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Green's function for Sturm-Liouville-type boundary value problems of fractional order impulsive differential equations and its application

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Abstract
In this paper, we discuss the expression and properties of Green's function for boundary value problems of nonlinear Sturm-Liouville-type fractional order impulsive differential equations. Its applications are also given. Our results are compared with some recent results by Bai and Lü.

Keywords: fractional differential equation; impulse; Green's function; existence; fixed point theorem

1 Introduction

Fractional differential equations arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, Bode's analysis of feedback amplifiers, capacitor theory, electrical circuits, electron-analytical chemistry, biology, control theory, fitting of experimental data, and so forth, and involves derivatives of fractional order. Fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. This is the main advantage of fractional differential equations in comparison with classical integer-order models. An excellent account in the study of fractional differential equations can be found in [1–5]. For the basic theory and recent development of the subject, we refer to a text by Lakshmikantham *et al.* [6]. For more details and examples, see [7–23] and the references therein.

Integer-order impulsive differential equations have become important in recent years as mathematical models of phenomena in both physical and social sciences. There has been a significant development in impulsive theory especially in the area of impulsive differential equations with fixed moments; see, for instance, [24–26]. Recently, the boundary value problems of impulsive differential equations of integer order have been studied extensively in the literature (see [27–36]).

On the other hand, impulsive differential equations of fractional order play an important role in theory and applications, see [37–46] and the references therein. However, as pointed out in [38, 39], the fractional impulsive differential equations have not been addressed so extensively and many aspects of these problems are yet to be explored. For example, the theory using Green's function to express the solution of fractional impulsive differential equations has not been investigated till now. Now, in this paper, we shall study the expression of the solution of fractional impulsive differential equations by using Green's function.

Consider the following nonlinear boundary value problem of fractional impulsive differential equations:

$$\begin{cases} {}^c\mathbf{D}_{0+}^q x(t) = \omega(t) f(t, x(t), x'(t)), & 1 < q \le 2, t \in J_1 = J \setminus \{t_1, t_2, \dots, t_n\}, \\ \Delta x|_{t=t_k} = \mathcal{I}_k(x(t_k)), \quad \Delta x'|_{t=t_k} = \mathcal{J}_k(x(t_k)), & t_k \in (0,1), k = 1, 2, \dots, n, \\ \alpha_1 x(0) - \beta_1 x'(0) = 0, \quad \alpha_2 x(1) + \beta_2 x'(1) = 0, \end{cases} \tag{1.1}$$

where ${}^c\mathbf{D}_{0+}^q$ is the Caputo fractional derivative, $J = [0,1]$, $\omega(t) : J \to R+$ is a continuous function, $f : J \times R \times R \to R$ is a continuous function, $\mathcal{I}_k, \mathcal{J}_k : R \to R$ are continuous functions, $\alpha_1, \alpha_2, \beta_1, \beta_2 \in R$ and $\alpha_1\alpha_2 + \alpha_1\beta_2 + \alpha_2\beta_1 \neq 0$. $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-)$ with $x(t_k^+) = \lim_{h\to 0^+} x(t_k + h)$, $x(t_k^-) = \lim_{h\to 0^-} x(t_k + h)$, $k = 1, 2, \dots, n$, $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = 1$. $\Delta x'|_{t=t_k}$ has a similar meaning for $x'(t)$.

Some special cases of (1.1) have been investigated. For example, Bai and Lü [13] considered problem (1.1) with $\mathcal{I}_k \equiv 0$ and $\mathcal{J}_k \equiv 0$. By using the fixed point theorem in cones, they proved some existence and multiplicity results of positive solutions of problem (1.1).

At the end of this section, it is worth mentioning that it is an important method to express the solution of differential equations by Green's function. According to the previous work, we find that the solution of impulsive differential equations with integer order can be expressed by Green's function of the case without impulse. For example, Green's function of the following boundary value problem

$$\begin{cases} -x''(t) = \sigma(t), & t \in (0,1), \\ x(0) = x(1) = 0 \end{cases} \tag{1.2}$$

can be expressed by

$$G(t,s) = \begin{cases} s(1-t), & 0 \le s \le t \le 1, \\ t(1-s), & 0 \le t \le s \le 1, \end{cases}$$

where $\sigma \in C[0,1]$. The solution of problem (1.2) can be expressed by

$$x(t) = \int_0^1 G(t,s)\sigma(s)\,ds.$$

If we consider impulsive differential equations

$$\begin{cases} -x''(t) = \sigma(t), & t \in (0,1), t \neq t_k, \\ \Delta x|_{t=t_k} = \mathcal{I}_k(x(t_k)), & k = 1, 2, \dots, n, \\ \Delta x'|_{t=t_k} = -\mathcal{L}_k(x(t_k)), & k = 1, 2, \dots, n, \\ x(0) = x(1) = 0, \end{cases} \tag{1.3}$$

then the solution of problem (1.3) can be expressed by

$$x(t) = \int_0^1 G(t,s)\sigma(s)\,ds + \sum_{k=1}^{n} G(t,t_k)\mathcal{L}_k(x_{t_k}) + \sum_{k=1}^{n} G'_s(t,t_k)\mathcal{I}_k(x_{t_k}),$$

where $\mathcal{I}_k, \mathcal{L}_k \in C[0,1]$, $k = 1,2,\dots,n$.

Naturally, one wishes to know whether or not the same result holds for the fractional order case. We first study the fractional order differential equations with Caputo derivatives

$$\begin{cases} -{}^c\mathbf{D}_{0+}^q x(t) = \sigma(t), & t \in (0,1), 1 < q \le 2, \\ x(0) = x(1) = 0, \end{cases} \tag{1.4}$$

where ${}^c\mathbf{D}_{0+}^q$ is the Caputo fractional derivative. Then

$$x(t) = \int_0^1 G(t,s)\sigma(s)\,ds,$$

where

$$G(t,s) = \frac{1}{\Gamma(q)} \begin{cases} t(1-s)^{q-1} - (t-s)^{q-1}, & 0 \le s \le t \le 1, \\ t(1-s)^{q-1}, & 0 \le t \le s \le 1. \end{cases}$$

Then we study whether the solution of fractional order impulsive differential equations with Caputo derivatives

$$\begin{cases} -{}^c\mathbf{D}_{0+}^q x(t) = \sigma(t), & t \in (0,1), 1 < q \le 2, \\ \Delta x|_{t=t_k} = \mathcal{I}_k(x(t_k)), & k = 1,2,\dots,n, \\ \Delta x'|_{t=t_k} = -\mathcal{L}_k(x(t_k)), & k = 1,2,\dots,n, \\ x(0) = x(1) = 0 \end{cases} \tag{1.5}$$

can be expressed by

$$x(t) = \int_0^1 G(t,s)\sigma(s)\,ds + \sum_{k=1}^{n} G(t,t_k)\mathcal{L}_k(x_{t_k}) + \sum_{k=1}^{n} G'_s(t,t_k)\mathcal{I}_k(x_{t_k}).$$

Thus, it is an interesting problem, and so it is worthwhile to study. We will give the answers in the following sections.

The organization of this paper is as follows. In Section 2, we present the expression and properties of Green's function associated with problem (1.1). In Section 3, we give some preliminaries about the operator and the fixed point theorem. In Section 4, we get some existence results for problem (1.1) by means of some standard fixed point theorems. The final section of the paper contains two examples to illustrate our main results.

2 Expression and properties of Green's function

Consider the following fractional impulsive boundary value problem:

$$\begin{cases} {}^c\mathbf{D}_{0+}^q x(t) = \sigma(t), \quad 1 < q \le 2, t \in J_1 = J \setminus \{t_1, t_2, \dots, t_n\}, \\ \Delta x|_{t=t_k} = \mathcal{I}_k(x(t_k)), \quad \Delta x'|_{t=t_k} = \mathcal{J}_k(x(t_k)), \quad t_k \in (0,1), k = 1, 2, \dots, n, \\ \alpha_1 x(0) - \beta_1 x'(0) = 0, \quad \alpha_2 x(1) + \beta_2 x'(1) = 0, \end{cases} \tag{2.1}$$

where ${}^c\mathbf{D}_{0+}^q$ is the Caputo fractional derivative, $\sigma(t) \in C[0,1]$, $\mathcal{I}_k, \mathcal{J}_k : R \to R$ are continuous functions, $\alpha_1, \alpha_2, \beta_1, \beta_2 \in R$ and $\alpha_1\alpha_2 + \alpha_1\beta_2 + \alpha_2\beta_1 \neq 0$. $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-)$ with $x(t_k^+) = \lim_{h\to 0^+} x(t_k + h)$, $x(t_k^-) = \lim_{h\to 0^-} x(t_k + h)$, $k = 1, 2, \dots, n$, $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = 1$. $\Delta x'|_{t=t_k}$ has a similar meaning for $x'(t)$.

Theorem 2.1 *The solution of problem* (2.1) *can be expressed by*

$$\begin{aligned} x(t) = {} & \int_{t_k}^t \frac{(t-s)^{q-1}}{\Gamma(q)} \sigma(s)\, ds + \sum_{i=1}^{n+1} G'_{1s}(t, t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{q-1}}{\Gamma(q)} \sigma(s)\, ds \\ & - \sum_{i=1}^{n} G_1(t, t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{q-2}}{\Gamma(q-1)} \sigma(s)\, ds + \sum_{i=1}^{n} G'_{1s}(t, t_i) \mathcal{I}_i\big(x(t_i)\big) \\ & - \sum_{i=1}^{n} G_1(t, t_i) \mathcal{J}_i\big(x(t_i)\big), \quad t \in (t_k, t_{k+1}], k = 0, 1, 2, \dots, n, t_0 = 0, t_{n+1} = 1, \end{aligned} \tag{2.2}$$

where

$$G_1(t,s) = -\frac{1}{\eta} \begin{cases} (\beta_1 + \alpha_1 t)(\alpha_2 + \beta_2 - \alpha_2 s), & t \le s, \\ (\beta_1 + \alpha_1 s)(\alpha_2 + \beta_2 - \alpha_2 t), & t \ge s, \end{cases} \tag{2.3}$$

and

$$\eta = \alpha_1\alpha_2 + \alpha_1\beta_2 + \alpha_2\beta_1. \tag{2.4}$$

Proof Suppose that x is a solution of (2.1). Then, for some constants $b_0, b_1 \in R$, we have

$$x(t) = I_{0^+}^q \sigma(t) - b_0 - b_1 t = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \sigma(s)\, ds - b_0 - b_1 t, \quad t \in [0, t_1]. \tag{2.5}$$

It follows from (2.5) that

$$x'(t) = \int_0^t \frac{(t-s)^{q-2}}{\Gamma(q-1)} \sigma(s)\, ds - b_1.$$

If $t \in (t_1, t_2]$, then, for some constants $c_0, c_1 \in R$, we can write

$$\begin{aligned} x(t) &= \int_{t_1}^t \frac{(t-s)^{q-1}}{\Gamma(q)} \sigma(s)\, ds - c_0 - c_1(t - t_1), \\ x'(t) &= \int_{t_1}^t \frac{(t-s)^{q-2}}{\Gamma(q-1)} \sigma(s)\, ds - c_1. \end{aligned}$$

Using the impulse conditions $\Delta x|_{t=t_1} = \mathcal{I}_1(x(t_1))$ and $\Delta x'|_{t=t_1} = \mathcal{J}_1(x(t_1))$, we find that

$$-c_0 = \int_0^{t_1} \frac{(t_1-s)^{q-1}}{\Gamma(q)}\sigma(s)\,ds - b_0 - b_1 t_1 + \mathcal{I}_1\bigl(x(t_1)\bigr),$$
$$-c_1 = \int_0^{t_1} \frac{(t_1-s)^{q-2}}{\Gamma(q-1)}\sigma(s)\,ds - b_1 + \mathcal{J}_1\bigl(x(t_1)\bigr).$$

Thus

$$\begin{aligned} x(t) &= \int_{t_1}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)}\sigma(s)\,ds + \int_0^{t_1}\left(\frac{(t-t_1)(t_1-s)^{q-2}}{\Gamma(q-1)} + \frac{(t_1-s)^{q-1}}{\Gamma(q)}\right)\sigma(s)\,ds \\ &\quad - b_0 - b_1 t + \mathcal{J}_1\bigl(x(t_1)\bigr)(t-t_1) + \mathcal{I}_1\bigl(x(t_1)\bigr), \\ x'(t) &= \int_{t_1}^{t} \frac{(t-s)^{q-2}}{\Gamma(q-1)}\sigma(s)\,ds + \int_0^{t_1} \frac{(t_1-s)^{q-2}}{\Gamma(q-1)}\sigma(s)\,ds - b_1 + \mathcal{J}_1\bigl(x(t_1)\bigr). \end{aligned}$$

If $t \in (t_k, t_{k+1}]$, repeating the above procedure, we obtain

$$\begin{aligned} x(t) &= \int_{t_k}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)}\sigma(s)\,ds + \sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\left(\frac{(t-t_i)(t_i-s)^{q-2}}{\Gamma(q-1)} + \frac{(t_i-s)^{q-1}}{\Gamma(q)}\right)\sigma(s)\,ds \\ &\quad - b_0 - b_1 t + \sum_{i=1}^{k}\mathcal{J}_i\bigl(x(t_i)\bigr)(t-t_i) + \sum_{i=1}^{k}\mathcal{I}_i\bigl(x(t_i)\bigr). \end{aligned} \tag{2.6}$$

It follows that $x(0) = -b_0$, $x'(0) = -b_1$ and

$$\begin{aligned} x(1) &= \int_{t_n}^{1} \frac{(1-s)^{q-1}}{\Gamma(q)}\sigma(s)\,ds + \sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\left(\frac{(1-t_i)(t_i-s)^{q-2}}{\Gamma(q-1)} + \frac{(t_i-s)^{q-1}}{\Gamma(q)}\right)\sigma(s)\,ds \\ &\quad - b_0 - b_1 + \sum_{i=1}^{n}\mathcal{J}_i\bigl(x(t_i)\bigr)(1-t_i) + \sum_{i=1}^{n}\mathcal{I}_i\bigl(x(t_i)\bigr), \\ x'(1) &= \int_{t_n}^{1} \frac{(1-s)^{q-2}}{\Gamma(q-1)}\sigma(s)\,ds + \sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}\sigma(s)\,ds - b_1 + \sum_{i=1}^{n}\mathcal{J}_i\bigl(x(t_i)\bigr). \end{aligned}$$

By the boundary conditions, we have

$$\begin{aligned} b_0 &= \frac{1}{\eta}\Biggl\{\alpha_2\beta_1\Biggl[\int_{t_n}^{1} \frac{(1-s)^{q-1}}{\Gamma(q)}\sigma(s)\,ds \\ &\quad + \sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\left(\frac{(1-t_i)(t_i-s)^{q-2}}{\Gamma(q-1)} + \frac{(t_i-s)^{q-1}}{\Gamma(q)}\right)\sigma(s)\,ds \\ &\quad + \sum_{i=1}^{n}\mathcal{J}_i\bigl(x(t_i)\bigr)(1-t_i) + \sum_{i=1}^{n}\mathcal{I}_i\bigl(x(t_i)\bigr)\Biggr] + \beta_1\beta_2\Biggl[\int_{t_n}^{1} \frac{(1-s)^{q-2}}{\Gamma(q-1)}\sigma(s)\,ds \\ &\quad + \sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}\sigma(s)\,ds + \sum_{i=1}^{n}\mathcal{J}_i\bigl(x(t_i)\bigr)\Biggr]\Biggr\}, \end{aligned} \tag{2.7}$$

$$b_1 = \frac{1}{\eta}\Biggl\{\alpha_1\alpha_2\Biggl[\int_{t_n}^{1} \frac{(1-s)^{q-1}}{\Gamma(q)}\sigma(s)\,ds + \sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\left(\frac{(1-t_i)(t_i-s)^{q-2}}{\Gamma(q-1)} + \frac{(t_i-s)^{q-1}}{\Gamma(q)}\right)\sigma(s)\,ds$$

$$+\sum_{i=1}^{n}\mathcal{J}_i\big(x(t_i)\big)(1-t_i)+\sum_{i=1}^{n}\mathcal{I}_i\big(x(t_i)\big)\Bigg]+\alpha_1\beta_2\Bigg[\int_{t_n}^{1}\frac{(1-s)^{q-2}}{\Gamma(q-1)}\sigma(s)\,ds$$
$$+\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}\sigma(s)\,ds+\sum_{i=1}^{n}\mathcal{J}_i\big(x(t_i)\big)\Bigg]\Bigg\}, \tag{2.8}$$

where η is defined in (2.4).

Substituting (2.7), (2.8) into (2.6), we obtain (2.2). This completes the proof. □

Remark 2.1 It is clear that $G_1(t,s)$ is Green's function of the boundary value problem

$$\begin{cases} x''(t)=\sigma(t), & t\in(0,1),\\ \alpha_1x(0)-\beta_1x'(0)=0, \\ \alpha_2x(1)+\beta_2x'(1)=0. \end{cases}$$

Remark 2.2 The expression of the solution of problem (2.1) is simpler than that of [37–46].

From (2.3), we can prove the following results.

Proposition 2.1 *For all $t,s\in J$, we have*

$$\big|G_1(t,s)\big|\le\frac{2(|\beta_1\alpha_2|+|\alpha_1\alpha_2|)+|\beta_1\beta_2|+|\alpha_1\beta_2|}{|\eta|}. \tag{2.9}$$

Proposition 2.2 *For all $t,s\in J$, we have*

$$\big|G'_{1s}(t,s)\big|\le\max\left\{\frac{|\alpha_2\beta_1|+|\alpha_1\alpha_2|}{|\eta|},\frac{|\alpha_1\beta_2|+2|\alpha_1\alpha_2|}{|\eta|}\right\},$$
$$\big|G'_{1t}(t,s)\big|\le\max\left\{\frac{|\alpha_1\beta_2|+2|\alpha_1\alpha_2|}{|\eta|},\frac{|\alpha_2\beta_1|+|\alpha_1\alpha_2|}{|\eta|}\right\}$$

and

$$\big|G''_{1st}(t,s)\big|=\frac{|\alpha_1\alpha_2|}{|\eta|},$$

where

$$G'_{1s}(t,s)=-\frac{1}{\eta}\begin{cases}-\alpha_2(\beta_1+\alpha_1t), & t\le s,\\ \alpha_1(\alpha_2+\beta_2-\alpha_2t), & t\ge s,\end{cases}$$
$$G'_{1t}(t,s)=-\frac{1}{\eta}\begin{cases}\alpha_1(\alpha_2+\beta_2-\alpha_2s), & t\le s,\\ -\alpha_2(\beta_1+\alpha_1s), & t\ge s,\end{cases}$$

and

$$G''_{1st}(t,s)=\frac{\alpha_1\alpha_2}{\eta}.$$

For the sake of convenience, let

$$\begin{aligned} c_1 &= \frac{2(|\beta_1\alpha_2| + |\alpha_1\alpha_2|) + |\beta_1\beta_2| + |\alpha_1\beta_2|}{|\eta|}, \\ c_2 &= \max\left\{\frac{|\alpha_2\beta_1| + |\alpha_1\alpha_2|}{|\eta|}, \frac{|\alpha_1\beta_2| + 2|\alpha_1\alpha_2|}{|\eta|}\right\}, \qquad c_3 = \frac{|\alpha_1\alpha_2|}{|\eta|}. \end{aligned} \tag{2.10}$$

Then it follows from (2.9) *and* (2.10) *that*

$$\left|G_1(t,s)\right| \le c_1, \qquad \left|G'_{1t}(t,s)\right| \le c_2, \qquad \left|G'_{1s}(t,s)\right| \le c_2, \qquad \left|G'_{1st}(t,s)\right| \le c_3. \tag{2.11}$$

From the proof of Theorem 2.1 we have the following results.

Proposition 2.3 *The solution of fractional impulsive differential equations can be expressed by Green's function, and it is not Green's function of the corresponding fractional differential equations, but Green's function of the corresponding integer order differential equations.*

3 Preliminaries

In this section, we give some preliminaries for discussing the solvability of problem (1.1) as follows.

Let $J' = [0,1]\backslash\{t_1, t_2, \dots, t_n\}$ and

$$\begin{aligned} PC[J,R] = \bigl\{&x : J \to R; x \in C\bigl((t_k, t_{k+1}), R\bigr), x\bigl(t_k^+\bigr) \text{ and } x\bigl(t_k^-\bigr) \text{ exist with } x\bigl(t_k^-\bigr) = x(t_k), \\ &k = 1, 2, \dots, n\bigr\}, \\ PC^1[J,R] = \bigl\{&x' \in PC[J,R]; x'\bigl(t_k^+\bigr), x\bigl(t_k^-\bigr) \text{ exist and } x' \text{ is left continuous at } t_k, \\ &k = 1, 2, \dots, n\bigr\}. \end{aligned}$$

Then $PC[J,R]$ is a Banach space with the norm

$$\|x\|_{PC} = \sup_{t\in J}\bigl|x(t)\bigr|,$$

$PC^1[J,R]$ is a Banach space with the norm

$$\|x\|_{PC^1} = \max\bigl\{\|x\|_{PC}, \bigl\|x'\bigr\|_{PC}\bigr\}.$$

Definition 3.1 A function $x \in PC^1[J,R] \cap C^2[J',R]$ with its Caputo derivative of order q existing on J is a solution of problem (1.1) if it satisfies (1.1).

We give the following hypotheses:

(H_1) $\omega : J \to [0, +\infty)$ is a continuous function, and there exists $t_0 \in J$ such that $\omega(t_0) > 0$;
(H_2) $f : J \times R \times R \to R$ is a continuous function;
(H_3) $\mathcal{I}_k, \mathcal{J}_k : R \to R$ are continuous functions.

It follows from Theorem 2.1 that:

Lemma 3.1 *If* (H_1)-(H_3) *hold, then a function* $x \in PC^1[J,R] \cap C^2[J',R]$ *is a solution of problem* (1.1) *if and only if* $x \in PC^1[J,R]$ *is a solution of the impulsive fractional integral equation*

$$\begin{aligned} x(t) = {} & \int_{t_k}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} \omega(s) f\bigl(s, x(s), x'(s)\bigr)\,ds \\ & + \sum_{i=1}^{n+1} G'_{1s}(t,t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-1}}{\Gamma(q)} \omega(s) f\bigl(s, x(s), x'(s)\bigr)\,ds \\ & - \sum_{i=1}^{n} G_1(t,t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-2}}{\Gamma(q-1)} \omega(s) f\bigl(s, x(s), x'(s)\bigr)\,ds \\ & + \sum_{i=1}^{n} G'_{1s}(t,t_i) \mathcal{I}_i\bigl(x(t_i)\bigr) - \sum_{i=1}^{n} G_1(t,t_i) \mathcal{J}_i\bigl(x(t_i)\bigr), \\ & t \in (t_k, t_{k+1}], k = 0,1,2,\ldots,n, t_0 = 0, t_{n+1} = 1. \end{aligned} \tag{3.1}$$

Define $T : PC^1[J,R] \to PC^1[J,R]$ by

$$\begin{aligned} (Tx)(t) = {} & \int_{t_k}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} \omega(s) f\bigl(s, x(s), x'(s)\bigr)\,ds \\ & + \sum_{i=1}^{n+1} G'_{1s}(t,t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-1}}{\Gamma(q)} \omega(s) f\bigl(s, x(s), x'(s)\bigr)\,ds \\ & - \sum_{i=1}^{n} G_1(t,t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-2}}{\Gamma(q-1)} \omega(s) f\bigl(s, x(s), x'(s)\bigr)\,ds \\ & + \sum_{i=1}^{n} G'_{1s}(t,t_i) \mathcal{I}_i\bigl(x(t_i)\bigr) - \sum_{i=1}^{n} G_1(t,t_i) \mathcal{J}_i\bigl(x(t_i)\bigr), \\ & t \in (t_k, t_{k+1}], k = 0,1,2,\ldots,n, t_0 = 0, t_{n+1} = 1. \end{aligned} \tag{3.2}$$

Using Lemma 3.1, problem (1.1) reduces to a fixed point problem $x = Tx$, where T is given by (3.2). Thus problem (1.1) has a solution if and only if the operator T has a fixed point.

From (3.2) and Lemma 3.1, it is easy to obtain the following result.

Lemma 3.2 *Assume that* (H_1)-(H_3) *hold. Then* $T : PC^1[J,R] \to PC^1[J,R]$ *is completely continuous.*

Proof Note that the continuity of f, ω, $\mathcal{I}_k$ and $\mathcal{J}_k$ together with $G_1(t,s)$ and $G'_{1s}(t,s)$ ensures the continuity of T.

Let $\Omega \subset PC^1[J,R]$ be bounded. Then there exist positive constants μ_1, μ_2 and μ_3 such that $|f(t,x(t),x'(t))| \le \mu_1$, $|\mathcal{I}_k(x)| \le \mu_2$ and $|\mathcal{J}_k(x)| \le \mu_3$, $\forall x \in \Omega$. Thus, $\forall x \in \Omega$, we have

$$\begin{aligned} \bigl|(Tx)(t)\bigr| \le {} & \int_{t_k}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} \omega(s) \bigl|f\bigl(s, x(s), x'(s)\bigr)\bigr|\,ds \\ & + \sum_{i=1}^{n+1} \bigl|G'_{1s}(t,t_i)\bigr| \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-1}}{\Gamma(q)} \omega(s) \bigl|f\bigl(s, x(s), x'(s)\bigr)\bigr|\,ds \end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{n}\bigl|G_1(t,t_i)\bigr|\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}\omega(s)\bigl|f\bigl(s,x(s),x'(s)\bigr)\bigr|\,ds\\
&+\sum_{i=1}^{n}\bigl|G'_{1s}(t,t_i)\bigr|\bigl|\mathcal{I}_i\bigl(x(t_i)\bigr)\bigr|+\sum_{i=1}^{n}\bigl|G_1(t,t_i)\bigr|\bigl|\mathcal{J}_i\bigl(x(t_i)\bigr)\bigr|\\
\le{}&\frac{\gamma\mu_1}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1}\,ds+\frac{\gamma\mu_1}{\Gamma(q)}\sum_{i=1}^{n+1}\bigl|G'_{1s}(t,t_i)\bigr|\int_{t_{i-1}}^{t_i}(t_i-s)^{q-1}\,ds\\
&+\frac{\gamma\mu_1}{\Gamma(q-1)}\sum_{i=1}^{n}\bigl|G_1(t,t_i)\bigr|\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}\,ds\\
&+\mu_2\sum_{i=1}^{n}\bigl|G'_{1s}(t,t_i)\bigr|+\mu_3\sum_{i=1}^{n}\bigl|G_1(t,t_i)\bigr|\\
={}&\frac{\gamma\mu_1}{\Gamma(q+1)}+\frac{\gamma\mu_1}{\Gamma(q+1)}\sum_{i=1}^{n+1}\bigl|G'_{1s}(t,t_i)\bigr|+\frac{\gamma\mu_1}{\Gamma(q)}\sum_{i=1}^{n}\bigl|G_1(t,t_i)\bigr|\\
&+\mu_2\sum_{i=1}^{n}\bigl|G'_{1s}(t,t_i)\bigr|+\mu_3\sum_{i=1}^{n}\bigl|G_1(t,t_i)\bigr|\\
\le{}&\gamma\mu_1\biggl[\frac{1}{\Gamma(q+1)}+\frac{1}{\Gamma(q+1)}(n+1)c_2+\frac{1}{\Gamma(q)}nc_1\biggr]\\
&+n(c_2\mu_2+c_1\mu_3):=\mu,
\end{aligned}
\tag{3.3}
$$

where

$$
\gamma=\max_{t\in J}\omega(t).
\tag{3.4}
$$

Furthermore, for any $t\in(t_k,t_{k+1}]$, $0\le k\le n$, we obtain

$$
\begin{aligned}
\bigl|(Tx)'(t)\bigr|\le{}&\int_{t_k}^{t}\frac{(t-s)^{q-2}}{\Gamma(q-1)}\omega(s)\bigl|f\bigl(s,x(s),x'(s)\bigr)\bigr|\,ds\\
&+\sum_{i=1}^{n+1}\bigl|G''_{1st}(t,t_i)\bigr|\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-1}}{\Gamma(q)}\omega(s)\bigl|f\bigl(s,x(s),x'(s)\bigr)\bigr|\,ds\\
&+\sum_{i=1}^{n}\bigl|G'_{1t}(t,t_i)\bigr|\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}\omega(s)\bigl|f\bigl(s,x(s),x'(s)\bigr)\bigr|\,ds\\
&+\sum_{i=1}^{n}\bigl|G''_{1st}(t,t_i)\bigr|\bigl|\mathcal{I}_i\bigl(x(t_i)\bigr)\bigr|+\sum_{i=1}^{n}\bigl|G'_{1t}(t,t_i)\bigr|\bigl|\mathcal{J}_i\bigl(x(t_i)\bigr)\bigr|\\
\le{}&\frac{\gamma\mu_1}{\Gamma(q-1)}\int_{t_k}^{t}(t-s)^{q-2}\,ds+\frac{\gamma\mu_1}{\Gamma(q)}\sum_{i=1}^{n+1}\bigl|G''_{1st}(t,t_i)\bigr|\int_{t_{i-1}}^{t_i}(t_i-s)^{q-1}\,ds\\
&+\frac{\gamma\mu_1}{\Gamma(q-1)}\sum_{i=1}^{n}\bigl|G'_{1t}(t,t_i)\bigr|\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}\,ds\\
&+\mu_2\sum_{i=1}^{n}\bigl|G''_{1st}(t,t_i)\bigr|+\mu_3\sum_{i=1}^{n}\bigl|G'_{1t}(t,t_i)\bigr|
\end{aligned}
$$

$$= \frac{\gamma\mu_1}{\Gamma(q)} + \frac{\gamma\mu_1}{\Gamma(q+1)}\sum_{i=1}^{n+1}\left|G''_{1st}(t,t_i)\right| + \frac{\gamma\mu_1}{\Gamma(q)}\sum_{i=1}^{n}\left|G'_{1t}(t,t_i)\right|$$
$$+ \mu_2\sum_{i=1}^{n}\left|G''_{1st}(t,t_i)\right| + \mu_3\sum_{i=1}^{n}\left|G'_{1t}(t,t_i)\right|$$
$$\le \gamma\mu_1\left[\frac{1}{\Gamma(q)} + \frac{1}{\Gamma(q+1)}(n+1)c_3 + \frac{1}{\Gamma(q)}nc_2\right] + nc_3\mu_2 + nc_2\mu_3 := \bar{\mu}. \tag{3.5}$$

On the other hand, from (3.5), for $t_1, t_2 \in (t_k, t_{k+1}]$ with $t_1 < t_2$, we have

$$\left|(Tx)'(t_2) - (Tx)'(t_1)\right| \le \int_{t_1}^{t_2}\left|(Tx)'(s)\right|ds \le \bar{\mu}(t_2 - t_1). \tag{3.6}$$

It follows from (3.3), (3.5) and (3.6) that T is equicontinuous on all subintervals $(t_k, t_{k+1}]$, $k = 1, 2, \dots, n$. Thus, by the Arzela-Ascoli theorem, the operator $T : PC^1[J,R] \to PC^1[J,R]$ is completely continuous. □

To prove our main results, we also need the following two lemmas.

Lemma 3.3 (See [47, 48]) (Schauder fixed point theorem) *Let D be a nonempty, closed, bounded, convex subset of a B-space X, and suppose that $T : D \to D$ is a completely continuous operator. Then T has a fixed point $x \in D$.*

Lemma 3.4 (See [47, 48]) (Leray-Schauder fixed point theorem) *Let X be a real Banach space and $T : X \to X$ be a completely continuous operator. If*

$$\{x : x \in X, x = \lambda Tx, 0 < \lambda < 1\}$$

is bounded, then T has a fixed point $x^* \in \Omega$, where

$$\Omega = \{x : x \in X, \|x\| \le l\}, \quad l = \sup\{x : x \in X, x = \lambda Tx, 0 < \lambda < 1\}.$$

4 Existence of solutions

In this section, we apply Lemma 3.3, Lemma 3.4 and the contraction mapping principle to establish the existence of solutions of problem (1.1). Let us begin by introducing some notation. Define

$$\xi = \overline{\lim_{|x|+|y|\to\infty}}\left(\max_{t\in J}\frac{|f(t,x,y)|}{|x|+|y|}\right),$$
$$\xi_1 = \overline{\lim_{|x|\to\infty}}\frac{|\mathrm{I}_k(x)|}{|x|}, \qquad \xi_2 = \overline{\lim_{|x|\to\infty}}\frac{|\mathcal{J}_k(x)|}{|x|}, \quad k = 1, 2, \dots, n.$$

Theorem 4.1 *Assume that* (H_1)-(H_3) *hold. Suppose further that*

$$\delta = \max\{\delta_1, \delta_2\} < 1, \tag{4.1}$$

where

$$\delta_1 = \gamma\left[\frac{2\xi(1 + (n+1)c_2)}{\Gamma(q+1)} + \frac{2\xi nc_1}{\Gamma(q)}\right] + n(\xi_1c_2 + \xi_2c_1)$$

and

$$\delta_2 = \gamma\left[\frac{2\xi(1+nc_2)}{\Gamma(q)} + \frac{2\xi(n+1)c_3}{\Gamma(q+1)}\right] + n(\xi_1 c_3 + \xi_2 c_2),$$

here γ *is defined in* (3.4). *Then problem* (1.1) *has at least one solution* $x \in PC^1[J,R] \cap C^2[J',R]$.

Proof We shall use Schauder's fixed point theorem to prove that T has a fixed point. First, recall that the operator $T : PC^1[J,R] \to PC^1[J,R]$ is completely continuous (see the proof of Lemma 3.2).

On account of (4.1), we can choose $\xi' > \xi$, $\xi_1' > \xi_1$ and $\xi_2' > \xi_2$ such that

$$\delta_1' = \gamma\left[\frac{2\xi'(1+(n+1)c_2)}{\Gamma(q+1)} + \frac{2\xi' nc_1}{\Gamma(q)}\right] + n(\xi_1' c_2 + \xi_2' c_1) < 1 \tag{4.2}$$

and

$$\delta_2' = \gamma\left[\frac{2\xi'(1+nc_2)}{\Gamma(q)} + \frac{2\xi'(n+1)c_3}{\Gamma(q+1)}\right] + n(\xi_1' c_3 + \xi_2' c_2) < 1. \tag{4.3}$$

By the definition of ξ, there exists $l > 0$ such that

$$\left|f(t,x,y)\right| < \xi'(|x| + |y|), \quad \forall t \in J, |x| + |y| > l,$$

so

$$\left|f(t,x,y)\right| < \xi'(|x| + |y|) + M, \quad \forall t \in J, x, y \in R, \tag{4.4}$$

where

$$M = \max_{t\in J, |x|+|y|\le l}\left|f(t,x,y)\right| < +\infty.$$

Similarly, we have

$$\left|\mathcal{I}_k(x)\right| < \xi_1'|x| + M_k, \quad \forall x \in R, k = 1,2,\ldots,n, \tag{4.5}$$

and

$$\left|\mathcal{J}_k(x)\right| \le \xi_2'|x| + \bar{M}_k, \quad \forall x \in R, k = 1,2,\ldots,n, \tag{4.6}$$

where M_k, $\bar{M}_k$ are positive constants.

It follows from (3.2) and (4.4)-(4.6) that

$$\begin{aligned}\left|(Tx)(t)\right| \le{}& \int_{t_k}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)}\omega(s)\left|f(s,x(s),y(s))\right| ds \\ &+ \sum_{i=1}^{n+1}\left|G_{1s}'(t,t_i)\right|\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-1}}{\Gamma(q)}\omega(s)\left|f(s,x(s),y(s))\right| ds\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{n}\left|G_1(t,t_i)\right|\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}\omega(s)\left|f\left(s,x(s),y(s)\right)\right|ds\\
&+\sum_{i=1}^{n}\left|G'_{1s}(t,t_i)\right|\left|\mathcal{I}_i\left(x(t_i)\right)\right|+\sum_{i=1}^{n}\left|G_1(t,t_i)\right|\left|\mathcal{J}_i\left(x(t_i)\right)\right|\\
\le{}&\gamma\left(\xi'\left(|x|+|y|\right)+M\right)\int_{t_k}^{t}\frac{(t-s)^{q-1}}{\Gamma(q)}\,ds\\
&+\gamma\left(\xi'\left(|x|+|y|\right)+M\right)\sum_{i=1}^{n+1}\left|G'_{1s}(t,t_i)\right|\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-1}}{\Gamma(q)}\,ds\\
&+\gamma\left(\xi'\left(|x|+|y|\right)+M\right)\sum_{i=1}^{n}\left|G_1(t,t_i)\right|\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}\,ds\\
&+\left(\xi'_1|x|+M_k\right)\sum_{i=1}^{n}\left|G'_{1s}(t,t_i)\right|+\left(\xi'_2|x|+\bar{M}_k\right)\sum_{i=1}^{n}\left|G_1(t,t_i)\right|\\
\le{}&\gamma\left[\frac{\xi'(|x|+|y|)+M}{\Gamma(q+1)}+\frac{\xi'(|x|+|y|)+M}{\Gamma(q+1)}(n+1)c_2+\frac{\xi'(|x|+|y|)+M}{\Gamma(q)}nc_1\right]\\
&+\left(\xi'_1|x|+M_k\right)nc_2+\left(\xi'_2|x|+\bar{M}_k\right)nc_1\\
\le{}&\gamma\left[\frac{2\xi'\|x\|_{PC^1}+M}{\Gamma(q+1)}+\frac{2\xi'\|x\|_{PC^1}+M}{\Gamma(q+1)}(n+1)c_2+\frac{2\xi'\|x\|_{PC^1}+M}{\Gamma(q)}nc_1\right]\\
&+\left(\xi'_1\|x\|_{PC^1}+M_k\right)nc_2+\left(\xi'_2\|x\|_{PC^1}+\bar{M}_k\right)nc_1\\
={}&\delta'_1\|x\|_{PC^1}+M^{(1)},
\end{aligned}
\tag{4.7}
$$

where δ'_1 is defined by (4.2) and $M^{(1)}$ is defined by

$$
M^{(1)}=\gamma\left[\frac{M(1+(n+1)c_2)}{\Gamma(q+1)}+\frac{Mnc_1}{\Gamma(q)}\right]+n(c_2M_k+c_1\bar{M}_k).
$$

Similarly, from (3.2) and (4.4)-(4.6), we get

$$
\begin{aligned}
\left|(Tx)'(t)\right|\le{}&\int_{t_k}^{t}\frac{(t-s)^{q-2}}{\Gamma(q-1)}\omega(s)\left|f\left(s,x(s),x'(s)\right)\right|ds\\
&+\sum_{i=1}^{n+1}\left|G''_{1st}(t,t_i)\right|\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-1}}{\Gamma(q)}\omega(s)\left|f\left(s,x(s),x'(s)\right)\right|ds\\
&+\sum_{i=1}^{n}\left|G'_{1t}(t,t_i)\right|\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}\omega(s)\left|f\left(s,x(s),x'(s)\right)\right|ds\\
&+\sum_{i=1}^{n}\left|G''_{1st}(t,t_i)\right|\left|\mathcal{I}_i\left(x(t_i)\right)\right|+\sum_{i=1}^{n}\left|G'_{1t}(t,t_i)\right|\left|\mathcal{J}_i\left(x(t_i)\right)\right|\\
\le{}&\gamma\left(\xi'\left(|x|+|y|\right)+M\right)\int_{t_k}^{t}\frac{(t-s)^{q-2}}{\Gamma(q-1)}\,ds\\
&+\gamma\left(\xi'\left(|x|+|y|\right)+M\right)\sum_{i=1}^{n+1}\left|G''_{1st}(t,t_i)\right|\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-1}}{\Gamma(q)}\,ds
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \gamma\big(\xi'\big(|x| + |y|\big) + M\big)\sum_{i=1}^{n}\big|G'_{1t}(t,t_i)\big|\int_{t_{i-1}}^{t_i}\frac{(t_i - s)^{q-2}}{\Gamma(q-1)}\,ds \\
&\quad + \big(\xi_1'|x| + M_k\big)\sum_{i=1}^{n}\big|G''_{1st}(t,t_i)\big| + \big(\xi_2'|x| + \bar{M}_k\big)\sum_{i=1}^{n}\big|G'_{1t}(t,t_i)\big| \\
&\le \gamma\big(2\xi'\|x\|_{PC^1} + M\big)\int_{t_k}^{t}\frac{(t-s)^{q-2}}{\Gamma(q-1)}\,ds \\
&\quad + \gamma\big(2\xi'\|x\|_{PC^1} + M\big)\sum_{i=1}^{n+1}\big|G''_{1st}(t,t_i)\big|\int_{t_{i-1}}^{t_i}\frac{(t_i - s)^{q-1}}{\Gamma(q)}\,ds \\
&\quad + \gamma\big(2\xi'\|x\|_{PC^1} + M\big)\sum_{i=1}^{n}\big|G'_{1t}(t,t_i)\big|\int_{t_{i-1}}^{t_i}\frac{(t_i - s)^{q-2}}{\Gamma(q-1)}\,ds \\
&\quad + \big(\xi_1'\|x\|_{PC^1} + M_k\big)\sum_{i=1}^{n}\big|G''_{1st}(t,t_i)\big| + \big(\xi_2'\|x\|_{PC^1} + \bar{M}_k\big)\sum_{i=1}^{n}\big|G'_{1t}(t,t_i)\big| \\
&\le \gamma\left[\frac{2\xi'\|x\|_{PC^1} + M}{\Gamma(q)} + \frac{2\xi'\|x\|_{PC^1} + M}{\Gamma(q+1)}(n+1)c_3 + \frac{2\xi'\|x\|_{PC^1} + M}{\Gamma(q)}nc_2\right] \\
&\quad + \big(\xi_1'\|x\|_{PC^1} + M_k\big)nc_3 + \big(\xi_2'\|x\|_{PC^1} + \bar{M}_k\big)nc_2 \\
&= \delta_2'\|x\|_{PC^1} + M^{(2)},
\end{aligned}
\tag{4.8}
$$

where δ_1' is defined by (4.3) and $M^{(2)}$ is defined by

$$
M^{(2)} = \gamma\left[\frac{M(1 + nc_2)}{\Gamma(q)} + \frac{(n+1)c_3M}{\Gamma(q+1)}\right] + n(M_kc_3 + \bar{M}_kc_2).
$$

It follows from (4.7) and (4.8) that

$$
\|Tx\|_{PC^1} \le \delta'\|x\|_{PC^1} + M', \quad \forall x \in PC^1[J,E],
$$

where

$$
\delta' = \max\big\{\delta_1', \delta_2'\big\} < 1, \qquad M' = \max\big\{M^{(1)}, M^{(2)}\big\}.
$$

Hence, we can choose a sufficiently large $r > 0$ such that $T(B_r) \subset B_r$, where

$$
B_r = \big\{x \in PC^1 : \|x\|_{PC^1} \le r\big\}.
$$

Consequently, Lemma 3.3 implies that T has a fixed point in B_r, and the proof is complete. □

Remark 4.1 Condition (4.1) is certainly satisfied if $|f(t,x,y)|/|x| + |y| \to 0$ uniformly in $t \in J$ as $|x| + |y| \to +\infty$, $|\mathcal{I}_k(x)|/|x| \to 0$ as $|x| \to +\infty$ and $|\mathcal{J}_k(x)|/|x| \to 0$ as $|x| \to +\infty$ $(k = 1, 2, \ldots, n)$.

Theorem 4.2 *Assume that* (H_1)-(H_3) *hold. In addition, let* f, $\mathcal{I}_k$ *and* $\mathcal{J}_k$ *satisfy the following conditions*:

(H_4) *There exists a nonnegative function* $\nu_1(t) \in C[0,1]$ *with* $\nu_1(t) > 0$ *on a subinterval of* $[0,1]$ *such that*

$$|f(t,x,y)| \le \nu_1(t), \quad \forall (t,x,y) \in J \times R \times R.$$

(H_5) *There exist constants* $\nu_2, \nu_3 > 0$ *such that*

$$|\mathcal{I}_k(x)| < \nu_2, \qquad |\mathcal{J}_k(x)| < \nu_3$$

for all $x \in R$, $k = 1,2,\ldots,n$.

Then problem (1.1) *has at least one solution.*

Proof It follows from Lemma 3.2 that the operator $T : PC^1[J,R] \to PC^1[J,R]$ is completely continuous.

Next, we show that the set

$$V = \{x \in PC^1(J,R) | x = \lambda Tx, 0 < \lambda < 1\}$$

is bounded.

Let $x \in V$. Then $x = \lambda Tx$ for $0 < \lambda < 1$. For any $t \in J$, we have

$$\begin{aligned} x(t) = \lambda \Bigg[& \int_{t_k}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} \omega(s) f\big(s, x(s), x'(s)\big)\, ds \\ & + \sum_{i=1}^{n+1} G'_{1s}(t,t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-1}}{\Gamma(q)} \omega(s) f\big(s, x(s), x'(s)\big)\, ds \\ & - \sum_{i=1}^{n} G_1(t,t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-2}}{\Gamma(q-1)} \omega(s) f\big(s, x(s), x'(s)\big)\, ds \\ & + \sum_{i=1}^{n} G'_{1s}(t,t_i) \mathcal{I}_i\big(x(t_i)\big) - \sum_{i=1}^{n} G_1(t,t_i) \mathcal{J}_i\big(x(t_i)\big) \Bigg]. \end{aligned} \tag{4.9}$$

It follows from (H_4), (H_5), (3.2) and (4.9) that

$$\begin{aligned} |x(t)| &= \lambda |(Tx)(t)| \\ &\le \int_{t_k}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} \omega(s) \big|f\big(s, x(s), y(s)\big)\big|\, ds \\ &\quad + \sum_{i=1}^{n+1} |G'_{1s}(t,t_i)| \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-1}}{\Gamma(q)} \omega(s) \big|f\big(s, x(s), y(s)\big)\big|\, ds \\ &\quad + \sum_{i=1}^{n} |G_1(t,t_i)| \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-2}}{\Gamma(q-1)} \omega(s) \big|f\big(s, x(s), y(s)\big)\big|\, ds \\ &\quad + \sum_{i=1}^{n} |G'_{1s}(t,t_i)| \big|\mathcal{I}_i\big(x(t_i)\big)\big| + \sum_{i=1}^{n} |G_1(t,t_i)| \big|\mathcal{J}_i\big(x(t_i)\big)\big| \end{aligned}$$

$$\begin{aligned}
&\le \gamma\tau \int_{t_k}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)}\,ds + \gamma\tau \sum_{i=1}^{n+1} \left|G'_{1s}(t,t_i)\right| \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-1}}{\Gamma(q)}\,ds \\
&\quad + \gamma\tau \sum_{i=1}^{n} \left|G_1(t,t_i)\right| \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-2}}{\Gamma(q-1)}\,ds \\
&\quad + \nu_2 \sum_{i=1}^{n} \left|G'_{1s}(t,t_i)\right| + \nu_3 \sum_{i=1}^{n} \left|G_1(t,t_i)\right| \\
&\le \gamma\tau \left[\frac{1}{\Gamma(q+1)} + \frac{1}{\Gamma(q+1)}(n+1)c_2 + \frac{1}{\Gamma(q)} n c_1\right] + \nu_2 n c_2 + \nu_3 n c_1,
\end{aligned} \tag{4.10}$$

where

$$\tau = \max_{t\in J} \nu_1(t).$$

It follows from (4.10) that

$$\|x\|_{PC} \le \gamma\tau \left[\frac{1 + c_2 + nc_2}{\Gamma(q+1)} + \frac{nc_1}{\Gamma(q)}\right] + n(\nu_2 c_2 + \nu_3 c_1). \tag{4.11}$$

Furthermore, for any $t \in (t_k, t_{k+1}]$, $0 \le k \le n$, we obtain

$$\begin{aligned}
\left|x'(t)\right| &= \lambda \left|(Tx)'(t)\right| \\
&\le \int_{t_k}^{t} \frac{(t-s)^{q-2}}{\Gamma(q-1)} \omega(s) \left|f\bigl(s, x(s), x'(s)\bigr)\right|\,ds \\
&\quad + \sum_{i=1}^{n+1} \left|G''_{1st}(t,t_i)\right| \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-1}}{\Gamma(q)} \omega(s) \left|f\bigl(s, x(s), x'(s)\bigr)\right|\,ds \\
&\quad + \sum_{i=1}^{n} \left|G'_{1t}(t,t_i)\right| \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-2}}{\Gamma(q-1)} \omega(s) \left|f\bigl(s, x(s), x'(s)\bigr)\right|\,ds \\
&\quad + \sum_{i=1}^{n} \left|G''_{1st}(t,t_i)\right| \left|\mathcal{I}_i\bigl(x(t_i)\bigr)\right| + \sum_{i=1}^{n} \left|G'_{1t}(t,t_i)\right| \left|\mathcal{J}_i\bigl(x(t_i)\bigr)\right| \\
&\le \gamma\tau \int_{t_k}^{t} \frac{(t-s)^{q-2}}{\Gamma(q-1)}\,ds + \gamma\tau \sum_{i=1}^{n+1} \left|G''_{1st}(t,t_i)\right| \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-1}}{\Gamma(q)}\,ds \\
&\quad + \gamma\tau \sum_{i=1}^{n} \left|G'_{1t}(t,t_i)\right| \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-2}}{\Gamma(q-1)}\,ds + \nu_2 \sum_{i=1}^{n} \left|G''_{1st}(t,t_i)\right| + \nu_3 \sum_{i=1}^{n} \left|G'_{1t}(t,t_i)\right| \\
&\le \gamma\tau \left[\frac{1}{\Gamma(q)} + \frac{1}{\Gamma(q+1)}(n+1)c_3 + \frac{1}{\Gamma(q)} n c_2\right] + \nu_2 n c_3 + \nu_3 n c_2,
\end{aligned}$$

which, for any $t \in J$, yields

$$\left\|x'\right\|_{PC} \le \gamma\tau \left[\frac{1 + nc_2}{\Gamma(q)} + \frac{(n+1)c_3}{\Gamma(q+1)}\right] + n(\nu_2 c_3 + \nu_3 c_2). \tag{4.12}$$

It follows from (4.11) and (4.12) that

$$\|x\|_{PC^1} \le \zeta, \quad \forall x \in PC^1(J,R), \tag{4.13}$$

where $\zeta \leq \max\{\zeta_1, \zeta_2\}$, here

$$\zeta_1 = \gamma\tau\left[\frac{1+c_2+nc_2}{\Gamma(q+1)} + \frac{nc_1}{\Gamma(q)}\right] + n(\nu_2 c_2 + \nu_3 c_1),$$
$$\zeta_2 = \gamma\tau\left[\frac{1+nc_2}{\Gamma(q)} + \frac{(n+1)c_3}{\Gamma(q+1)}\right] + n(\nu_2 c_3 + \nu_3 c_2).$$

So it follows from (4.13) that the set V is bounded. Thus, as a consequence of Lemma 3.4, the operator T has at least one fixed point. Consequently, the problem (1.1) has at least one solution. This finishes the proof. □

Finally we consider the existence of a unique solution for problem (1.1) by applying the contraction mapping principle.

Theorem 4.3 *Assume that* (H_1)-(H_3) *hold. In addition, let* f, $\mathcal{I}_k$ *and* $\mathcal{J}_k$ *satisfy the following conditions*:

(H_6) *There exists a constant* $L_1 > 0$ *such that*

$$\left|f(t,x,y) - f(t,\bar{x},\bar{y})\right| \leq L_1\left(|x-\bar{x}| + |y-\bar{y}|\right)$$

for each $t \in J$ *and all* $x, y, \bar{x}, \bar{y} \in R$.

(H_7) *There exist constants* $L_2, L_3 > 0$ *such that*

$$\left|\mathcal{I}_k(x) - \mathcal{I}_k(y)\right| < L_2|x-y|, \qquad \left|\mathcal{J}_k(x) - \mathcal{I}_k(y)\right| < L_3|x-y|$$

for all $x, y \in R$, $k = 1, 2, \ldots, n$.

If

$$\Lambda < \max\{\Lambda_1, \Lambda_2\} < 1, \tag{4.14}$$

where

$$\Lambda_1 = \frac{2L_1(1+c_1+nc_1)}{\Gamma(q+1)}\gamma + \frac{2L_1 nc_1}{\Gamma(q)}\gamma + n(c_2L_2 + c_1L_3),$$
$$\Lambda_2 = \frac{2L_1(1+nc_2)}{\Gamma(q)}\gamma + \frac{2L_1(n+1)c_3}{\Gamma(q+1)}\gamma + n(c_3L_2 + c_2L_3),$$

here c_1 *and* c_2 *are defined in* (2.10), *then problem* (1.1) *has a unique solution*.

Proof Let $x, y \in PC^1[J,R]$. Then, for each $t \in J$, it follows from (H_6), (H_7) and (3.2) that

$$\begin{aligned}
&\left|(Tx)(t) - (Ty)(t)\right| \\
&\quad\leq \int_{t_k}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)}\omega(s)\left|f\left(s,x(s),\bar{x}(s)\right) - f\left(s,y(s),\bar{y}(s)\right)\right| ds \\
&\qquad+ \sum_{i=1}^{n+1}\left|G'_{1s}(t,t_i)\right| \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-1}}{\Gamma(q)}\omega(s)\left|f\left(s,x(s),\bar{x}(s)\right) - f\left(s,y(s),\bar{y}(s)\right)\right| ds
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{i=1}^{n} \bigl|G_1(t,t_i)\bigr| \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-2}}{\Gamma(q-1)} \omega(s) \bigl|f\bigl(s,x(s),\bar{x}(s)\bigr) - f\bigl(s,y(s),\bar{y}(s)\bigr)\bigr|\,ds \\
&+ \sum_{i=1}^{n} \bigl|G'_{1s}(t,t_i)\bigr| \bigl|\mathcal{I}_i\bigl(x(t_i)\bigr) - \mathcal{I}_i\bigl(y(t_i)\bigr)\bigr| + \sum_{i=1}^{n} \bigl|G_1(t,t_i)\bigr| \bigl|\mathcal{J}_i\bigl(x(t_i)\bigr) - \mathcal{J}_i\bigl(y(t_i)\bigr)\bigr| \\
\le{}& \frac{2L_1\|x-y\|_{PC^1}}{\Gamma(q)} \gamma \int_{t_k}^{t} (t-s)^{q-1}\,ds + \frac{2L_1\|x-y\|_{PC^1}}{\Gamma(q)} \gamma \sum_{i=1}^{n+1} \bigl|G_1(t,t_i)\bigr| \int_{t_{i-1}}^{t_i} (t_i-s)^{q-2}\,ds \\
&+ \frac{2L_1\|x-y\|_{PC^1}}{\Gamma(q-1)} \gamma \sum_{i=1}^{n} \bigl|G'_{1s}(t,t_i)\bigr| \int_{t_{i-1}}^{t_i} (t_i-s)^{q-1}\,ds \\
&+ L_2\|x-y\|_{PC^1} \sum_{i=1}^{n} \bigl|G'_{1s}(t,t_i)\bigr| + L_3\|x-y\|_{PC^1} \sum_{i=1}^{n} \bigl|G_1(t,t_i)\bigr| \\
\le{}& \frac{2L_1\|x-y\|_{PC^1}}{\Gamma(q+1)} \gamma + \frac{2L_1\|x-y\|_{PC^1}}{\Gamma(q+1)} \gamma (n+1)c_1 + \frac{2L_1\|x-y\|_{PC^1}}{\Gamma(q)} \gamma n c_2 \\
&+ nc_2L_2\|x-y\|_{PC^1} + nc_1L_3\|x-y\|_{PC^1} \\
={}& \biggl[\frac{2L_1(1+c_1+nc_1)}{\Gamma(q+1)} \gamma + \frac{2L_1 nc_1}{\Gamma(q)} \gamma + n(c_2L_2 + c_1L_3)\biggr] \|x-y\|_{PC^1}
\end{aligned}
$$

and

$$
\begin{aligned}
&\bigl|(Tx)'(t) - (Ty)'(t)\bigr| \\
&\quad\le \int_{t_k}^{t} \frac{(t-s)^{q-2}}{\Gamma(q-1)} \omega(s) \bigl|f\bigl(s,x(s),\bar{x}(s)\bigr) - f\bigl(s,y(s),\bar{y}(s)\bigr)\bigr|\,ds \\
&\qquad+ \sum_{i=1}^{n+1} \bigl|G''_{1st}(t,t_i)\bigr| \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-1}}{\Gamma(q)} \omega(s) \bigl|f\bigl(s,x(s),\bar{x}(s)\bigr) - f\bigl(s,y(s),\bar{y}(s)\bigr)\bigr|\,ds \\
&\qquad+ \sum_{i=1}^{n} \bigl|G'_{1t}(t,t_i)\bigr| \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-2}}{\Gamma(q-1)} \omega(s) \bigl|f\bigl(s,x(s),\bar{x}(s)\bigr) - f\bigl(s,y(s),\bar{y}(s)\bigr)\bigr|\,ds \\
&\qquad+ \sum_{i=1}^{n} \bigl|G''_{1st}(t,t_i)\bigr| \bigl|\mathcal{I}_i\bigl(x(t_i)\bigr) - \mathcal{I}_i\bigl(y(t_i)\bigr)\bigr| + \sum_{i=1}^{n} \bigl|G'_{1t}(t,t_i)\bigr| \bigl|\mathcal{J}_i\bigl(x(t_i)\bigr) - \mathcal{J}_i\bigl(y(t_i)\bigr)\bigr| \\
&\quad\le \frac{2L_1\|x-y\|_{PC^1}}{\Gamma(q-1)} \gamma \int_{t_k}^{t} (t-s)^{q-2}\,ds + \frac{2L_1\|x-y\|_{PC^1}}{\Gamma(q)} \gamma \sum_{i=1}^{n+1} \bigl|G''_{1st}(t,t_i)\bigr| \int_{t_{i-1}}^{t_i} (t_i-s)^{q-1}\,ds \\
&\qquad+ \frac{2L_1\|x-y\|_{PC^1}}{\Gamma(q-1)} \gamma \sum_{i=1}^{n} \bigl|G'_{1t}(t,t_i)\bigr| \int_{t_{i-1}}^{t_i} (t_i-s)^{q-2}\,ds \\
&\qquad+ L_2\|x-y\|_{PC^1} \sum_{i=1}^{n} \bigl|G''_{1st}(t,t_i)\bigr| + L_3\|x-y\|_{PC^1} \sum_{i=1}^{n} \bigl|G'_{1t}(t,t_i)\bigr| \\
&\quad\le \frac{2L_1\|x-y\|_{PC^1}}{\Gamma(q)} \gamma + \frac{2L_1\|x-y\|_{PC^1}}{\Gamma(q+1)} \gamma (n+1)c_3 + \frac{2L_1\|x-y\|_{PC^1}}{\Gamma(q)} \gamma n c_2 \\
&\qquad+ nc_3L_2\|x-y\|_{PC^1} + nc_2L_3\|x-y\|_{PC^1} \\
&\quad= \biggl[\frac{2L_1(1+nc_2)}{\Gamma(q)} \gamma + \frac{2L_1(n+1)c_3}{\Gamma(q+1)} \gamma + n(c_3L_2 + c_2L_3)\biggr] \|x-y\|_{PC^1}.
\end{aligned}
$$

Consequently, we have $\|Tx - Ty\|_{PC^1} \le \Lambda\|x - y\|_{PC^1}$, where Λ is defined by (4.14). As $\Lambda < 1$, therefore T is a contraction. Thus, the conclusion of the theorem follows by the contraction mapping principle. The proof is complete. □

5 Examples

To illustrate how our main results can be used in practice, we present two examples.

Example 5.1 Let $q = \frac{3}{2}$, $n = 1$. We consider the following boundary value problem:

$$\begin{cases} {}^c\mathbf{D}_{0+}^{\frac{3}{2}}x(t) = d_0 t^{\frac{1}{3}}[d_1\sqrt[5]{d_2 t - x + x'} - \frac{1}{20}x' - d_3\ln(1 + x^2)], & t \in J, t \neq \frac{1}{2}, \\ \Delta x|_{t_1=\frac{1}{2}} = \frac{1}{10}x(\frac{1}{2}), \qquad \Delta x'|_{t_1=\frac{1}{2}} = \frac{1}{6}x(\frac{1}{2}), \\ x(0) = x(1) = 0, \end{cases} \tag{5.1}$$

where d_0, d_1, d_2 and d_3 are positive real numbers.

Conclusion Problem (5.1) has at least one solution in $PC^1[J,R] \cap C^2[J',R]$, where $J' = [0,\frac{1}{2}) \cup (\frac{1}{2},1]$.

Proof It follows from (5.1) that

$$\omega(t) = dt^{\frac{1}{3}},$$

$$f(t,x,x') = d_1\sqrt[5]{d_2 t - x + x'} - \frac{1}{20}x' - d_3\ln(1 + x^2), \tag{5.2}$$

$$t_1 = \frac{1}{2}, \qquad \mathcal{I}_1(x) = \frac{1}{10}x, \qquad \mathcal{J}_1(x) = \frac{1}{6}x, \tag{5.3}$$

$$\alpha_1 = \alpha_2 = 1, \qquad \beta_1 = \beta_2 = 0.$$

From the definition of $\omega, f, \mathcal{I}_1$ and $\mathcal{J}_1$, it is easy to see that (H_1)-(H_3) hold.

On the other hand, it follows from (5.2) and (5.3) that

$$|f(t,x,y)| \le d_1\sqrt[5]{d_2 + |x| + |y|} + \frac{1}{20}|y| + d_3\ln(1 + |x|^2)$$

and

$$|\mathcal{I}_1(x)| = \frac{1}{10}|x|, \qquad |\mathcal{J}_1(x)| = \frac{1}{6}x, \quad \forall t \in J, x, y \in R.$$

So $\xi \le \frac{1}{20}$, $\xi_1 \le \frac{1}{10}$, $\xi_2 \le \frac{1}{6}$. Noticing that $\alpha_1 = \alpha_2 = 1$, $\beta_1 = \beta_2 = 0$, we have $\eta = 1$ and

$$G_1(t,s) = -\begin{cases} s(1-t), & 0 \le s \le t \le 1, \\ t(1-s), & 0 \le t \le s \le 1, \end{cases} \qquad G'_{1t}(t,s) = \begin{cases} s, & 0 \le s \le t \le 1, \\ s-1, & 0 \le t \le s \le 1, \end{cases}$$

$$G'_{1s}(t,s) = \begin{cases} t-1, & 0 \le s \le t \le 1, \\ t, & 0 \le t \le s \le 1, \end{cases} \qquad G'_{1st}(t,s) = 1.$$

Therefore, $c_1 = \frac{1}{4}$, $c_2 = c_3 = 1$, and therefore (4.1) is satisfied because

$$\delta_1 \le \frac{9}{20\sqrt{\pi}} + \frac{11}{240}, \qquad \delta_2 \le \frac{13}{30\sqrt{\pi}} + \frac{4}{15}, \qquad \delta \le \frac{13}{30\sqrt{\pi}} + \frac{4}{15} < 1.$$

Thus, our conclusion follows from Theorem 4.1. □

Example 5.2 Let $q = \frac{5}{4}$, $n = 1$. We consider the following boundary value problem:

$$\begin{cases} {}^c\mathbf{D}_{0+}^{\frac{5}{4}}x(t) = (\sin t + 1)\frac{(d_4e^t+d_5)(d_6+tx^2\sin^2 x)}{d_6+t^2+x^2+y^2}, & t \in J, t \neq \frac{1}{3}, \\ \Delta x|_{t_1=\frac{1}{3}} = d_7e^{-x^2(\frac{1}{3})} + d_8\sin^2 x(\frac{1}{3}), \\ \Delta x'|_{t_1=\frac{1}{3}} = \frac{d_9+d_{10}x^2(\frac{1}{3})}{1+x^2(\frac{1}{3})}, \\ x(0) = x(1) = 0, \end{cases} \tag{5.4}$$

where d_4, d_5, d_6, d_7, d_8, d_9 and d_{10} are positive real numbers with $d_{10} > d_9$.

Conclusion Problem (5.4) has at least one solution in $PC^1[J,R] \cap C^2[J',R]$, where $J' = [0,\frac{1}{3}) \cup (\frac{1}{3},1]$.

Proof It follows from (5.4) that

$$\omega(t) = \sin t + 1,$$

$$f(t,x,y) = (d_4e^t + d_5)\frac{d_6 + tx^2\sin^2 x}{d_6 + t^2 + x^2 + y^2}, \tag{5.5}$$

$$t_1 = \frac{1}{3}, \qquad \mathcal{I}_1(x) = d_7e^{-x^2} + d_8\sin^2 x, \qquad \mathcal{J}_1(x) = \frac{d_9 + d_{10}x^2}{1 + x^2}, \tag{5.6}$$

$$\alpha_1 = \alpha_2 = 1, \qquad \beta_1 = \beta_2 = 0.$$

From the definition of $\omega, f, \mathcal{I}_1$ and $\mathcal{J}_1$, we can obtain that (H_1)-(H_3) hold.

On the other hand, from (5.5) and (5.6) we have

$$\left|f(t,x,y)\right| \leq d_4e^t + d_5 := \nu_1(t),$$

and

$$\left|\mathcal{I}_1(x)\right| \leq \frac{d_7}{e} + d_8, \qquad \left|\mathcal{J}_1(x)\right| \leq d_{10}, \quad \forall t \in J, x, y \in R.$$

Therefore, the conditions (H_4) and (H_5) of Theorem 4.2 are satisfied. Thus, Theorem 4.2 gives our conclusion. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All results belong to JZ and MF. All authors read and approved the final manuscript.

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