convergence of the series in (1.8) is determined by the decay of $|\Phi(\lambda)|$. But the decay of an entire function of exponential type cannot be as fast as $e^{-c|x|}$ as $|x| \to \infty$ for some positive c [16]. In [17], Qian has introduced the following regularized sampling formula. For $h \in (0, \pi/\sigma]$, $N \in \mathbb{N}$ and $r > 0$, Qian defined the operator [17]

$$(G_{h,N}\mathfrak{g})(\lambda) = \sum_{n\in Z_N(\lambda)} \mathfrak{g}(nh)\operatorname{sinc}\left(h^{-1}\pi\lambda - n\pi\right)G\left(\frac{\lambda - nh}{\sqrt{2}rh}\right), \quad \lambda \in \mathbb{R}, \tag{1.9}$$

where $G(t) := \exp(-t^2)$, which is called the Gaussian function, $Z_N(x) := \{n \in \mathbb{Z} : |[h^{-1}x] - n| \leq N\}$ and $[x]$ denotes the integer part of $x \in \mathbb{R}$; see also [18, 19]. Qian also derived the following error bound. If $\mathfrak{g} \in \mathcal{B}^2_\sigma$, $h \in (0, \pi/\sigma]$ and $a := \min\{r(\pi - h\sigma), (N-2)/r\} \geq 1$, then [17, 18]

$$\left|\mathfrak{g}(\lambda) - (G_{h,N}\mathfrak{g})(\lambda)\right| \leq \frac{2\sqrt{\sigma\pi}\,\|\mathfrak{g}\|_2}{\pi^2 a^2}\left(\sqrt{2\pi}a + e^{3/2r^2}\right)e^{-a^2/2}, \quad \lambda \in \mathbb{R}. \tag{1.10}$$

In [16] Schmeisser and Stenger extended the operator (1.9) to the complex domain $\mathbb{C}$. For $\sigma > 0$, $h \in (0, \pi/\sigma]$ and $\omega := (\pi - h\sigma)/2$, they defined the operator [16]

$$(\mathcal{G}_{h,N}\mathfrak{g})(\lambda) := \sum_{n\in\mathbb{Z}_N(\lambda)} \mathfrak{g}(nh) S_n\left(\frac{\pi\lambda}{h}\right) G\left(\frac{\sqrt{\omega}(\lambda - nh)}{\sqrt{N}h}\right), \tag{1.11}$$

where $\mathbb{Z}_N(\lambda) := \{n \in \mathbb{Z} : |[h^{-1}\Re\lambda + 1/2] - n| \le N\}$ and $N \in \mathbb{N}$. Note that the summation limits in (1.11) depend on the real part of λ. Schmeisser and Stenger [16] proved that if $\mathfrak{g}$ is an entire function such that

$$\bigl|\mathfrak{g}(\xi + i\eta)\bigr| \le \phi\bigl(|\xi|\bigr)e^{\sigma|\eta|}, \quad \xi, \eta \in \mathbb{R}, \tag{1.12}$$

where ϕ is a non-decreasing, non-negative function on $[0, \infty)$ and $\sigma \ge 0$, then for $h \in (0, \pi/\sigma)$, $\omega := (\pi - h\sigma)/2$, $N \in \mathbb{N}$, $|\Im\lambda| < N$, we have

$$\begin{aligned} &\bigl|\mathfrak{g}(\lambda) - (\mathcal{G}_{h,N}\mathfrak{g})(\lambda)\bigr| \\ &\quad \le 2\bigl|\sin\bigl(h^{-1}\pi\lambda\bigr)\bigr|\phi\bigl(|\Re\lambda| + h(N+1)\bigr)\frac{e^{-\omega N}}{\sqrt{\pi\omega N}}\beta_N\bigl(h^{-1}\Im\lambda\bigr), \quad \lambda \in \mathbb{C}, \end{aligned} \tag{1.13}$$

where

$$\beta_N(t) := \cosh(2\omega t) + \frac{2e^{\omega t^2/N}}{\sqrt{\pi\omega N}[1 - (t/N)^2]} + \frac{1}{2}\left[\frac{e^{2\omega t}}{e^{2\pi(N-t)} - 1} + \frac{e^{-2\omega t}}{e^{2\pi(N+t)} - 1}\right]. \tag{1.14}$$

The amplitude error arises when the exact values $\mathfrak{g}(nh)$ of (1.11) are replaced by the approximations $\widetilde{\mathfrak{g}}(nh)$. We assume that $\widetilde{\mathfrak{g}}(nh)$ are close to $\mathfrak{g}(nh)$, *i.e.*, there is $\varepsilon > 0$ sufficiently small such that

$$\sup_{n\in\mathbb{Z}_n(\lambda)} \bigl|\mathfrak{g}(nh) - \widetilde{\mathfrak{g}}(nh)\bigr| < \varepsilon. \tag{1.15}$$

Let $h \in (0, \pi/\sigma)$, $\omega := (\pi - h\sigma)/2$ and $N \in \mathbb{N}$ be fixed numbers. The authors in [20] proved that if (1.15) is held, then for $|\Im\lambda| < N$, we have

$$\bigl|(\mathcal{G}_{h,N}\mathfrak{g})(\lambda) - (\mathcal{G}_{h,N}\widetilde{\mathfrak{g}})(\lambda)\bigr| \le A_{\varepsilon,N}(\Im\lambda), \tag{1.16}$$

where

$$A_{\varepsilon,N}(\Im\lambda) = 2\varepsilon e^{-\omega/4N}\bigl(1 + \sqrt{N/\omega\pi}\bigr)\exp\bigl((\omega + \pi)h^{-1}|\Im\lambda|\bigr). \tag{1.17}$$

Without eigenparameter appearing in any of boundary conditions, in [21] and [12] Tharwat *et al.* approximately computed the eigenvalues of the discontinuous Dirac system which is studied in the monographs of [22] by Hermite interpolations and regularized sinc-methods, respectively. In the regularized sinc-method, also the same in the Hermite interpolations method, the basic idea is as follows: The eigenvalues are characterized as the zeros of an analytic function $F(\lambda)$ which can be written in the form $F(\lambda) = f_0(\lambda) + f(\lambda)$, where $f_0(\lambda)$ is a known part. The ingenuity of the approach is in trying to choose the function $F(\lambda)$ so that $f(\lambda) \in \mathcal{B}^2_\sigma$ (unknown part) and can be approximated by the WKS sampling theorem if its values at some equally spaced points are known; see [9–12]. Recall that, in regularized sinc and Hermite interpolations methods, it is necessary that $f(\lambda)$ is an L^2-function. In this paper we will use the sinc-Gaussian sampling formula (1.11) to compute eigenvalues of (1.1)-(1.5) numerically. As is expected, the new method reduced the error bounds remarkably (see the examples in Section 4). Also here, the basic idea is to write the function of eigenvalues as the sum of two terms, one known and the other unknown but an entire function of exponential type which satisfies (1.12). In other words, the unknown term is not necessarily an L^2-function. Then we approximate the unknown part using (1.11) and obtain better results. We would like to mention that the papers in computing eigenvalues by the sinc-Gaussian method are few; see [20, 23–25]. In Sections 2, 3 we derive the sinc-Gaussian technique to compute the eigenvalues of (1.1)-(1.5) with error estimates. The last section involves some illustrative examples.

2 Preliminaries

In this section we derive approximate values of the eigenvalues of problem (1.1)-(1.5). Recall that problem (1.1)-(1.5) has a denumerable set of real and simple eigenvalues, *cf.* [26]; see also [22, 27–29]. Let

$$\mathfrak{y}(\cdot,\lambda) = \begin{pmatrix} \mathfrak{y}_1(\cdot,\lambda) \\ \mathfrak{y}_2(\cdot,\lambda) \end{pmatrix}, \qquad \mathfrak{y}_i(x,\lambda) = \begin{cases} \mathfrak{y}_{i1}(x,\lambda), & x \in [-1,0), \\ \mathfrak{y}_{i2}(x,\lambda), & x \in (0,1], \end{cases} \qquad i = 1,2, \tag{2.1}$$

be the solution of (1.1) satisfying the following initial conditions:

$$\begin{pmatrix} \mathfrak{y}_{11}(-1,\lambda) & \mathfrak{y}_{12}(0^+,\lambda) \\ \mathfrak{y}_{21}(-1,\lambda) & \mathfrak{y}_{22}(0^+,\lambda) \end{pmatrix} = \begin{pmatrix} \cos\alpha & \delta^{-1}\mathfrak{y}_{11}(0^-,\lambda) \\ \sin\alpha & \delta^{-1}\mathfrak{y}_{21}(0^-,\lambda) \end{pmatrix}. \tag{2.2}$$

In [26], Tharwat proved the existence and uniqueness of (2.2). Since $\mathfrak{y}(\cdot,\lambda)$ satisfies (1.2), then the eigenvalues of problem (1.1)-(1.5) are the zeros of the function (see Lemma 2.4 of [26, p.8])

$$\Delta(\lambda) = \delta^2\big((a_1 + \lambda\sin\beta)\mathfrak{y}_{12}(1,\lambda) - (a_2 + \lambda\cos\beta)\mathfrak{y}_{22}(1,\lambda)\big). \tag{2.3}$$

Notice that both $\mathfrak{y}(\cdot,\lambda)$ and $\Delta(\lambda)$ are entire functions of λ, and $\mathfrak{y}(\cdot,\lambda)$ satisfies the system of integral equations (*cf.* [26])

$$\mathfrak{y}_{11}(x,\lambda) = \cos\big(\lambda(x+1)+\alpha\big) - \mathcal{S}_{-1,1}\mathfrak{y}_{11}(x,\lambda) - \widetilde{\mathcal{S}}_{-1,2}\mathfrak{y}_{21}(x,\lambda), \tag{2.4}$$

$$\mathfrak{y}_{21}(x,\lambda) = \sin\big(\lambda(x+1)+\alpha\big) + \widetilde{\mathcal{S}}_{-1,1}\mathfrak{y}_{11}(x,\lambda) - \mathcal{S}_{-1,2}\mathfrak{y}_{21}(x,\lambda), \tag{2.5}$$

$$\begin{aligned} \mathfrak{y}_{12}(x,\lambda) = {} & \frac{1}{\delta}\mathfrak{y}_{11}\big(0^-,\lambda\big)\cos(\lambda x) - \frac{1}{\delta}\mathfrak{y}_{21}\big(0^-,\lambda\big)\sin(\lambda x) \\ & - \mathcal{S}_{0,1}\mathfrak{y}_{12}(x,\lambda) - \widetilde{\mathcal{S}}_{0,2}\mathfrak{y}_{22}(x,\lambda), \end{aligned} \tag{2.6}$$

$$\begin{aligned} \mathfrak{y}_{22}(x,\lambda) = {} & \frac{1}{\delta}\mathfrak{y}_{11}\big(0^-,\lambda\big)\sin(\lambda x) + \frac{1}{\delta}\mathfrak{y}_{21}\big(0^-,\lambda\big)\cos(\lambda x) \\ & + \widetilde{\mathcal{S}}_{0,1}\mathfrak{y}_{12}(x,\lambda) - \mathcal{S}_{0,2}\mathfrak{y}_{22}(x,\lambda), \end{aligned} \tag{2.7}$$

where $\mathcal{S}_{-1,i}$, $\widetilde{\mathcal{S}}_{-1,i}$, $\mathcal{S}_{0,i}$ and $\widetilde{\mathcal{S}}_{0,i}$, $i=1,2$, are the Volterra integral operators defined by

$$\begin{aligned}
\mathcal{S}_{-1,i}\varphi(x,\lambda) &:= \int_{-1}^{x} \sin\lambda(x-t)r_i(t)\varphi(t,\lambda)\,dt,\\
\widetilde{\mathcal{S}}_{-1,i}\varphi(x,\lambda) &:= \int_{-1}^{x} \cos\lambda(x-t)r_i(t)\varphi(t,\lambda)\,dt,\\
\mathcal{S}_{0,i}\varphi(x,\lambda) &:= \int_{0}^{x} \sin\lambda(x-t)r_i(t)\varphi(t,\lambda)\,dt,\\
\widetilde{\mathcal{S}}_{0,i}\varphi(x,\lambda) &:= \int_{0}^{x} \cos\lambda(x-t)r_i(t)\varphi(t,\lambda)\,dt.
\end{aligned}$$

For convenience, we define the constants

$$\begin{aligned}
&c_1 := \int_{-1}^{0}\bigl[\bigl|r_1(t)\bigr| + \bigl|r_2(t)\bigr|\bigr]\,dt, \qquad c_2 := c_1\exp(c_1),\\
&c_3 := \int_{0}^{1}\bigl[\bigl|r_1(t)\bigr| + \bigl|r_2(t)\bigr|\bigr]\,dt, \qquad c_4 := c_2 + \frac{2}{|\delta|}(1+c_2),\\
&c_5 := \max\bigl\{|a_1| + |a_2|, |\sin\beta| + |\cos\beta|\bigr\}.
\end{aligned} \tag{2.8}$$

Define $\mathfrak{z}_{-1,i}(\cdot,\lambda)$ and $\mathfrak{z}_{0,i}(\cdot,\lambda)$, $i=1,2$, to be

$$\begin{aligned}
\mathfrak{z}_{-1,1}(x,\lambda) &:= \mathcal{S}_{-1,1}\mathfrak{y}_{11}(x,\lambda) + \widetilde{\mathcal{S}}_{-1,2}\mathfrak{y}_{21}(x,\lambda),\\
\mathfrak{z}_{-1,2}(x,\lambda) &:= \widetilde{\mathcal{S}}_{-1,1}\mathfrak{y}_{11}(x,\lambda) - \mathcal{S}_{-1,2}\mathfrak{y}_{21}(x,\lambda),
\end{aligned} \tag{2.9}$$

$$\begin{aligned}
\mathfrak{z}_{0,1}(x,\lambda) &:= \mathcal{S}_{0,1}\mathfrak{y}_{12}(x,\lambda) + \widetilde{\mathcal{S}}_{0,2}\mathfrak{y}_{22}(x,\lambda),\\
\mathfrak{z}_{0,2}(x,\lambda) &:= \widetilde{\mathcal{S}}_{0,1}\mathfrak{y}_{12}(x,\lambda) - \mathcal{S}_{0,2}\mathfrak{y}_{22}(x,\lambda).
\end{aligned} \tag{2.10}$$

Lemma 2.1 *The functions $\mathfrak{z}_{-1,1}(x,\lambda)$ and $\mathfrak{z}_{-1,2}(x,\lambda)$ are entire in λ for any fixed $x\in[-1,0)$ and satisfy the growth condition*

$$\bigl|\mathfrak{z}_{-1,1}(x,\lambda)\bigr|, \bigl|\mathfrak{z}_{-1,2}(x,\lambda)\bigr| \le 2c_2 e^{|\Im\lambda|(x+1)}, \quad \lambda\in\mathbb{C}. \tag{2.11}$$

Proof Since $\mathfrak{z}_{-1,1}(x,\lambda) = \mathcal{S}_{-1,1}\mathfrak{y}_{11}(x,\lambda) + \widetilde{\mathcal{S}}_{-1,2}\mathfrak{y}_{21}(x,\lambda)$, then from (2.4) and (2.5) we obtain

$$\begin{aligned}
\mathfrak{z}_{-1,1}(x,\lambda) = {}& \mathcal{S}_{-1,1}\cos\bigl(\lambda(x+1)+\alpha\bigr) + \widetilde{\mathcal{S}}_{-1,2}\sin\bigl(\lambda(x+1)+\alpha\bigr)\\
& - \mathcal{S}_{-1,1}\mathfrak{z}_{-1,1}(x,\lambda) + \widetilde{\mathcal{S}}_{-1,2}\mathfrak{z}_{-1,2}(x,\lambda).
\end{aligned}$$

Using the inequalities $|\sin z|\le e^{|\Im z|}$ and $|\cos z|\le e^{|\Im z|}$ for $z\in\mathbb{C}$ leads for $\lambda\in\mathbb{C}$ to

$$\begin{aligned}
\bigl|\mathfrak{z}_{-1,1}(x,\lambda)\bigr| \le{}& \bigl|\mathcal{S}_{-1,1}\cos\bigl(\lambda(x+1)+\alpha\bigr)\bigr| + \bigl|\widetilde{\mathcal{S}}_{-1,2}\sin\bigl(\lambda(x+1)+\alpha\bigr)\bigr|\\
& + \bigl|\mathcal{S}_{-1,1}\mathfrak{z}_{-1,1}(x,\lambda)\bigr| + \bigl|\widetilde{\mathcal{S}}_{-1,2}\mathfrak{z}_{-1,2}(x,\lambda)\bigr|\\
\le{}& e^{|\Im\lambda|(x+1)}\int_{-1}^{x}\bigl[\bigl|r_1(t)\bigr|\bigl|\mathfrak{z}_{-1,1}(t,\lambda)\bigr| + \bigl|r_2(t)\bigr|\bigl|\mathfrak{z}_{-1,2}(t,\lambda)\bigr|\bigr]e^{-|\Im\lambda|(t+1)}\,dt\\
& + 2e^{|\Im\lambda|(x+1)}\int_{-1}^{x}\bigl[\bigl|r_1(t)\bigr| + \bigl|r_2(t)\bigr|\bigr]\,dt
\end{aligned}$$

$$\begin{aligned}&\leq 2c_1 e^{|\Im\lambda|(x+1)}\\&\quad+e^{|\Im\lambda|(x+1)}\int_{-1}^{x}\bigl[|r_1(t)||\mathfrak{z}_{-1,1}(t,\lambda)|+|r_2(t)||\mathfrak{z}_{-1,2}(t,\lambda)|\bigr]e^{-|\Im\lambda|(t+1)}\,dt.\end{aligned}$$

The above inequality can be reduced to

$$\begin{aligned}&e^{-|\Im\lambda|(x+1)}|\mathfrak{z}_{-1,1}(x,\lambda)|\\&\quad\leq 2c_1+\int_{-1}^{x}\bigl[|r_1(t)||\mathfrak{z}_{-1,1}(t,\lambda)|+|r_2(t)||\mathfrak{z}_{-1,2}(t,\lambda)|\bigr]e^{-|\Im\lambda|(t+1)}\,dt.\end{aligned}\tag{2.12}$$

Similarly, we can prove that

$$\begin{aligned}&e^{-|\Im\lambda|(x+1)}|\mathfrak{z}_{-1,2}(x,\lambda)|\\&\quad\leq 2c_1+\int_{-1}^{x}\bigl[|r_1(t)||\mathfrak{z}_{-1,1}(t,\lambda)|+|r_2(t)||\mathfrak{z}_{-1,2}(t,\lambda)|\bigr]e^{-|\Im\lambda|(t+1)}\,dt.\end{aligned}\tag{2.13}$$

Then from (2.12), (2.13) and Lemma 3.1 of [28, p.204], we obtain (2.11). □

In a similar manner, we will prove the following lemma for $\mathfrak{z}_{0,1}(\cdot,\lambda)$ and $\mathfrak{z}_{0,2}(\cdot,\lambda)$.

Lemma 2.2 *The functions $\mathfrak{z}_{0,1}(x,\lambda)$ and $\mathfrak{z}_{0,2}(x,\lambda)$ are entire in λ for any fixed $x\in(0,1]$ and satisfy the growth condition*

$$|\mathfrak{z}_{0,1}(x,\lambda)|,|\mathfrak{z}_{0,2}(x,\lambda)|\leq 2c_3c_4e^{|\Im\lambda|(x+1)},\quad \lambda\in\mathbb{C}.\tag{2.14}$$

Proof Since $\mathfrak{z}_{0,1}(x,\lambda)=\mathcal{S}_{0,1}\mathfrak{y}_{12}(x,\lambda)+\widetilde{\mathcal{S}}_{0,2}\mathfrak{y}_{22}(x,\lambda)$, then from (2.6) and (2.7) we obtain

$$\begin{aligned}\mathfrak{z}_{0,1}(x,\lambda)={}&\frac{1}{\delta}\mathfrak{y}_{11}(0^-,\lambda)\mathcal{S}_{0,1}\cos(\lambda x)-\frac{1}{\delta}\mathfrak{y}_{21}(0^-,\lambda)\mathcal{S}_{0,1}\sin(\lambda x)-\mathcal{S}_{0,1}\mathfrak{z}_{-1,2}(x,\lambda)\\&+\frac{1}{\delta}\mathfrak{y}_{11}(0^-,\lambda)\widetilde{\mathcal{S}}_{0,2}\sin(\lambda x)+\frac{1}{\delta}\mathfrak{y}_{21}(0^-,\lambda)\widetilde{\mathcal{S}}_{0,2}\cos(\lambda x)+\widetilde{\mathcal{S}}_{0,2}\mathfrak{z}_{-1,2}(x,\lambda).\end{aligned}$$

Then from (2.4) and (2.5) and Lemma 2.1, we get

$$\begin{aligned}|\mathfrak{z}_{0,1}(x,\lambda)|\leq{}&\frac{1}{|\delta|}|\mathfrak{y}_{11}(0^-,\lambda)||\mathcal{S}_{0,1}\cos(\lambda x)|+\frac{1}{|\delta|}|\mathfrak{y}_{21}(0^-,\lambda)||\mathcal{S}_{0,1}\sin(\lambda x)|\\&+|\mathcal{S}_{0,1}\mathfrak{z}_{-1,2}(x,\lambda)|+\frac{1}{|\delta|}|\mathfrak{y}_{11}(0^-,\lambda)||\widetilde{\mathcal{S}}_{0,2}\sin(\lambda x)|\\&+\frac{1}{|\delta|}|\mathfrak{y}_{21}(0^-,\lambda)||\widetilde{\mathcal{S}}_{0,2}\cos(\lambda x)|+|\widetilde{\mathcal{S}}_{0,2}\mathfrak{z}_{-1,2}(x,\lambda)|\\\leq{}&2\left(c_2+\frac{2}{|\delta|}(1+c_2)\right)c_3e^{|\Im\lambda|(x+1)}\\={}&2c_3c_4e^{|\Im\lambda|(x+1)}.\end{aligned}$$

Similarly, we can prove that

$$|\mathfrak{z}_{0,2}(x,\lambda)|\leq 2c_3c_4e^{|\Im\lambda|(x+1)}.$$

□

3 The numerical scheme

In this section we derive the method of computing eigenvalues of problem (1.1)-(1.5) numerically. The basic idea of the scheme is to split $\Delta(\lambda)$ into two parts a known part $\mathcal{K}(\lambda)$ and an unknown one $\mathcal{U}(\lambda)$. Then we approximate $\mathcal{U}(\lambda)$ using (1.11) to get the approximate $\Delta(\lambda)$ and then compute the approximate zeros. We first split $\Delta(\lambda)$ into two parts as follows:

$$\Delta(\lambda) := \mathcal{K}(\lambda) + \mathcal{U}(\lambda), \tag{3.1}$$

where $\mathcal{U}(\lambda)$ is the unknown part involving integral operators

$$\begin{aligned}\mathcal{U}(\lambda) := {}& \delta\big[a_2 \sin\lambda - a_1 \cos\lambda + \lambda \sin(\lambda - \beta)\big]\mathfrak{z}_{-1,1}\big(0^-, \lambda\big) \\ & - \delta\big[a_1 \sin\lambda + a_2 \cos\lambda + \lambda \cos(\lambda - \beta)\big]\mathfrak{z}_{-1,2}\big(0^-, \lambda\big) \\ & + \delta^2\big[-(a_1 + \lambda \sin\beta)\mathfrak{z}_{0,1}(1,\lambda) + (a_2 + \lambda\cos\beta)\mathfrak{z}_{0,2}(1,\lambda)\big]\end{aligned} \tag{3.2}$$

and $\mathcal{K}(\lambda)$ is the known part

$$\mathcal{K}(\lambda) := \delta\big[a_1 \cos(2\lambda + \alpha) - a_2 \sin(2\lambda + \alpha) - \lambda \sin(2\lambda + \alpha - \beta)\big]. \tag{3.3}$$

Then, from Lemma 2.1 and Lemma 2.2, we have the following result.

Lemma 3.1 *The function $\mathcal{U}(\lambda)$ is entire in λ and the following estimate holds*:

$$\big|\mathcal{U}(\lambda)\big| \le \phi(\lambda) e^{2|\Im\lambda|}, \tag{3.4}$$

where

$$\phi(\lambda) =: M\big(1 + |\lambda|\big), \qquad M := 2|\delta| c_5\big(c_2 + |\delta| c_3 c_4\big). \tag{3.5}$$

Proof From (3.2) we have

$$\begin{aligned}\big|\mathcal{U}(\lambda)\big| \le {}& |\delta|\big[|a_2||\sin\lambda| + |a_1||\cos\lambda| + |\lambda|\big|\sin(\lambda-\beta)\big|\big]\big|\mathfrak{z}_{-1,1}\big(0^-,\lambda\big)\big| \\ & + |\delta|\big[|a_1||\sin\lambda| + |a_2||\cos\lambda| + |\lambda|\big|\cos(\lambda-\beta)\big|\big]\big|\mathfrak{z}_{-1,2}\big(0^-,\lambda\big)\big| \\ & + \delta^2\big[\big(|a_1| + |\lambda||\sin\beta|\big)\big|\mathfrak{z}_{0,1}(1,\lambda)\big| + \big(|a_2| + |\lambda||\cos\beta|\big)\big|\mathfrak{z}_{0,2}(1,\lambda)\big|\big].\end{aligned}$$

Using the inequalities $|\sin\lambda| \le e^{|\Im\lambda|}$ and $|\cos\lambda| \le e^{|\Im\lambda|}$ for $\lambda \in \mathbb{C}$, Lemma 2.1 and Lemma 2.2 imply (3.4). □

Thus $\mathcal{U}(\lambda)$ is an entire function of exponential type $\sigma = 2$. In the following we let $\lambda \in \mathbb{R}$ since all eigenvalues are real. Now we approximate the function $\mathcal{U}(\lambda)$ using the operator (1.11) where $h \in (0, \pi/2)$ and $\omega := (\pi - 2h)/2$ and then, from (1.13), we obtain

$$\big|\mathcal{U}(\lambda) - (\mathcal{G}_{h,N}\mathcal{U})(\lambda)\big| \le T_{h,N}(\lambda), \tag{3.6}$$

where

$$T_{h,N}(\lambda) := 2\big|\sin\big(h^{-1}\pi\lambda\big)\big|\phi\big(|\Re\lambda| + h(N+1)\big)\frac{e^{-\omega N}}{\sqrt{\pi\omega N}}\beta_N(0), \quad \lambda \in \mathbb{R}. \tag{3.7}$$

The samples $\mathcal{U}(nh) = \Delta(nh) - \mathcal{K}(nh)$, $n \in \mathbb{Z}_N(\lambda)$ cannot be computed explicitly in the general case. We approximate these samples numerically by solving the initial value problems defined by (1.1) and (2.2) to obtain the approximate values $\widetilde{\mathcal{U}}(nh)$, $n \in \mathbb{Z}_N(\lambda)$, *i.e.*, $\widetilde{\mathcal{U}}(nh) = \widetilde{\Delta}(nh) - \mathcal{K}(nh)$. Here we use the computer algebra system MATHEMATICA to obtain approximate solutions with the required accuracy. However, a separate study for the effect of different numerical schemes and the computational costs would be interesting. Accordingly, we have the explicit expansion

$$(\mathcal{G}_{h,N}\widetilde{\mathcal{U}})(\lambda) := \sum_{n\in\mathbb{Z}_N(\lambda)} \widetilde{\mathcal{U}}(nh)\operatorname{sinc}\bigl(h^{-1}\pi\lambda - n\pi\bigr)G\biggl(\frac{\sqrt{\omega}(\lambda - nh)}{\sqrt{N}h}\biggr). \tag{3.8}$$

Therefore we get (*cf.* (1.16))

$$\bigl|(\mathcal{G}_{h,N}\mathcal{U})(\lambda) - (\mathcal{G}_{h,N}\widetilde{\mathcal{U}})(\lambda)\bigr| \le A_{\varepsilon,N}(0), \quad \lambda \in \mathbb{R}. \tag{3.9}$$

Now let $\widetilde{\Delta}_N(\lambda) := \mathcal{K}(\lambda) + (\mathcal{G}_{h,N}\widetilde{\mathcal{U}})(\lambda)$. From (3.6) and (3.9) we obtain

$$\bigl|\Delta(\lambda) - \widetilde{\Delta}_N(\lambda)\bigr| \le T_{h,N}(\lambda) + A_{\varepsilon,N}(0), \quad \lambda \in \mathbb{R}. \tag{3.10}$$

Let λ^* be an eigenvalue and λ_N be its desired approximation, *i.e.*, $\Delta(\lambda^*) = 0$ and $\widetilde{\Delta}_N(\lambda_N) = 0$. From (3.10) we have $|\widetilde{\Delta}_N(\lambda^*)| \le T_{h,N}(\lambda^*) + A_{\varepsilon,N}(0)$. Define the curves

$$a_\pm(\lambda) = \widetilde{\Delta}_N(\lambda) \pm T_{h,N}(\lambda) + A_{\varepsilon,N}(0). \tag{3.11}$$

The curves $a_+(\lambda)$, $a_-(\lambda)$ enclose the curve of $\Delta(\lambda)$ for suitably large N. Hence the closure interval is determined by solving $a_\pm(\lambda) = 0$, which gives an interval

$$I_{\varepsilon,N} := [a_-, a_+].$$

It is worthwhile to mention that the simplicity of the eigenvalues guarantees the existence of approximate eigenvalues, *i.e.*, the λ_N for which $\widetilde{\Delta}_N(\lambda_N) = 0$. Next we estimate the error $|\lambda^* - \lambda_N|$ for the eigenvalue λ^*.

Theorem 3.2 *Let λ^* be an eigenvalue of* (1.1)-(1.5) *and let λ_N be its approximation. Then, for $\lambda \in \mathbb{R}$, we have the following estimate*:

$$\bigl|\lambda^* - \lambda_N\bigr| < \frac{T_{h,N}(\lambda_N) + A_{\varepsilon,N}(0)}{\inf_{\zeta\in I_{\varepsilon,N}} |\Delta'(\zeta)|}, \tag{3.12}$$

where the interval $I_{\varepsilon,N}$ is defined above.

Proof Replacing λ by λ_N in (3.10), we obtain

$$\bigl|\Delta(\lambda_N) - \Delta\bigl(\lambda^*\bigr)\bigr| < T_{h,N}(\lambda_N) + A_{\varepsilon,N}(0), \tag{3.13}$$

where we have used $\widetilde{\Delta}_N(\lambda_N) = \Delta(\lambda^*) = 0$. Using the mean value theorem yields that for some $\zeta \in J_{\varepsilon,N} := [\min(\lambda^*, \lambda_N), \max(\lambda^*, \lambda_N)]$,

$$\bigl|\bigl(\lambda^* - \lambda_N\bigr)\Delta'(\zeta)\bigr| \le T_{h,N}(\lambda_N) + A_{\varepsilon,N}(0), \quad \zeta \in J_{\varepsilon,N} \subset I_{\varepsilon,N}. \tag{3.14}$$

Since λ^* is simple and N is sufficiently large, then $\inf_{\zeta \in I_{\varepsilon,N}} |\Delta'(\zeta)| > 0$ and we get (3.12). □

4 Numerical examples

This section includes two examples illustrating the sinc-Gaussian method. It is clearly seen that the sinc-Gaussian method gives remarkably better results. We indicate in these two examples the effect of the amplitude error in the method by determining enclosure intervals for different values of ε. We also indicate the effect of N and h by several choices. We would like to mention that MATHEMATICA has been used to obtain the exact values for these examples where eigenvalues cannot be computed concretely. MATHEMATICA is also used in rounding off the exact eigenvalues, which are square roots. Each example is presented via figures that accurately illustrate the procedure near some of the approximated eigenvalues. More explanations are given below.

Example 4.1 Consider the system

$$y_2'(x) - r(x)y_1(x) = \lambda y_1(x), \qquad y_1'(x) + r(x)y_2(x) = -\lambda y_2(x), \quad x \in [-1,0) \cup (0,1], \tag{4.1}$$

$$y_1(-1) = 0, \qquad (1+\lambda)y_1(1) + y_2(1) = 0, \tag{4.2}$$

$$y_1(0^-) - 2y_1(0^+) = 0, \qquad y_2(0^-) - 2y_2(0^+) = 0. \tag{4.3}$$

Here

$$r_1(x) = r_2(x) = r(x) = \begin{cases} x, & x \in [-1,0), \\ x^2, & (0,1], \end{cases} \tag{4.4}$$

$\alpha = \beta = \frac{\pi}{2}$, $a_1 = 1$, $a_2 = -1$ and $\delta = 2$. Direct calculations give

$$\mathcal{K}(\lambda) = 2\bigl(\cos[2\lambda] - (1+\lambda)\sin[2\lambda]\bigr) \tag{4.5}$$

and

$$\Delta(\lambda) = 2\left(\cos\left[\frac{1}{6} - 2\lambda\right] + (1+\lambda)\sin\left[\frac{1}{6} - 2\lambda\right]\right). \tag{4.6}$$

As is clearly seen, the eigenvalues cannot be computed explicitly. The following three tables (Tables 1, 2, 3) indicate the application of our technique to this problem and the effect of ε. By exact we mean the zeros of $\Delta(\lambda)$ computed by MATHEMATICA.

Figures 1 and 2 illustrate the enclosure intervals dominating λ_{-2} for $N = 20$, $h = 0.2$ and $\varepsilon = 10^{-2}$, $\varepsilon = 10^{-5}$, respectively. The middle curve represents $\Delta(\lambda)$, while the upper and

Table 1 The approximation $\lambda_{k,N}$ and the exact solution λ_k for different choices of h and N

λ_k			λ_{-2}	λ_{-1}	λ_0	λ_1
Exact λ_k			−1.9050594725435388	−0.8005149927957496	0.3944055848645847	1.8242788740449205
$\lambda_{k,N}$	$h = 0.8$,	$N = 10$	−1.9050945328700728	−0.8005149844410676	0.3943794190610962	1.8242617833701285
	$\omega = 0.7714$	$N = 20$	−1.9050594925575182	−0.8005149927903844	0.39440557044475477	1.8242788645444055
	$h = 0.2$,	$N = 10$	−1.9050594937724747	−0.8005149927866473	0.3944055855507727	1.8242788693330168
	$\omega = 1.3714$	$N = 20$	−1.905059472543563	−0.8005149927957529	0.39440558486458616	1.824278874044914

Table 2 Absolute error $|\lambda_k - \lambda_{k,N}|$

λ_k		λ_{-2}	λ_{-1}	λ_0	λ_1
$h = 0.8$	$N = 10$	3.50603×10^{-5}	8.35468×10^{-9}	2.61658×10^{-5}	1.70907×10^{-5}
	$N = 20$	2.0014×10^{-8}	5.36515×10^{-12}	1.44198×10^{-8}	9.50052×10^{-9}
$h = 0.2$	$N = 10$	2.12289×10^{-8}	9.10227×10^{-12}	6.86188×10^{-10}	4.7119×10^{-9}
	$N = 20$	2.42029×10^{-14}	3.33067×10^{-15}	3.33067×10^{-15}	6.43929×10^{-15}

Table 3 For $N = 20$ and $h = 0.2$, the exact solutions λ_k are all inside the interval $[a_-, a_+]$ for different values of ε

λ_k	λ_{-2}	λ_{-1}	λ_0	λ_1
Exact λ_k	−1.9050594725435388	−0.8005149927957496	0.3944055848645847	1.8242788740449205
$I_{\varepsilon,N}$, $\varepsilon = 10^{-2}$	[−1.9270913, −1.8826416]	[−0.8259079, −0.7752883]	[0.3752245, 0.4132930]	[1.8121593, 1.8363114]
$I_{\varepsilon,N}$, $\varepsilon = 10^{-5}$	[−1.9050816, −1.9050372]	[−0.8005402, −0.8004897]	[0.3944246, 0.3944055]	[1.8242667, 1.8242909]

Figure 1 The enclosure interval dominating λ_{-2} for $h = 0.2$, $N = 20$ and $\varepsilon = 10^{-2}$.

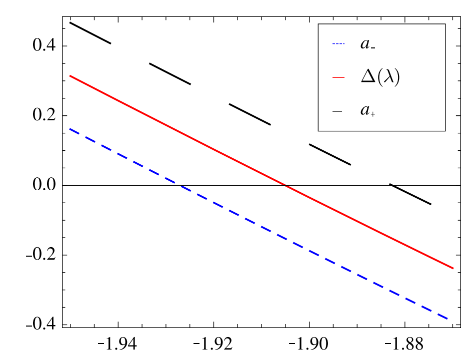
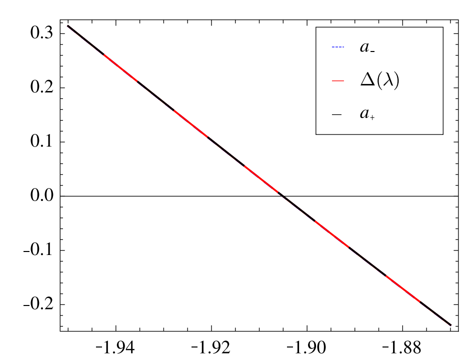


Figure 2 The enclosure interval dominating λ_{-2} for $h = 0.2$, $N = 20$ and $\varepsilon = 10^{-5}$.



lower curves represent the curves of $a_+(\lambda)$, $a_-(\lambda)$, respectively. We notice that when $\varepsilon = 10^{-5}$, the two curves are almost identical. Similarly, Figures 3 and 4 illustrate the enclosure intervals dominating λ_{-1} for $h = 0.2$, $N = 20$ and $\varepsilon = 10^{-2}$, $\varepsilon = 10^{-5}$, respectively.

Example 4.2 In this example we consider the system

$$y_2'(x) - r(x)y_1(x) = \lambda y_1(x), \qquad y_1'(x) + r(x)y_2(x) = -\lambda y_2(x), \qquad x \in [-1,0) \cup (0,1], \tag{4.7}$$

$$\sqrt{3}y_1(-1) - y_2(-1) = 0, \qquad \left(1 + \frac{1}{2}\lambda\right)y_1(1) - \left(1 + \frac{\sqrt{3}}{2}\lambda\right)y_2(1) = 0, \tag{4.8}$$

$$y_1(0^-) - 3y_1(0^+) = 0, \qquad y_2(0^-) - 3y_2(0^+) = 0, \tag{4.9}$$

Figure 3 **The enclosure interval dominating λ_{-1} for $h = 0.2$, $N = 20$ and $\varepsilon = 10^{-2}$.**

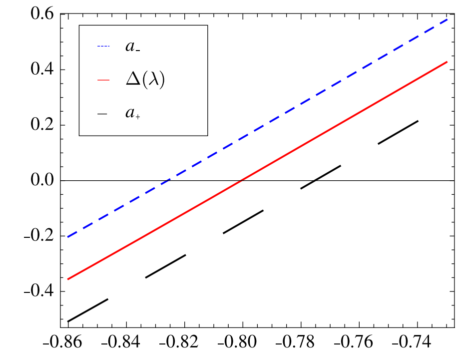


Figure 4 **The enclosure interval dominating λ_{-1} for $h = 0.2$, $N = 20$ and $\varepsilon = 10^{-5}$.**

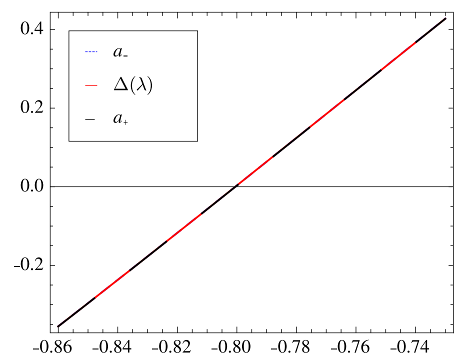


Table 4 **The approximation $\lambda_{k,N}$ and the exact solution λ_k for different choices of h and N**

λ_k			λ_{-2}	λ_{-1}	λ_0	λ_1
Exact λ_k			−1.443241990338957	−0.5507950329405884	0.8894376317278696	2.427882996831557
$\lambda_{k,N}$	$h = 0.6$,	$N = 10$	−1.4432116741528003	−0.5507870771754422	0.8894392796056301	2.4278845029050586
	$\omega = 0.9714$	$N = 20$	−1.4432419877352867	−0.5507950322921764	0.8894376316344037	2.427882996941257
	$h = 0.1$,	$N = 10$	−1.443241954240034	−0.5507950143369327	0.8894376262695777	2.4278830194325765
	$\omega = 1.4714$	$N = 20$	−1.4432419903389377	−0.5507950329405837	0.88943763172786576	2.4278829968315647

where

$$r_1(x) = r_2(x) = r(x) = \begin{cases} x + 1, & x \in [-1, 0), \\ x, & (0, 1], \end{cases} \tag{4.10}$$

$a_1 = a_2 = 1$, $\alpha = \frac{\pi}{3}$, $\beta = \frac{\pi}{6}$ and $\delta = 3$. Direct calculations give

$$\mathcal{K}(\lambda) = 3\left[\cos\left[\frac{\pi}{3} + 2\lambda\right] - \lambda \sin\left[\frac{\pi}{6} + 2\lambda\right] - \sin\left[\frac{\pi}{3} + 2\lambda\right]\right] \tag{4.11}$$

and

$$\Delta(\lambda) = -\frac{3}{2}\bigl[(-1 + \sqrt{3} + \lambda)\cos[1 + 2\lambda] + (1 + \sqrt{3} + \sqrt{3}\lambda)\sin[1 + 2\lambda]\bigr]. \tag{4.12}$$

Tables 4, 5, give the exact eigenvalues $\{\lambda_k\}_{k=-2}^{1}$ and their approximate ones for different values of h, N, ε. In Table 6, we give the absolute error for different values of h and N.

Here Figures 5, 6, 7, 8 illustrate the enclosure intervals dominating λ_0 and λ_1 for $h = 0.1$, $N = 20$ and $\varepsilon = 10^{-2}$, $\varepsilon = 10^{-5}$, respectively.

Table 5 For $N = 20$ and $h = 0.1$, the exact solutions λ_k are all inside the interval $[a_-, a_+]$ for different values of ε

λ_k	λ_{-2}	λ_{-1}	λ_0	λ_1
Exact λ_k	−1.443241990338957	−0.5507950329405884	0.8894376317278696	2.427882996831557
$I_{\varepsilon,N}$, $\varepsilon = 10^{-2}$	[−1.4716489, −1.4144426]	[−0.5736938, −0.5287366]	[0.8789632, 0.8998212]	[2.4214822, 2.4342626]
$I_{\varepsilon,N}$, $\varepsilon = 10^{-5}$	[−1.4432705, −1.4432134]	[−0.5508174, −0.5507725]	[0.8894272, 0.8894480]	[2.4278766, 2.4278893]

Table 6 Absolute error $|\lambda_k - \lambda_{k,N}|$

λ_k		λ_{-2}	λ_{-1}	λ_0	λ_1
$h = 0.6$	$N = 10$	3.03162×10^{-5}	7.95577×10^{-6}	1.64788×10^{-6}	1.50607×10^{-6}
	$N = 20$	2.60367×10^{-9}	6.48412×10^{-10}	9.34659×10^{-11}	1.097×10^{-10}
$h = 0.1$	$N = 10$	3.60989×10^{-8}	1.86037×10^{-8}	5.45829×10^{-9}	2.2601×10^{-8}
	$N = 20$	1.93179×10^{-14}	4.66294×10^{-15}	3.88578×10^{-15}	7.54952×10^{-15}

Figure 5 The enclosure interval dominating λ_0 for $h = 0.1$, $N = 20$ and $\varepsilon = 10^{-2}$.

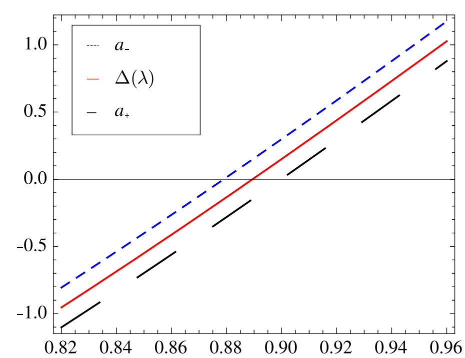


Figure 6 The enclosure interval dominating λ_0 for $h = 0.1$, $N = 20$ and $\varepsilon = 10^{-5}$.

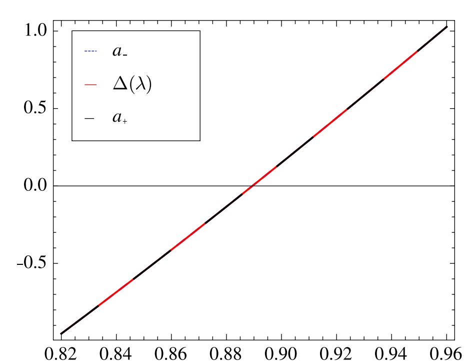


Figure 7 The enclosure interval dominating λ_1 for $h = 0.1$, $N = 20$ and $\varepsilon = 10^{-2}$.

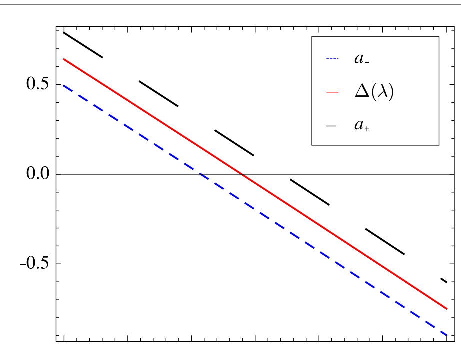
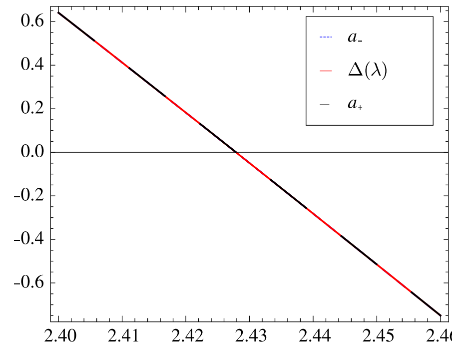


Figure 8 The enclosure interval dominating λ_1 for $h = 0.1$, $N = 20$ and $\varepsilon = 10^{-5}$.



Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors have equal contributions to each part of this article. All the authors read and approved the final manuscript.

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