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Symmetric positive solutions of higher-order boundary value problems

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Abstract

We study the higher-order boundary value problems. The existence of symmetric positive solutions of the problem is discussed. Our results extend some recent work in the literature. The analysis of this paper mainly relies on the monotone iterative technique.

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Keywords: higher-order boundary value problems; symmetric positive solutions; existence; monotone iterative technique

1 Introduction

We study the boundary value problem (BVP)

$$\begin{cases} (-1)^n u^{(2n)}(t) = f(t, u), & t \in (0, 1), \\ \alpha_i u^{(2i-2)}(j) + (-1)^{j+1} \beta_i u^{(2i-1)}(j) = 0, & i = 1, \dots, n, j = 0, 1, \end{cases} \quad (1.1)$$

where $n \geq 1$ is an integer, $f : (0, 1) \times (0, \infty) \rightarrow [0, \infty)$ is continuous, α_i, β_i are nonnegative constants, $\alpha_i^2 + 2\alpha_i\beta_i > 0$, $i = 1, \dots, n$. $f(t, u)$ may be singular at $u = 0$, $t = 0$ (and/or $t = 1$). If a function $u : [0, 1] \rightarrow \mathbb{R}$ is continuous and satisfies $u(t) = u(1 - t)$ for $t \in [0, 1]$, then we say that $u(t)$ is symmetric on $[0, 1]$. By a symmetric positive solution of BVP (1.1) we mean a symmetric function $u \in C^{2n}[0, 1]$ such that $u(t) > 0$ for $t \in (0, 1)$ and $u(t)$ satisfies (1.1).

In recent years, many authors have studied BVP (1.1), they only considered that f is non-decreasing or nonincreasing in u , or the boundary condition depends only on derivatives of even orders; see [1–8] and references cited therein. To the best of the author's knowledge, there is no such results involving (1.1). In this note, we intend to fill in such gaps in the literature.

The organization of this paper is as follows. After this introduction, in Section 2, we state the assumptions and some preliminary lemmas. By applying the monotone iterative technique, we discuss the existence of symmetric positive solutions for (1.1) and obtain the main results in Section 3.

2 Preliminaries

For convenience, in this paper we let $E = C^{(2n)}[0, 1]$, $\Delta_i = \alpha_i^2 + 2\alpha_i\beta_i$,

$$G_i(t, s) = \begin{cases} \frac{1}{\Delta_i}(\beta_i + \alpha_i s)(\beta_i + \alpha_i(1 - t)), & 0 \leq s < t \leq 1, \\ \frac{1}{\Delta_i}(\beta_i + \alpha_i t)(\beta_i + \alpha_i(1 - s)), & 0 \leq t \leq s \leq 1, \end{cases} \quad (2.1)$$

$$e_i(t) = G_i(t, t) = \frac{1}{\Delta_i} (\beta_i + \alpha_i t)(\beta_i + \alpha_i(1 - t)), \quad t \in [0, 1], \tag{2.2}$$

where $i = 1, \dots, n$, and define

$$P = \left\{ u \in E : u(0) \geq 0, u(t) > 0 \text{ for } t \in (0, 1), u(t) = u(1 - t) \text{ and there exists constant } l_u \in (0, 1) \text{ satisfying } l_u e_1(t) \leq u(t) \leq l_u^{-1} e_1(t) \text{ for } t \in [0, 1] \right\}.$$

Remark 2.1 The set P is not a cone as it is not closed.

Throughout this paper, we assume the following:

(H1) α_i, β_i are nonnegative constants, $\alpha_i^2 + 2\alpha_i\beta_i > 0, i = 1, \dots, n. f : (0, 1) \times (0, \infty) \rightarrow [0, \infty)$ is continuous and symmetric in t , i.e., f satisfies

$$f(1 - t, u) = f(t, u), \quad t \in (0, 1). \tag{2.3}$$

(H2) For $(t, u) \in (0, 1) \times (0, \infty), f(t, u)$ is nondecreasing in u and there exists a constant $\lambda \in (0, 1)$ such that if $\sigma \in (0, 1]$, then

$$\sigma^\lambda f(t, u) \leq f(t, \sigma u). \tag{2.4}$$

(H2') For $(t, u) \in (0, 1) \times (0, \infty), f(t, u)$ is nonincreasing in u and there exists a constant $\lambda \in (0, 1)$ such that if $\sigma \in (0, 1]$, then

$$\sigma^{-\lambda} f(t, u) \geq f(t, \sigma u). \tag{2.5}$$

(H3) $0 < \int_0^1 f(t, e_1(t)) dt < \infty.$

Example 2.1 Consider the equation

$$f(t, u) = |2t - 1|u^{\frac{1}{4}}, \quad (t, u) \in (0, 1) \times (0, \infty).$$

It is easy to see that the function f satisfies assumptions (H1) and (H2). In fact, if $\sigma \in (0, 1]$, there exists constant λ with $\frac{1}{4} \leq \lambda < 1$ such that $f(t, \sigma u) \geq \sigma^\lambda f(t, u).$

Remark 2.2 It is easy to see that (H2) implies that if $\sigma \in [1, \infty),$ then

$$f(t, \sigma u) \leq \sigma^\lambda f(t, u), \tag{2.6}$$

and (H2') implies that if $\sigma \in [1, \infty),$ then

$$f(t, \sigma u) \geq \sigma^{-\lambda} f(t, u). \tag{2.7}$$

Now, we present several lemmas that will be used in the proof of our results. By routine calculations we have the following results.

Lemma 2.1 *Let v be integrable on $(0,1)$, then the BVP*

$$\begin{cases} (-1)^n u^{(2n)}(t) = v(t), & t \in (0,1), \\ \alpha_i u^{(2i-2)}(j) + (-1)^{j+1} \beta_i u^{(2i-1)}(j) = 0, & i = 1, \dots, n, j = 0, 1 \end{cases}$$

has a unique solution

$$u(t) = \int_0^1 \cdots \int_0^1 G_1(t, s_{n-1}) G_2(s_{n-1}, s_{n-2}) \cdots G_n(s_1, s) v(s) ds ds_1 \cdots ds_{n-1},$$

where $G_i(t, s)$ are defined by (2.1).

Lemma 2.2 *For any $t, s \in [0, 1]$, we have*

$$\varepsilon_i e_i(t) e_i(s) \leq G_i(t, s) \leq e_i(t), \tag{2.8}$$

$$G_i(1-t, 1-s) = G_i(t, s), \tag{2.9}$$

where $\varepsilon_i = \frac{\Delta_i}{(\alpha_i + \beta_i)^2}$, $i = 1, \dots, n$.

3 Main results

Define the operator $T : E \rightarrow E$ by

$$Tu(t) = \int_0^1 \cdots \int_0^1 G_1(t, s_{n-1}) G_2(s_{n-1}, s_{n-2}) \cdots G_n(s_1, s) f(s, u(s)) ds ds_1 \cdots ds_{n-1}, \tag{3.1}$$

where $G_i(t, s)$ are defined by (2.1). It is clear that u is a solution of (1.1) if and only if u is a fixed point of T .

Theorem 3.1 *Assume (H1)-(H3) hold. Then BVP (1.1) has at least one symmetric positive solution.*

Proof

Claim 3.1 $T : P \rightarrow P$ is completely continuous and nondecreasing.

In fact, for $u \in P$, it is obvious that $Tu \in E$, $Tu(t) > 0$ for $t \in (0, 1)$ and $Tu(0) \geq 0$. (2.3), (2.9) and a change of variables imply

$$\begin{aligned} & Tu(1-t) \\ &= \int_0^1 \cdots \int_0^1 G_1(1-t, s_{n-1}) G_2(s_{n-1}, s_{n-2}) \cdots G_n(s_1, s) f(s, u(s)) ds ds_1 \cdots ds_{n-1} \\ &= \int_0^1 \cdots \int_0^1 G_1(1-t, 1-s_{n-1}) G_2(1-s_{n-1}, s_{n-2}) \cdots G_n(s_1, s) f(s, u(s)) ds \cdots ds_{n-1} \\ &= \cdots = \int_0^1 \cdots \int_0^1 G_1(t, s_{n-1}) G_2(s_{n-1}, s_{n-2}) \cdots G_n(s_1, s) f(s, u(s)) ds ds_1 \cdots ds_{n-1} \\ &= Tu(t), \quad t \in [0, 1]. \end{aligned} \tag{3.2}$$

For any $u \in P$, from (2.4), (2.6), (2.8), and (H3), we have

$$\begin{aligned} Tu(t) &\leq e_1(t) \int_0^1 \cdots \int_0^1 G_2(s_{n-1}, s_{n-2}) \cdots G_n(s_1, s) f(s, l_u^{-1} e_1(s)) ds ds_1 \cdots ds_{n-1} \\ &\leq l_u^{-\lambda} e_1(t) \int_0^1 \cdots \int_0^1 G_2(s_{n-1}, s_{n-2}) \cdots G_n(s_1, s) f(s, e_1(s)) ds ds_1 \cdots ds_{n-1} \\ &\leq l_{Tu}^{-1} e_1(t), \end{aligned} \tag{3.3}$$

$$\begin{aligned} Tu(t) &\geq \varepsilon_1 e_1(t) \int_0^1 \cdots \int_0^1 e_1(s_{n-1}) G_2(s_{n-1}, s_{n-2}) \cdots G_n(s_1, s) f(s, l_u e_1(s)) ds ds_1 \cdots ds_{n-1} \\ &\geq l_u^\lambda \varepsilon_1 e_1(t) \int_0^1 \cdots \int_0^1 e_1(s_{n-1}) G_2(s_{n-1}, s_{n-2}) \cdots G_n(s_1, s) f(s, e_1(s)) ds ds_1 \cdots ds_{n-1} \\ &\geq l_{Tu} e_1(t) \end{aligned} \tag{3.4}$$

for $t \in [0, 1]$, where l_{Tu} satisfies

$$\begin{aligned} 0 &< l_{Tu} \\ &< \min \left\{ 1, \frac{l_u^\lambda}{\int_0^1 \cdots \int_0^1 G_2(s_{n-1}, s_{n-2}) \cdots G_n(s_1, s) f(s, e_1(s)) ds ds_1 \cdots ds_{n-1}}, \right. \\ &\quad \left. l_u^\lambda \varepsilon_1 \int_0^1 \cdots \int_0^1 e_1(s_{n-1}) G_2(s_{n-1}, s_{n-2}) \cdots G_n(s_1, s) f(s, e_1(s)) ds ds_1 \cdots ds_{n-1} \right\}. \end{aligned}$$

Thus, it follows from (3.3) and (3.4) that $Tu(t) \in P$, and so $T : P \rightarrow P$. Next by a standard method and the Ascoli-Arzela theorem one can prove that $T : P \rightarrow P$ is completely continuous, we omit it here. From (H2), it is easy to see that T is nondecreasing in u . Hence, Claim 3.1 holds.

Claim 3.2 *Let $0 < \delta < 1$ be fixed number satisfying*

$$0 < \delta \leq l_{Te_1}^{1/(1-\lambda)}, \tag{3.5}$$

where λ is defined in (H2) in which $\sigma = \delta$, and assume

$$u_0 = \delta e_1(t), \quad v_0 = \delta^{-1} e_1(t), \tag{3.6}$$

$$u_n = Tu_{n-1}, \quad v_n = Tv_{n-1}, \quad n = 1, 2, \dots \tag{3.7}$$

Then

$$u_0 \leq u_1 \leq \cdots \leq u_n \leq \cdots \leq v_n \leq \cdots \leq v_1 \leq v_0, \tag{3.8}$$

and there exists $u^* \in P$ such that

$$u_n(t) \rightarrow u^*(t), \quad v_n(t) \rightarrow u^*(t), \quad \text{uniformly on } [0, 1]. \tag{3.9}$$

In fact, $0 < l_{Te_1} < 1$ since $Te_1 \in P$. So, from (3.5) and noting that $0 < \lambda < 1$, $0 < \delta < 1$. From (3.6), we have $u_0, v_0 \in P$ and $u_0 \leq v_0$.

On the other hand, from (2.4) and (2.6), we have

$$\begin{aligned} u_1 &= Tu_0(t) \\ &= \int_0^1 \cdots \int_0^1 G_1(t, s_{n-1})G_2(s_{n-1}, s_{n-2}) \cdots G_n(s_1, s)f(s, \delta e_1(s)) ds ds_1 \cdots ds_{n-1} \\ &\geq \delta^\lambda \int_0^1 \cdots \int_0^1 G_1(t, s_{n-1})G_2(s_{n-1}, s_{n-2}) \cdots G_n(s_1, s)f(s, e_1(s)) ds ds_1 \cdots ds_{n-1} \\ &= \delta^\lambda Te_1 \geq \delta^\lambda l_{Te_1} e_1(t) \geq \delta^\lambda \delta^{1-\lambda} e_1(t) = u_0, \\ v_1 &= Tv_0(t) \\ &= \int_0^1 \cdots \int_0^1 G_1(t, s_{n-1})G_2(s_{n-1}, s_{n-2}) \cdots G_n(s_1, s)f(s, \delta^{-1} e_1(s)) ds ds_1 \cdots ds_{n-1} \\ &\leq \delta^{-\lambda} \int_0^1 \cdots \int_0^1 G_1(t, s_{n-1})G_2(s_{n-1}, s_{n-2}) \cdots G_n(s_1, s)f(s, e_1(s)) ds ds_1 \cdots ds_{n-1} \\ &= \delta^{-\lambda} Te_1 \leq \delta^{-\lambda} l_{Te_1}^{-1} e_1(t) \leq \delta^{-\lambda} \delta^{-(1-\lambda)} e_1(t) = v_0. \end{aligned}$$

Since $u_0 \leq v_0$ and T is nondecreasing, by induction, (3.8) holds.

Let $\varsigma = \delta^2$, then $0 < \varsigma < 1$. It follows from

$$T(cu) \geq c^\lambda Tu, \quad \text{if } 0 < c < 1, u \in P$$

that, for any natural number n ,

$$u_n = Tu_{n-1} = T^n u_0 = T^n(\delta e_1(t)) = T^n(\varsigma \delta^{-1} e_1(t)) \geq \varsigma^{\lambda^n} T^n(\delta^{-1} e_1(t)) = \varsigma^{\lambda^n} v_n.$$

Thus, for all natural numbers n and p , we have

$$0 \leq u_{n+p} - u_n \leq v_n - u_n \leq (1 - \varsigma^{\lambda^n}) v_n \leq (1 - \varsigma^{\lambda^n}) \delta^{-1} e_1(t),$$

which implies that there exists $u^* \in P$ such that (3.9) holds, and Claim 3.2 holds.

Letting $n \rightarrow \infty$ in (3.7), we obtain $u^*(t) = Tu^*(t)$, which is a symmetric positive solution of BVP (1.1), and this completes the proof of the theorem. \square

Theorem 3.2 *Assume (H1), (H2') and (H3) hold. Then BVP (1.1) has at least one symmetric positive solution.*

Proof

Claim 3.3 $T : P \rightarrow P$ is completely continuous and nonincreasing.

The proof of Claim 3.3 is similar to the proof of Claim 3.1, so this is omitted.

Claim 3.4 Let $0 < \xi < 1$ be fixed number, $\eta > 1$ be sufficiently large constant satisfying

$$(\xi^{-\lambda+1})^\eta \leq l_{Te_1}, \tag{3.10}$$

where λ is defined in (H2') in which $\sigma = \xi^\eta$, and assume

$$u_0 = \xi^\eta e_1(t), \quad u_n = Tu_{n-1}, \quad n = 1, 2, \dots \tag{3.11}$$

Then

$$u_0 \leq u_2 \leq \dots \leq u_{2n} \leq \dots \leq u_{2n+1} \leq \dots \leq u_3 \leq u_1, \tag{3.12}$$

and there exists $u^* \in P$ such that

$$u_{2n}(t) \rightarrow u^*(t), \quad u_{2n+1}(t) \rightarrow u^*(t), \quad \text{uniformly on } [0, 1]. \tag{3.13}$$

In fact, $0 < \xi^\eta < 1$ since $0 < \xi < 1$ and $\eta > 1$. So from (3.11),

$$u_0(t) \leq e_1(t). \tag{3.14}$$

From (2.5), (3.10), (3.14), and noting that T is nonincreasing in u , we have

$$u_1 = Tu_0(t) \geq Te_1(t) \geq l_{Te_1} e_1(t) \geq (\xi^{-\lambda+1})^\eta e_1(t) = (\xi^\lambda)^{-\eta} \xi^\eta e_1(t) \geq u_0, \tag{3.15}$$

$$\begin{aligned} u_1 &= Tu_0(t) \\ &= \int_0^1 \dots \int_0^1 G_1(t, s_{n-1}) G_2(s_{n-1}, s_{n-2}) \dots G_n(s_1, s) f(s, \xi^\eta e_1(s)) ds ds_1 \dots ds_{n-1} \\ &\leq (\xi^\eta)^{-\lambda} \int_0^1 \dots \int_0^1 G_1(t, s_{n-1}) G_2(s_{n-1}, s_{n-2}) \dots G_n(s_1, s) f(s, e_1(s)) ds ds_1 \dots ds_{n-1} \\ &= \xi^{-\eta\lambda} Te_1 \leq \xi^{-\eta\lambda} l_{Te_1}^{-1} e_1(t) \leq \xi^{-\eta} e_1(t). \end{aligned} \tag{3.16}$$

Therefore,

$$u_2 = Tu_1(t) \leq Tu_0(t) = u_1, \tag{3.17}$$

$$\begin{aligned} u_2 &= Tu_1(t) \geq T(\xi^{-\eta} e_1(t)) \\ &= \int_0^1 \dots \int_0^1 G_1(t, s_{n-1}) G_2(s_{n-1}, s_{n-2}) \dots G_n(s_1, s) f(s, \xi^{-\eta} e_1(s)) ds ds_1 \dots ds_{n-1} \\ &\geq (\xi^{-\eta})^{-\lambda} Te_1(t) \geq \xi^{\eta\lambda} l_{Te_1} e_1(t) \geq \xi^\eta e_1(t) = u_0. \end{aligned} \tag{3.18}$$

From (3.15), (3.17), (3.18), and noting that T^2 is nondecreasing, by induction, (3.12) holds.

On the other hand, from (2.5) and (2.7), for $0 < \sigma \leq 1$,

$$T^2(\sigma u) \geq T(\sigma^{-\lambda} Tu) \geq (\sigma^{-\lambda})^{-\lambda} T^2 u = \sigma^{\lambda^2} T^2 u. \tag{3.19}$$

Then from (3.16) and (3.19), we have

$$\begin{aligned} u_{2n} &= Tu_{2n-1} = T^{2n} u_0 = T^{2n} (\xi^\eta e_1(t)) \\ &= T^{2n-2} T^2 (\xi^{2\eta} \xi^{-\eta} e_1(t)) \geq T^{2n-2} T^2 (\xi^{2\eta} u_1(t)) \end{aligned}$$

$$\begin{aligned} &\geq T^{2n-2}((\xi^{2\eta})^{\lambda^2} T^2 u_1(t)) \geq \dots \\ &\geq (\xi^{2\eta})^{\lambda^{2n}} T^{2n} u_1(t) = (\xi^{2\eta})^{\lambda^{2n}} T^{2n+1} u_0(t) = (\xi^{2\eta})^{\lambda^{2n}} u_{2n+1}, \end{aligned}$$

and thus

$$(\xi^{2\eta})^{\lambda^{2n}} u_{2n+1}(t) \leq u_{2n}(t) \leq u_{2n+1}(t).$$

Therefore, for all natural numbers n and p , we have

$$\begin{aligned} 0 &\leq u_{2n+2p}(t) - u_{2n}(t) \leq u_{2n+1}(t) - u_{2n}(t) \\ &\leq (1 - (\xi^{2\eta})^{\lambda^{2n}}) u_{2n+1}(t) \leq (1 - (\xi^{2\eta})^{\lambda^{2n}}) u_1(t), \end{aligned} \tag{3.20}$$

$$0 \leq u_{2n+1}(t) - u_{2n+2p+1}(t) \leq u_{2n+1}(t) - u_{2n}(t) \leq (1 - (\xi^{2\eta})^{\lambda^{2n}}) u_1(t). \tag{3.21}$$

From (3.20) and (3.21), there exists $u^* \in P$ such that (3.13) holds, and Claim 3.4 holds.

Letting $n \rightarrow \infty$ in (3.11), we obtain $u^*(t) = Tu^*(t)$, which is a symmetric positive solution of BVP (1.1), and this completes the proof of the theorem. \square

Remark 3.1 [3, 5] only considered that f is nondecreasing or nonincreasing in u , and $\alpha_i = 1, \beta_i = 0$ in (1.1), so our results extend the work in the literature.

Example 3.1 Consider the BVP

$$\begin{cases} -u^{(6)}(t) = f(t, u), & t \in (0, 1), \\ \alpha_i u^{(2i-2)}(j) + (-1)^{j+1} \beta_i u^{(2i-1)}(j) = 0, & i = 1, 2, 3, j = 0, 1, \end{cases} \tag{3.22}$$

where $f(t, u) = u^\alpha \sin \pi t$ for $(t, u) \in (0, 1) \times (0, \infty)$, $0 < \alpha < 1, \alpha_i = \beta_i = 1, i = 1, 2, 3$.

It is easy to see that function $f(t, u)$ satisfies (H1) and (H3). If $\sigma \in (0, 1]$, there exists constant λ with $0 < \alpha \leq \lambda < 1$ such that $f(t, \sigma u) \geq \sigma^\lambda f(t, u)$, so (H2) is also satisfied. Therefore, from Theorem 3.1, (3.22) has at least one symmetric positive solution.

Competing interests

The author declares that she has no competing interests.

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