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Boundary regularity for quasilinear elliptic systems with super quadratic controllable growth condition

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Abstract
We consider the boundary regularity for weak solutions to quasilinear elliptic systems under a super quadratic controllable growth condition, and we obtain a general criterion for a weak solution to be regular in the neighborhood of a given boundary point. Combined with existing results on the interior partial regularity, this result yields an upper bound on the Hausdorff dimension of a singular set at the boundary.

Keywords: quasilinear elliptic systems; controllable growth condition; *A*-harmonic approximation technique; boundary partial regularity

1 Introduction

In this paper we are concerned with partial regularity for weak solutions of quasilinear elliptic systems:

$$-D_\alpha\left(A_{ij}^{\alpha\beta}(x,u)D_\beta u^j\right) = B_i(x,u,Du), \tag{1.1}$$

where Ω is a bounded domain in R^n, $n \geq 2$, $N > 1$, and u and B_i take values in R^N. Here each $A_{ij}^{\alpha\beta}$ maps $\Omega \times R^N$ into R, and each B_i maps $\Omega \times R^N \times R^{nN}$ into R. For $m > 2$, we have the following.

(H1) There exists $L > 0$ such that

$$A_{ij}^{\alpha\beta}(x,\xi)(\nu,\tilde{\nu}) \leq L\left(1+|\xi|^2\right)^{\frac{m-2}{2}}|\nu||\tilde{\nu}| \quad \text{for all } (x,\xi) \in \overline{\Omega} \times R^N, \nu,\tilde{\nu} \in R^{nN}.$$

(H2) $A_{ij}^{\alpha\beta}(x,\xi)$ is uniformly strongly elliptic, that is, for some $\lambda > 0$ we have

$$A_{ij}^{\alpha\beta}(x,\xi)(\nu,\nu) \geq \lambda\left(1+|\xi|^2\right)^{\frac{m-2}{2}}|\nu|^2 \quad \text{for all } (x,\xi) \in \overline{\Omega} \times R^N, \nu \in R^{nN}.$$

(H3) There exists a monotone nondecreasing concave function $\omega(t,s) : [0,\infty) \to [0,\infty)$ with $\omega(t,0) = 0$, continuous at 0, such that

$$\left|A_{ij}^{\alpha\beta}(x,u) - A_{ij}^{\alpha\beta}(y,v)\right| \leq \omega\left(|x-y|^m + |u-v|^m\right)$$

for all $x,y \in \overline{\Omega}$, $u,v \in R^N$.

(H4) The B_i fulfill the following controllable growth condition:

$$\left|B_i(x,\xi,\nu)\right| \le C\left(|\nu|^{m(1-\frac{1}{r})} + |\xi|^{r-1} + 1\right),$$

where $r = \frac{nm}{n-m}$ if $n > m$, or any exponent if $n = m$; for all $x \in \overline{\Omega}$, $\xi \in R^N$ and $\nu \in R^{nN}$.

(H5) There exist s with $s > n$ and a function $g \in H^{1,s}(\Omega, R^N)$, such that we have

$$u|_{\partial\Omega} = g|_{\partial\Omega}.$$

Note that we trivially have $g \in H^{1,m}(\Omega, R^N)$. Further, by Sobolev's embedding theorem we have $g \in C^{0,\kappa}(\Omega, R^N)$ for any $\kappa \in [0, 1-\frac{n}{s}]$. If $g|_{\partial\Omega} \equiv 0$, we will take $g \equiv 0$ on Ω.

If the domain we consider is an upper half unit ball B^+, the boundary condition is the following.

(H5)′ There exist s with $s > n$ and a function $g \in H^{1,s}(B^+, R^N)$, such that we have

$$u|_D = g|_D.$$

Here we write $B_\rho(x_0) = \{x \in R^n : |x - x_0| < \rho\}$, and further $B_\rho = B_\rho(0)$, $B = B_1$. For $x_0 \in R^{n-1} \times \{0\}$ we write $B_\rho^+(x_0)$ for $\{x \in R^n : x_n > 0, |x - x_0| < \rho\}$, and we set $B_\rho^+ = B_\rho^+(0)$, $B^+ = B_1^+$. We further write $D_\rho(x_0) = \{x \in R^n : x_n = 0, |x - x_0| < \rho\}$, and we set $D_\rho = D_\rho(0)$, $D = D_1$. For bounded $X \subset R^n$ with $L^n(X) > 0$ we denote the average of a given function $g \in L^1(X)$ by $⨍_X g\,dx$, *i.e.* $⨍_X g\,dx = \frac{1}{L^n(X)} \int_X g\,dx$. For $\nu \in L^1(\partial\Omega)$, $x_0 \in \partial\Omega$ we set $\nu'_{x_0,R} = ⨍_{\partial\Omega \cap \overline{B}_R(x_0)} \nu\,dH^{n-1}$. In particular, for $\nu \in L^1(D_\rho(x_0))$, $x_0 \in D$, we write $\nu'_{x_0,\rho} = ⨍_{D_\rho(x_0)} \nu\,dH^{n-1}$.

Now we can definite weak solutions to systems (1.1). Because there is a very large literature on the existence of weak solutions [1, 2], we assume that a weak solution exists [3] and deal with the problem of regularity directly.

Definition 1.1 By a weak solution of (1.1) we mean a vector-valued function $u \in W^{1,m}(\overline{\Omega}, R^N)$ such that

$$\int_\Omega A_{ij}^{\alpha\beta}(x,u)\left(D_\beta u^j, D_\alpha \varphi^i\right) dx = \int_\Omega B_i(x,u,Du) \cdot \varphi^i\,dx, \tag{1.2}$$

holds for all test-functions $\varphi \in C_0^\infty(\overline{\Omega}, R^N)$ and, by approximation, for all $\varphi \in W_0^{1,m}(\overline{\Omega}, R^N)$, where we have introduced the notation

$$A_{ij}^{\alpha\beta}(x,\xi)(\nu,\tilde{\nu}) = \left(A_{ij}^{\alpha\beta}(x,\xi)\nu\right) \cdot \tilde{\nu}. \tag{1.3}$$

In the current situation, Sobolev's embedding theorem yields the existence of a constant C_s depending only on s, n, and N such that we have

$$\sup_{B_\rho^+(x_0)} \left|g - g'_{x_0,\rho}\right| \le C_s \rho^{1-\frac{n}{s}} \|g\|_{H^{1,s}(B_\rho^+(x_0),R^N)} \tag{1.4}$$

for $x_0 \in D$, $\rho \le 1 - |x_0|$. Obviously, the inequality remains true if we replace $\|g\|_{H^{1,s}(B_\rho^+(x_0),R^N)}$ by $\|g\|_{H^{1,s}(B^+,R^N)}$, which we will henceforth abbreviate simply as $\|g\|_{H^{1,s}}$.

We also note here that Poincaré's inequality in this setting yields

$$\int_{B_\rho^+(x_0)} \left|g - g'_{x_0,\rho}\right|^m dx \le C_p \rho^m \int_{B_\rho^+(x_0)} |Dg|^m dx \tag{1.5}$$

for a constant C_p depending only on n.

Finally, we fix an exponent $\sigma \in (0,1)$ as follows: if $g \equiv 0$, σ can be chosen arbitrary (but henceforth fixed); otherwise we take σ fixed in $(0, 1 - \frac{n}{s}]$.

Under such assumptions, one cannot expect that weak solutions to (1.1) will be classical [4]. This was first shown by De Giorgi [5]. Thus, our goal is to establish a partial regularity for weak solutions of systems (1.1).

There are some previous partial regularity results at boundary for inhomogeneous quasilinear systems. Arkhipova has studied regularity up to the boundary for nonlinear and quasilinear systems [6–8]. For systems in diagonal form, boundary regularity was first established by Wiegner [9], and the proof was generalized and extended by Hildebrandt-Widman [10]. Jost-Meier [11] established full regularity in a neighborhood of boundary for minima of functionals with the form $\int_\Omega A(x,u)|Du|^2\,dx$.

The results which are most closely related to that given here were shown in [12] and [13]. In this paper, we would get the desired conclusions by the method of A-harmonic approximation. The A-harmonic approximation technique is a natural extension of harmonic approximation technique. In [14] Simon used harmonic approximation method to simplify Allard's [15] regularity theorem and later on Schoen and Uhlenbeck's [16] regularity result for harmonic maps. The idea was generalized to more general linear operators by Duzaar and Steffen [17], in order to deal with the regularity of almost minimizers to elliptic variational integrals in the setting of geometric measure theory. As a by-product Duzaar and Grotowski [18] were able to use the idea of A-harmonic approximation to deal with elliptic systems under quadratic growth, even to the boundary points for nonlinear elliptic systems [19] and variational problems [20].

In this context, we use an A-harmonic approximation method to establish boundary regularity results.

Theorem 1.1 *Let Ω be a bounded domain in R^N, with boundary of class C^1. Let u be a weak solution of* (1.1) *satisfying the structural conditions* (H1)-(H5). *Consider a fixed $\gamma \in (0,\sigma]$. Then there exist positive R_1 and ε_0 (depending only on n, N, λ, L, $\omega(\cdot)$ and γ) with the property that*

$$⨍_{B_R(x_0)\cap\Omega} \left[\left|u - u'_{x_0,R}\right|^2 + \frac{|u - u'_{x_0,R}|^m}{R^{m-2}}\right] dx + \left[\|g\|^2_{H^{1,s}} + \|g\|^m_{H^{1,s}}\right] R^{2(1-\frac{n}{s})} + R^2 \le \varepsilon_0^2$$

for some $R \in (0, R_1]$, $x_0 \in \partial\Omega$, which implies $u \in C^{0,\gamma}(\overline{B}_{\frac{R}{2}}(x_0) \cap \overline{\Omega}, R^N)$.

Note in particular that the boundary condition (H5) means that $u'_{x_0,R}$ makes sense: in fact, we have $u'_{x_0,R} = g'_{x_0,R}$.

A standard covering argument [3] allows us to obtain the following.

Corollary 1.1 *Under the assumptions of Theorem* 1.1 *the singular set of the weak solution u has $(n-m)$-dimensional Hausdorff measure zero in $\overline{\Omega}$.*

If the domain of the main step in proving Theorem 1.1 is a half ball, the result then is the following.

Theorem 1.2 *Consider a weak solution of* (1.1) *on the upper half unit ball B^+ which satisfies the structural conditions* (H1)-(H4) *and* (H5)$'$. *Then there exist positive R_0 and ε_0* (*depending only on n, N, λ, L, $\omega(\cdot)$ and γ*) *with the property that*

$$\fint_{B_R(x_0)\cap\Omega}\left[\left|u-u'_{x_0,R}\right|^2+\frac{|u-u'_{x_0,R}|^m}{R^{m-2}}\right]dx+\left[\|g\|^2_{H^{1,s}}+\|g\|^m_{H^{1,s}}\right]R^{2(1-\frac{n}{s})}+R^2\le\varepsilon_0^2$$

for some $R\in(0,R_0]$, $x_0\in\partial\Omega$, which implies $u\in C^{0,\gamma}(\overline{B}_{\frac{R}{2}}(x_0),R^N)$.

Analogously to above, the boundary condition (H5)$'$ ensures that $u'_{x_0,R}$ exists, and $u'_{x_0,R}=g'_{x_0,R}$.

2 The *A*-harmonic approximation technique

In this section we present an A-harmonic approximation lemma [12], and some standard results due to Campanato [21, 22].

Lemma 2.1 (A-harmonic approximation lemma) *Consider fixed positive λ and L, and $n,N\in N$ with $n\ge 2$. Then for any given $\varepsilon>0$ there exists $\delta=\delta(n,N,\lambda,L,\varepsilon)\in(0,1]$ with the following property: for any $A\in\mathrm{Bil}(R^{nN})$ satisfying*

$$A(v,v)\ge\lambda|v|^2\quad\text{for all } v\in R^{nN}\tag{2.1}$$

and

$$\left|A(v,\overline{v})\right|\le L|v||\overline{v}|\quad\text{for all } v,\overline{v}\in R^{nN}\tag{2.2}$$

for any $w\in H^{1,2}(B^+_\rho(x_0),R^N)$ (*for some $\rho>0$, $x_0\in R^n$*) *satisfying*

$$\rho^{2-n}\int_{B^+_\rho(x_0)}|Dw|^2\,dx\le 1\tag{2.3}$$

and

$$\left|\rho^{2-n}\int_{B^+_\rho(x_0)}A(Dg,D\varphi)\,dx\right|\le\delta\rho\sup_{B^+_\rho(x_0)}|D\varphi|,\tag{2.4}$$

and

$$w|_{D_\rho(x_0)}=0\tag{2.5}$$

for all $\varphi\in C^1_0(B^+_\rho(x_0),R^N)$, there exists an A-harmonic function

$$v\in\tilde{H}=\left\{\tilde{w}\in H^{1,2}\big(B^+_\rho(x_0),R^N\big)\,\middle|\,\rho^{2-n}\int_{B^+_\rho(x_0)}|D\tilde{w}|^2\,dx\le 1,\tilde{w}|_{D_\rho(x_0)}\equiv 0\right\},$$

with

$$\rho^{-n}\int_{B^+_\rho(x_0)}|v-w|^2\,dx\le\varepsilon.\tag{2.6}$$

Next we recall a characterization of Hölder continuous functions with a slight modification [21].

Lemma 2.2 *Consider $n \in N$, $n \geq 2$, and $x_0 \in R^{n-1} \times \{0\}$. Suppose that there are positive constants κ and α, with $\alpha \in (0,1]$ such that, for some $v \in L^2(B^+_{6R}(x_0))$, we have the following*:

$$\inf_{\mu \in R} \left\{ \rho^{-n} \int_{B^+_\rho(y)} |v - \mu|^2 \, dx \right\} \leq \kappa^2 \left(\frac{\rho}{R} \right)^{2\alpha} \tag{2.7}$$

for all $y \in D_{2R}(x_0)$ and $\rho \leq 4R$; and

$$\inf_{\mu \in R} \rho^{-n} \left\{ \int_{B_\rho(y)} |v - \mu|^2 \, dx \right\} \leq \kappa^2 \left(\frac{\rho}{R} \right)^{2\alpha} \tag{2.8}$$

for all $y \in B^+_{2R}(x_0)$ and $B_\rho(y) \subset B^+_{2R}(x_0)$.

Then there exists a Hölder continuous representative of the L^2-class of v on $\overline{B}^+_R(x_0)$, and for this representative $\overline{v}$ we have

$$\left| \overline{v}(x) - \overline{v}(z) \right| \leq C_\kappa \left(\frac{|x - z|}{R} \right)^{\alpha} \tag{2.9}$$

for all $x, z \in \overline{B}^+_R(x_0)$, with the constant C_κ depending only on n and α.

We close this section by a standard estimate for the solutions to homogeneous second order elliptic systems with constant coefficients, due originally to Campanato [22].

Lemma 2.3 *Consider fixed positive λ and L, and $n, N \in N$ with $n \geq 2$. Then there exists C_0 depending only on n, N, λ, and L (without loss of generality we take $C_0 \geq 1$) such that for $A \in \mathrm{Bil}(R^{nN})$ satisfying (2.1) and (2.2), any A-harmonic function h on $B^+_\rho(x_0)$ with $h|_{D_\rho(x_0)} \equiv 0$ satisfies*

$$\rho^2 \sup_{B^+_{\frac{\rho}{2}}(x_0)} |Dh|^2 \leq C_0 \rho^{2-n} \int_{B^+_\rho(x_0)} |Dh|^2 \, dx.$$

3 Caccioppoli inequality

In this section we prove Caccioppoli's inequality.

Theorem 3.1 (Caccioppoli inequality) *Let $u \in W^{1,m}(\overline{\Omega}, R^N)$ be a weak solution of systems* (1.1) *under the conditions* (H1)-(H5). *Then there exists $\rho_0 > 0$ depending only on L, s, and $\|g\|_{H^{1,s}}$, such that for all $B^+_\rho(x_0) \subset B^+_R(x_0)$, with $x_0 \in D^+$, $\rho < R < \rho_0$, we have*

$$\begin{aligned} &\int_{B^+_\rho(x_0)} \left(|Du|^2 + \left| D|u|^{\frac{m}{2}} \right|^2 \right) dx \\ &\quad \leq C_1 \int_{B^+_R(x_0)} \left[\frac{1}{(R-\rho)^2} \left| u(x) - u'_{x_0,R} \right|^2 + \frac{1}{(R-\rho)^m} \left| u(x) - u'_{x_0,R} \right|^m \right] dx \\ &\qquad + C_2 \left[\|g\|^2_{H^{1,s}} R^{n(1-\frac{2}{s})} + \|g\|^m_{H^{1,s}} R^{n(1-\frac{m}{s})} \right] + C_3 \alpha_n R^n, \end{aligned} \tag{3.1}$$

where C_1, C_2, C_3 depend only on λ, L, m, and $\|g\|_{L^\infty(B,R^N)}$, and C_2 additionally on C_p, C_s, and also on s.

Proof Now we consider a cut off function $\eta \in C_0^\infty(B_\rho^+(x_0))$, satisfying $0 \le \eta \le 1$, $\eta \equiv 0$ on $B_\rho^+(x_0)$, and $|\nabla\eta| < \frac{1}{R-\rho}$. Then the function $\varphi = (u-g)\eta^2$ is in $W_0^{1,m}(B_\rho^+(x_0), R^N)$, and thus it can be taken as a test-function.

Using conditions (H1), (H4), (H5), and the definition of weak solutions (1.2), we have

$$\begin{aligned}
&\int_{B_R^+(x_0)} A_{ij}^{\alpha\beta}(\cdot,u)(Du,Du)\eta^2\,dx \\
&\quad\le 2L\int_{B_R^+(x_0)} \bigl(1+|u|^2\bigr)^{\frac{m-2}{2}}|Du||D\eta||u-g|\eta\,dx + L\int_{B_R^+(x_0)} \bigl(1+|u|^2\bigr)^{\frac{m-2}{2}}|Du||Dg|\eta^2\,dx \\
&\qquad + C\int_{B_R^+(x_0)} \bigl(|Du|^{m(1-\frac{1}{r})} + |u|^{r-1} + 1\bigr)|\varphi|\,dx.
\end{aligned} \tag{3.2}$$

By Young's, Hölder's and then Sobolev's inequalities, together with the estimate inequalities (1.4), (1.5), and the fact that $u'_{x_0,R} = g'_{x_0,R}$, yields

$$\begin{aligned}
&\int_{B_R^+(x_0)} A_{ij}^{\alpha\beta}(x,u)(Du,Du)\eta^2\,dx \\
&\quad\le 2\varepsilon\int_{B_R^+(x_0)} \bigl(1+|u|^2\bigr)^{\frac{m-2}{2}}|Du|^2\eta^2\,dx + \frac{2L}{\varepsilon}\int_{B_R^+(x_0)} \bigl(1+|u|^2\bigr)^{\frac{m-2}{2}}|D\eta|^2\bigl|u-u'_{x_0,R}\bigr|^2\,dx \\
&\qquad + \frac{2L^2}{\varepsilon}\int_{B_R^+(x_0)} \bigl(1+|u|^2\bigr)^{\frac{m-2}{2}}|D\eta|^2\bigl|g-g'_{x_0,R}\bigr|^2\,dx \\
&\qquad + \frac{L^2}{\varepsilon}\int_{B_R^+(x_0)} \bigl(1+|u|^2\bigr)^{\frac{m-2}{2}}|Dg|^2\eta^2\,dx \\
&\qquad + C\biggl(\int_{B_R^+(x_0)} \bigl(|Du|^m + |u|^r + 1\bigr)\,dx\biggr)^{1-\frac{1}{r}}\biggl(\int_{B_R^+(x_0)} |D\varphi|^m\,dx\biggr)^{\frac{1}{m}}.
\end{aligned} \tag{3.3}$$

Using Young's inequality again,

$$\begin{aligned}
&\int_{B_R^+(x_0)} \bigl(1+|u|^2\bigr)^{\frac{m-2}{2}}|D\eta|^2\bigl|u-u'_{x_0,R}\bigr|^2\,dx \\
&\quad= \int_{B_R^+(x_0)} \bigl(1+\bigl|u-u'_{x_0,R}+u'_{x_0,R}\bigr|^2\bigr)^{\frac{m-2}{2}}|D\eta|^2\bigl|u-u'_{x_0,R}\bigr|^2\,dx \\
&\quad\le 2^{\frac{m-2}{2}}\int_{B_R^+(x_0)} \bigl(1+\bigl|u'_{x_0,R}\bigr|^2\bigr)^{\frac{m-2}{2}}|D\eta|^2\bigl|u-u'_{x_0,R}\bigr|^2\,dx + 2^{\frac{m-2}{2}}\int_{B_R^+(x_0)} \frac{|u-u'_{x_0,R}|^m}{(R-\rho)^m}\,dx,
\end{aligned}$$

here we have used the fact that $|D\eta| < \frac{1}{R-\rho}$ and $0 < \rho < R < 1$.

Similarly, we have

$$\begin{aligned}
&\int_{B_R^+(x_0)} \bigl(1+|u|^2\bigr)^{\frac{m-2}{2}}|D\eta|^2\bigl|g-g'_{x_0,R}\bigr|^2\,dx \\
&\quad= \int_{B_R^+(x_0)} \bigl(1+\bigl|u-u'_{x_0,R}+u'_{x_0,R}\bigr|^2\bigr)^{\frac{m-2}{2}}|D\eta|^2\bigl|g-g'_{x_0,R}\bigr|^2\,dx
\end{aligned}$$

$$
\begin{aligned}
&\le 2^{\frac{m-2}{2}} \int_{B_R^+(x_0)} \left(1 + \left|u'_{x_0,R}\right|^2\right)^{\frac{m-2}{2}} |D\eta|^2 \left|g - g'_{x_0,R}\right|^2 \, dx \\
&\quad + 2^{\frac{m-2}{2}} \int_{B_R^+(x_0)} \left|u - u'_{x_0,R}\right|^{m-2} |D\eta|^2 \left|g - g'_{x_0,R}\right|^2 \, dx \\
&\le 2^{\frac{m-2}{2}} \int_{B_R^+(x_0)} \left(1 + \left|u'_{x_0,R}\right|^2\right)^{\frac{m-2}{2}} |D\eta|^2 \left|g - g'_{x_0,R}\right|^2 \, dx \\
&\quad + 2^{\frac{m-2}{2}} \int_{B_R^+(x_0)} \left|u - u'_{x_0,R}\right|^{m} \, dx + 2^{\frac{m-2}{2}} \int_{B_R^+(x_0)} |D\eta|^m \left|g - g'_{x_0,R}\right|^m \, dx \\
&\le 2^{\frac{m-2}{2}} \left(1 + \left|u'_{x_0,R}\right|^2\right)^{\frac{m-2}{2}} C_p \int_{B_R^+(x_0)} |Dg|^2 \, dx + 2^{\frac{m-2}{2}} C_p \int_{B_R^+(x_0)} |Dg|^m \, dx \\
&\quad + 2^{\frac{m-2}{2}} \int_{B_R^+(x_0)} \left|u - u'_{x_0,R}\right|^{m} \, dx,
\end{aligned}
$$

$$
\begin{aligned}
&\int_{B_R^+(x_0)} \left(1 + |u|^2\right)^{\frac{m-2}{2}} \eta^2 |Dg|^2 \, dx \\
&\quad \le 2^{\frac{m-2}{2}} \int_{B_R^+(x_0)} \left(1 + \left|u'_{x_0,R}\right|^2\right)^{\frac{m-2}{2}} \eta^2 |Dg|^2 \, dx \\
&\qquad + 2^{\frac{m-2}{2}} \int_{B_R^+(x_0)} \left|u - u'_{x_0,R}\right|^{m-2} \eta^2 |Dg|^2 \, dx \\
&\quad \le 2^{\frac{m-2}{2}} \int_{B_R^+(x_0)} \left(1 + \left|u'_{x_0,R}\right|^2\right)^{\frac{m-2}{2}} \eta^2 |Dg|^2 \, dx + 2^{\frac{m-2}{2}} \int_{B_R^+(x_0)} |Dg|^m \eta^m \, dx \\
&\qquad + 2^{\frac{m-2}{2}} \int_{B_R^+(x_0)} \left|u - u'_{x_0,R}\right|^{m} \, dx
\end{aligned}
$$

and

$$
\begin{aligned}
&\left(\int_{B_R^+(x_0)} \left(|Du|^m + |u|^r + 1\right) dx\right)^{1-\frac{1}{r}} \left(\int_{B_R^+(x_0)} |D\varphi|^m \, dx\right)^{\frac{1}{m}} \\
&\quad \le \varepsilon \int_{B_R^+(x_0)} |D\varphi|^m \, dx + C(\varepsilon) \left(\int_{B_R^+(x_0)} \left(|Du|^m + |u|^r + 1\right) dx\right)^{(1-\frac{1}{r})(\frac{m}{m-1})}.
\end{aligned}
$$

Combining these estimates in (3.3), we have

$$
\begin{aligned}
&\int_{B_R^+(x_0)} A_{ij}^{\alpha\beta}(x,u)(Du, Du)\eta^2 \, dx \\
&\quad \le 2\varepsilon \int_{B_R^+(x_0)} \left(1 + |u|^2\right)^{\frac{m-2}{2}} |Du|^2 \eta^2 \, dx \\
&\qquad + 2^{\frac{m-2}{2}} \int_{B_R^+(x_0)} \left(1 + \left|u'_{x_0,R}\right|^2\right)^{\frac{m-2}{2}} |D\eta|^2 \left|u - u'_{x_0,R}\right|^2 \, dx \\
&\qquad + 2^{\frac{m+2}{2}} \int_{B_R^+(x_0)} \frac{|u - u'_{x_0,R}|^m}{(R-\rho)^m} \, dx + 2^{\frac{m-2}{2}} \int_{B_R^+(x_0)} \left(1 + \left|u'_{x_0,R}\right|^2\right)^{\frac{m-2}{2}} (1 + C_p) |Dg|^2 \, dx \\
&\qquad + 2^{\frac{m-2}{2}} (1 + C_p) \int_{B_R^+(x_0)} |Dg|^m \, dx + \varepsilon \int_{B_R^+(x_0)} |D\varphi|^m \, dx \\
&\qquad + C(\varepsilon) \left(\int_{B_R^+(x_0)} \left(|Du|^m + |u|^r + 1\right) dx\right)^{(1-\frac{1}{r})(\frac{m}{m-1})}.
\end{aligned}
$$

Noting that $u \in W^{1,m}(\overline{B_R^+(x_0)}, R^N)$, we have

$$\begin{aligned}
&\left(\int_{B_R^+(x_0)} \left(|Du|^m + |u|^r + 1\right) dx\right)^{\frac{m}{m-1}(1-\frac{1}{r})} \\
&\quad \le \left(\int_{B_R^+(x_0)} |Du|^m \, dx\right)^{\frac{m}{m-1}(1-\frac{1}{r})} + \left(\int_{B_R^+(x_0)} \left|u - u'_{x_0,R}\right|^r dx\right)^{\frac{m}{m-1}(1-\frac{1}{r})} \\
&\qquad + \left(\left|u'_{x_0,R}\right|^{r-1} + 1\right)\left(\alpha_n R^n/2\right)^{\frac{m}{m-1}(1-\frac{1}{r})} \\
&\quad \le \left(\int_{B_R^+(x_0)} |Du|^m \, dx\right)^{\frac{m}{m-1}(1-\frac{1}{r})} + \left(\int_{B_R^+(x_0)} |Du|^m \, dx\right)^{\frac{r-1}{m-1}} \\
&\qquad + \left(\left|u'_{x_0,R}\right|^{r-1} + 1\right)\left(\alpha_n R^n/2\right)^{\frac{m}{m-1}(1-\frac{1}{r})} \\
&\quad \le C\left(\varepsilon, \|u\|_{W^{1,m}}\right)\left(\left|u'_{x_0,R}\right|^{r-1} + 1\right)\left(\alpha_n R^n/2\right).
\end{aligned} \tag{3.4}$$

Using (H2), we thus have

$$\begin{aligned}
&(\lambda - 2\varepsilon) \int_{B_R^+(x_0)} \left[|Du|^2 + \left|D|u|^{\frac{m}{2}}\right|^2\right] dx \\
&\quad \le (\lambda - 2\varepsilon) \int_{B_R^+(x_0)} \left(1 + |u|^2\right)^{\frac{m-2}{2}} |Du|^2 \, dx \\
&\quad \le C(\varepsilon, m, L) \int_{B_R^+(x_0)} \left(|D\eta|^2 \left|u - u'_{x_0,R}\right|^2 + |D\eta|^m \left|u - u'_{x_0,R}\right|^m\right) dx \\
&\qquad + C(m, L, \varepsilon) \int_{B_R^+(x_0)} \left(|Dg|^2 + |Dg|^m\right) dx + C\left(\varepsilon, \|u\|_{W^{1,m}}\right)\left(\alpha_n R^n/2\right).
\end{aligned}$$

Recalling that $u'_{x_0,R} = g'_{x_0,R}$ and $\varphi = (u - g)\eta^2$, we get

$$\begin{aligned}
&(\lambda - 2\varepsilon) \int_{B_R^+(x_0)} \left[|Du|^2 + \left|D|u|^{\frac{m}{2}}\right|^2\right] dx \\
&\quad \le C(\varepsilon, L, m) \int_{B_R^+(x_0)} \left[\frac{1}{(R-\rho)^2}\left|u - u'_{x_0,R}\right|^2 + \frac{1}{(R-\rho)^m}\left|u - u'_{x_0,R}\right|^m\right] dx \\
&\qquad + C(\varepsilon, L, m, C_s, C_p)\left[\|g\|_{H^{1,s}}^2 \left(\alpha_n R^n/2\right) R^{(1-\frac{2}{s})} + \left(\frac{\alpha_n}{2} R^n\right)^{1-\frac{m}{s}} \|g\|_{H^{1,s}}^m\right] \\
&\qquad + C\left(\varepsilon, \|u\|_{W^{1,m}}\right)\left(\alpha_n R^n/2\right).
\end{aligned}$$

Fixing ε small enough yields the desired inequality immediately. □

4 Proof of the main theorem

In this section we proceed to the proof of partial regularity result.

Lemma 4.1 *Consider* $u \in W^{1,m}(\overline{\Omega}, R^N)$ *to be a weak solution of* (1.1), $x_0 \in D$, $y \in D_R(x_0)$, $D_\rho(y) \subset D_R(x_0)$, $R < 1 - |x_0|$, *and* $\varphi \in C_0^\infty(B^+_{\frac{\rho}{2}}(y), R^N)$ *with* $\sup_{B^+_\rho(y)} |D\varphi| \le 1$. *Then*

$$\left|\left(\frac{\rho}{2}\right)^{2-n} \int_{B^+_{\frac{\rho}{2}}(y)} A\left(y, u^+_{y,\rho}\right)(Du, D\varphi)\, dx\right| \le C_5 \sqrt{I}\left[\sqrt{I} + \omega(I)\right]\rho \sup_{B^+_{\frac{\rho}{2}}(x_0)} |D\varphi|, \tag{4.1}$$

where

$$I(z,r_0) = \fint_{B^+_{r_0}(z)} \left[\left|u - u'_{z,r_0}\right|^2 + \frac{1}{{r_0}^{m-2}} \left|u - u'_{z,r_0}\right|^m \right] dx + \left[\|g\|^2_{H^{1,s}} + \|g\|^m_{H^{1,s}} \right] {r_0}^{2(1-\frac{n}{s})} + {r_0}^2$$

for $z \in D$, $r \in (0, 1 - |z|)$.

Proof From the definition of weak solution we have

$$\begin{aligned}
&\int_{B^+_{\frac{\rho}{2}}(y)} A^{\alpha\beta}_{ij}\left(y, u'_{y,\rho}\right)(Du, D\varphi)\,dx \\
&\quad \le C\left[\int_{B^+_{\frac{\rho}{2}}(y)} \left(|Du|^{m(1-\frac{1}{r})} + |u|^{r-1} + 1\right) dx\right] \rho \sup_{B^+_{\frac{\rho}{2}}(y)} |D\varphi| \\
&\qquad + \int_{B^+_{\frac{\rho}{2}}(y)} \left|A^{\alpha\beta}_{ij}\left(y, u'_{y,\rho}\right) - A^{\alpha\beta}_{ij}(x,u)\right| \cdot |Du|\,dx \sup_{B^+_{\frac{\rho}{2}}(y)} |D\varphi|.
\end{aligned} \tag{4.2}$$

Similarly as (3.4), by Hölder's inequality and Sobolev's embedding theorem, we have

$$C \int_{B^+_{\frac{\rho}{2}}(y)} \left(|Du|^{m(1-\frac{1}{r})} + |u|^{r-1} + 1\right) dx \le C\left(\left|u'_{y,\rho/2}\right|^{r-1} + 1\right)\alpha_n \rho^n / 2.$$

Henceforth we restrict ρ to be sufficiently small. Applying in turn Young's inequality, (H3), Caccioppoli's inequality (Theorem 3.1) and then Jensen's inequality we calculate that

$$\begin{aligned}
&\int_{B^+_{\frac{\rho}{2}}(y)} A^{\alpha\beta}_{ij}\left(y, u'_{y,\rho}\right)(Du, D\varphi)\,dx \\
&\quad \le C\left(\left|u'_{y,\rho/2}\right|^{r-1} + 1\right)\alpha_n \rho^n/2 \cdot \rho \sup_{B^+_{\frac{\rho}{2}}(x_0)} |D\varphi| \\
&\qquad + \omega\left(\rho^m + \fint_{B^+_\rho(y)} \left|u - u'_{y,\rho}\right|^m dx\right)\left(\alpha_n \rho^n/2\right)\left(\fint_{B^+_\rho(y)} |Du|^2\,dx\right)^{\frac{1}{2}} \rho \sup_{B^+_{\frac{\rho}{2}}(x_0)} |D\varphi|.
\end{aligned} \tag{4.3}$$

For $z \in D$, $r_0 \in (0, 1 - |z|)$, we introduce the notation

$$I(z,r_0) = \fint_{B^+_{r_0}(y)} \left[\left|u - u'_{y,r_0}\right|^2 + \frac{1}{r_0^{m-2}} \left|u - u'_{y,r_0}\right|^m \right] dx + \left[\|g\|^2_{H^{1,s}} + \|g\|^m_{H^{1,s}} \right] r_0^{2(1-\frac{n}{s})} + r_0^2,$$

and further write I for $I(y,\rho)$. Defining the constant C_4 by $C_4 = max\{C_1, C_2, C_3\}$, from (4.3) and Theorem 3.1, we have

$$\int_{B^+_{\frac{\rho}{2}}(y)} A^{\alpha\beta}_{ij}\left(y, u'_{y,\rho}\right)(Du, D\varphi)\,dx \le \frac{1}{2}\alpha_n \rho^{n-1} C_4 I + \sqrt{C_4}\alpha_n \rho^{n-1}/2\omega(I)\sqrt{I}.$$

For arbitrary $\varphi \in C^\infty_0(\Omega, R^N)$ we thus have

$$\int_{B^+_{\frac{\rho}{2}}(y)} A^{\alpha\beta}_{ij}\left(y, u'_{y,\rho}\right)(Du, D\varphi)\,dx \le \frac{1}{2}\alpha_n \rho^{n-1} \max\{C_4, \sqrt{C_4}\}\sqrt{I}\left(\sqrt{I} + \omega(I)\right) \sup_{B^+_{\frac{\rho}{2}}(x_0)} |D\varphi|.$$

Multiplying through by $(\frac{\rho}{2})^{2-n}$, this yields

$$\left|\left(\frac{\rho}{2}\right)^{2-n}\int_{B^+_{\frac{\rho}{2}}(y)} A^{\alpha\beta}_{ij}\big(y,u'_{y,\rho}\big)(Du,D\varphi)\,dx\right| \le C_5\sqrt{I}\big(\sqrt{I}+\omega(I)\big)\rho\sup_{B^+_{\frac{\rho}{2}}(x_0)}|D\varphi| \tag{4.4}$$

for C_5 defined by $C_5 = 2^{n-2}\max\{C_4,\sqrt{C_4}\}$. □

Lemma 4.2 *Consider u satisfying the conditions of Theorem* 1.1 *and σ fixed, then we can find δ and s_0 together with positive constant C_{11} such that the smallness conditions $0 < \omega(|u|,s_0) \le \frac{\delta}{2}$ and $I(x_0,R) \le C_9^{-1}\min\{\frac{\delta^2}{4},s_0\}$ together imply the growth condition*

$$I(y,\theta\rho) \le \theta^{2\sigma} I(y,\rho).$$

Proof Set $w = u - g$, using in turn (H1), Young's inequality and Hölder's inequality, from (4.4) we can see that for $C_6 = C_5 + L(\frac{\alpha_n}{2})^{1-\frac{1}{s}}(1+|u'_{y,\rho}|^2)^{\frac{m-2}{2}}$:

$$\left|\left(\frac{\rho}{2}\right)^{2-n}\int_{B^+_{\frac{\rho}{2}}(y)} A^{\alpha\beta}_{ij}\big(y,u'_{y,\rho}\big)(Dw,D\varphi)\,dx\right| \le C_6\sqrt{I}\big(\sqrt{I}+\omega(I)\big)\rho\sup_{B^+_{\frac{\rho}{2}}(x_0)}|D\varphi|. \tag{4.5}$$

We now set $v = \frac{w}{\gamma}$, for $\gamma = C_6\sqrt{I}$. From (4.5) yields

$$\left|\left(\frac{\rho}{2}\right)^{2-n}\int_{B^+_{\frac{\rho}{2}}(y)} A^{\alpha\beta}_{ij}\big(y,u'_{y,\rho}\big)(Dv,D\varphi)\,dx\right| \le \big(\sqrt{I}+\omega(I)\big)\rho\sup_{B^+_{\frac{\rho}{2}}(x_0)}|D\varphi|, \tag{4.6}$$

and we observe from the definition of γ, (4.6) means that

$$\left(\frac{\rho}{2}\right)^{2-n}\int_{B^+_{\frac{\rho}{2}}(y)}|Dv|^2\,dx < 1. \tag{4.7}$$

Further we note that

$$v|_{D_\rho(y)} = \frac{1}{\gamma}w|_{D_\rho(y)} = \frac{1}{\gamma}(u-g)|_{D_\rho(y)} \equiv 0. \tag{4.8}$$

For $\varepsilon > 0$ we take $\delta = \delta(n,N,\lambda,L,\varepsilon)$ to be the corresponding δ from the A-harmonic approximation lemma. Suppose that we could ensure that the smallness condition

$$\sqrt{I}+\omega(I) \le \delta \tag{4.9}$$

holds. Then in view of (4.6), (4.7), (4.8) we would be able to apply A-harmonic approximation lemma to conclude that the existence of a function $h \in H^{1,2}(B^+_{\frac{\rho}{2}}(y),R^N)$, which is $A^{\alpha\beta}_{ij}(y,u'_{y,\rho})$-harmonic, with $h|_{D_{\frac{\rho}{2}}}(y) \equiv 0$ such that

$$\left(\frac{\rho}{2}\right)^{2-n}\int_{B^+_{\frac{\rho}{2}}(y)}|Dh|^2\,dx \le 1, \tag{4.10}$$

$$\left(\frac{\rho}{2}\right)^{-n}\int_{B^+_{\frac{\rho}{2}}(y)}|v-h|^2\,dx\le\varepsilon. \tag{4.11}$$

For $\theta\in(0,\frac{1}{4}]$ arbitrary (to be fixed later), from Lemma 2.3 and (4.10), recalling also that $h(y)=0$, we have

$$\sup_{B^+_{\theta\rho}(y)}|h|^2\le\theta^2\rho^2\sup_{B^+_{\frac{\rho}{4}}(y)}|Dh|^2\le 4C_0\theta^2. \tag{4.12}$$

Using (4.11) and (4.12) we observe

$$(\theta\rho)^{-n}\int_{B^+_{\theta\rho}(y)}|v|^2\,dx\le 2(\theta\rho)^{-n}\left[\left(\frac{\rho}{2}\right)^{-n}\varepsilon+\frac{1}{2}\alpha_n(\theta\rho)^n\sup_{B^+_{\theta\rho}(y)}|h|^2\right]\le 2^{1-n}\theta^{-n}\varepsilon+4\alpha_nC_0\theta^2$$

and hence, on multiplying this through by γ^2, we obtain the estimate

$$(\theta\rho)^{-n}\int_{B^+_{\theta\rho}(y)}|w|^2\,dx\le\left(2^{1-n}\theta^{-n}\varepsilon+4\alpha_nC_0\theta^2\right)\gamma^2. \tag{4.13}$$

For the time being, we restrict ourselves to the case that g does not vanish identically. Recalling that $w=u-g$, (4.13) yields, using in turn Poincaré's, Sobolev's and then Hölder's inequalities, noting also that $u'_{y,\theta\rho}=g'_{y,\theta\rho}$:

$$\begin{aligned}
&(\theta\rho)^{-n}\int_{B^+_{\theta\rho}(y)}\left|u-u'_{y,\theta\rho}\right|^2dx\\
&\quad\le 2\left(2^{1-n}\theta^{-n}\varepsilon+4\alpha_nC_0\theta^2\right)\gamma^2+2C_p(\theta\rho)^{2-n}\left[\frac{1}{2}\alpha_n(\theta\rho)^n\right]^{1-\frac{2}{s}}\|g\|^2_{H^{1,s}}\\
&\quad\le C_7\left(\theta^{-n}\varepsilon+\theta^2\right)I+C_7\theta^{2(1-\frac{n}{s})}I
\end{aligned} \tag{4.14}$$

for $C_7=\max\{8\alpha_nC_0C_6^2,2^{\frac{2}{s}}C_p\alpha_n^{1-\frac{2}{s}}\}$ and provided $\varepsilon=\theta^{n+2}$ together with $\theta^2\le\theta^{2(1-\frac{n}{s})}$, we have

$$(\theta\rho)^{-n}\int_{B^+_{\theta\rho}(y)}\left|u-u'_{y,\theta\rho}\right|^2dx\le 3C_7\theta^{2(1-\frac{n}{s})}I. \tag{4.15}$$

For $2<m<n$ ($n\ge 3$) we have $2<m<r$, where $r=\begin{cases}\frac{mn}{n-m}, & \text{if } n>m,\\ \text{any number}>m, & \text{if } m\ge n,\end{cases}$ with $\frac{1}{r}<\frac{1}{m}<\frac{1}{2}$. Therefore we can find $t\in[0,1]$ such that $\frac{1}{m}=\frac{1-t}{2}+\frac{t}{r}$.

Using Sobolev's, Caccioppoli's, and Young's inequalities together with (4.14), we have

$$\begin{aligned}
&(\theta\rho)^{-n}(\theta\rho)^{-m+2}\int_{B^+_{\theta\rho}(y)}\left|u-u'_{y,\theta\rho}\right|^m dx\\
&\quad\le\frac{(\theta\rho)^{-n}}{(\theta\rho)^{m-2}}\left(\int_{B^+_{\theta\rho}(y)}\left|u-u'_{y,\theta\rho}\right|^2dx\right)^{\frac{(1-t)m}{2}}\left(\int_{B^+_{\theta\rho}(y)}\left|u-u'_{y,\theta\rho}\right|^r dx\right)^{\frac{mt}{r}}\\
&\quad\le\frac{(\theta\rho)^{-n}}{(\theta\rho)^{m-2}}\left(C_7(\theta\rho)^n\left(\theta^{-n}\varepsilon+\theta^2+\theta^{2(1-\frac{n}{s})}\right)I\right)^{\frac{(1-t)m}{2}}\left((\theta\rho)^m\int_{B^+_{\theta\rho}(y)}|Du|^m\,dx\right)^t
\end{aligned}$$

$$\begin{aligned}
&\le \frac{(\theta\rho)^{-n}}{(\theta\rho)^{m-2}}\bigl[C_7(\theta\rho)^n\bigl(\theta^{-n}\varepsilon+\theta^2+\theta^{2(1-\frac{n}{s})}\bigr)I\bigr]^{\frac{(1-t)m}{2}}\left[\frac{\alpha_n}{2}(\theta\rho)^{(m+n)}C\right]^t \\
&\le (\theta\rho)^{-n}\left(\frac{(1-t)m}{2}(\theta\rho)^nC_7\bigl(\theta^{-n}\varepsilon+\theta^2+\theta^{2(1-\frac{n}{s})}\bigr)I+\frac{mt}{r}C^{\frac{r}{mt}}\left(\frac{\alpha_n}{2}\right)^{\frac{r}{m}}(\theta\rho)^{(mt+nt)\cdot\frac{r}{mt}}\right) \\
&\le C_8\bigl(\theta^{-n}\varepsilon+\theta^2+\theta^{2(1-\frac{n}{s})}\bigr)I
\end{aligned}\tag{4.16}$$

for $C_8 = \frac{(1-t)m}{2}C_7 + \frac{mt}{r}C^{\frac{r}{mt}}(\frac{\alpha_n}{2})^{\frac{r}{m}}$.

We then fix $\varepsilon = \theta^{n+2}$, note that this also fixes δ. Since $\rho \le 1$, we see from the definition of γ: $[\|g\|^2_{H^{1,s}} + \|g\|^m_{H^{1,s}}](\theta\rho)^{2(1-\frac{n}{s})} \le \theta^{2(1-\frac{n}{s})}I$, and further $(\theta\rho)^2 \le \theta^2 I$.

Combining these estimates with (4.15) and (4.16), we have

$$I(y,\theta\rho) \le 3(C_7+C_8+1)\theta^{2(1-\frac{n}{s})}I. \tag{4.17}$$

We choose $\theta \in (0,\frac{1}{4}]$ small enough, such that we have $3(C_7+C_8+1)\theta^{2(1-\frac{n}{s})} \le \theta^{2\sigma}$.

Thus from (4.17) we can see

$$I(y,\theta\rho) \le \theta^{2\sigma}I(y,\rho). \tag{4.18}$$

We now choose $s_0 > 0$ such that $0 < \omega(s_0) \le \frac{\delta}{2}$, and we define C_9 by $C_9 = 2^{n+2m+1}(2C_s^2 + 2^mC_s^m)$. Suppose that

$$I(x_0,R) \le C_9^{-1}\min\left\{\frac{\delta^2}{4}, s_0\right\} \tag{4.19}$$

for some $R \in (0,R_0]$, where $R_0 = \min\{\sqrt{2s_0}, 1-|x_0|\}$.

For any $y \in D_{\frac{R}{2}}(x_0)$ we use Sobolev's inequality to calculate

$$\begin{aligned}
\frac{\alpha_nR^n}{2^{n+1}}\bigl|u'_{x_0,R}-u'_{y,\frac{R}{2}}\bigr|^2 &= \int_{B^+_{\frac{R}{2}}(y)}\bigl|u'_{x_0,R}-u'_{y,\frac{R}{2}}\bigr|^2\,dx \\
&= \int_{B^+_{\frac{R}{2}}(y)}\bigl|g'_{x_0,R}-g'_{y,\frac{R}{2}}\bigr|^2\,dx \\
&\le 2\int_{B^+_{\frac{R}{2}}(y)}\bigl|g-g'_{x_0,R}\bigr|^2\,dx + 2\int_{B^+_{\frac{R}{2}}(y)}\bigl|g-g'_{y,\frac{R}{2}}\bigr|^2\,dx \\
&\le 2\alpha_nC_s^2\|g\|^2_{H^{1,s}}R^{n+2(1-\frac{n}{s})}
\end{aligned}\tag{4.20}$$

and

$$\begin{aligned}
\frac{\alpha_nR^n}{2^{n+1}}\bigl|u'_{x_0,R}-u'_{y,\frac{R}{2}}\bigr|^m &= \int_{B^+_{\frac{R}{2}}(y)}\bigl|u'_{x_0,R}-u'_{y,\frac{R}{2}}\bigr|^m\,dx \\
&= \int_{B^+_{\frac{R}{2}}(y)}\bigl|g'_{x_0,R}-g'_{y,\frac{R}{2}}\bigr|^m\,dx \\
&\le 2^{m-1}\int_{B^+_{\frac{R}{2}}(y)}\bigl|g-g'_{x_0,R}\bigr|^m\,dx + 2^{m-1}\int_{B^+_{\frac{R}{2}}(y)}\bigl|g-g'_{y,\frac{R}{2}}\bigr|^m\,dx
\end{aligned}$$

$$\begin{aligned}&\le 2^m\alpha_n C_s^m\|g\|_{H^{1,s}}^m R^{n+m(1-\frac{n}{s})}\\&\le 2^m\alpha_n C_s^m\|g\|_{H^{1,s}}^m R^{n+2(1-\frac{n}{s})}.\end{aligned}\tag{4.21}$$

Using these estimations, we can obtain

$$\begin{aligned}I\bigl(y,(1/2)R\bigr)\le{}& 2^{n+2m+1}\biggl[\fint_{B_R^+(y)}\biggl(\bigl|u-u'_{x_0,R}\bigr|^2+\frac{1}{R^{m-2}}\bigl|u-u'_{x_0,R}\bigr|^m\biggr)\,dx\biggr]+\frac{1}{4}R^2\\&+\bigl(2C_s^2+2^mC_s^m\bigr)\bigl[\|g\|_{H^{1,s}}^2+\|g\|_{H^{1,s}}^m\bigr]R^{2(1-\frac{n}{s})}\le C_{10}I(x_0,R).\end{aligned}\tag{4.22}$$

Thus we have

$$\sqrt{I\bigl(y,(1/2)R\bigr)}+\omega\bigl(I\bigl(y,(1/2)R\bigr)\bigr)\le\frac{\delta}{2}+\omega(s_0)\le\frac{\delta}{2}+\frac{\delta}{2}\le\delta,$$

which means that the condition (4.18) is sufficient to guarantee the smallness condition (4.9) for $\rho=R/2$, for all $y\in D_{R/2}(x_0)$. Thus, we can conclude that (4.17) holds in this situation. From (4.18) we thus have

$$\sqrt{I\bigl(y,(1/2)\theta R\bigr)}+\omega\bigl(I\bigl(y,(1/2)\theta R\bigr)\bigr)\le\frac{\delta}{2}+\omega(s_0)\le\frac{\delta}{2}+\frac{\delta}{2}\le\delta,$$

which means that we can apply (4.18) on $B^+_{\theta R/2}(y)$ as well, and this yields

$$I\bigl(y,\theta^2R/2\bigr)\le\theta^{4\sigma}I\bigl(y,(1/2)R\bigr),$$

and inductively we find

$$I\bigl(y,\theta^kR/2\bigr)\le\theta^{2\sigma k}I\bigl(y,(1/2)R\bigr).\tag{4.23}$$

The next step is to go from a discrete to a continuous version of the decay estimate. Given $\rho\in(0,R/2]$, we can find $k\in N_0$ such that $\theta^{k+1}R/2<\rho<\theta^kR/2$. Then we calculate in a similar manner to above. Firstly, we use Sobolev's inequality (1.4) to see that

$$\int_{B_\rho^+(y)}\bigl|u'_{y,\rho}-u'_{y,\theta^kR/2}\bigr|^2\,dx\le 2\alpha_nC_s^2\|g\|_{H^{1,s}}^2\biggl(\frac{1}{2}\theta^kR\biggr)^{n+2(1-\frac{n}{s})}$$

and

$$\int_{B_\rho^+(y)}\bigl|u'_{y,\rho}-u'_{y,\theta^kR/2}\bigr|^m\,dx\le 2^m\alpha_nC_s^m\|g\|_{H^{1,s}}^m\biggl(\frac{1}{2}\theta^kR\biggr)^{n+m(1-\frac{n}{s})},$$

which allows us to deduce that

$$\begin{aligned}&\int_{B_\rho^+(y)}\biggl(\bigl|u-u'_{y,\rho}\bigr|^2+\frac{1}{\rho^{m-2}}\bigl|u-u'_{y,\rho}\bigr|^m\biggr)\,dx\\&\quad\le 2^{2m-3}\biggl[\int_{B_\rho^+(y)}\biggl(\bigl|u-u'_{y,\theta^kR/2}\bigr|^2+\frac{1}{(\theta^kR/2)^{m-2}}\bigl|u-u'_{y,\theta^kR/2}\bigr|^m\biggr)\,dx\biggr]\\&\qquad+4\alpha_nC_s^2\|g\|_{H^{1,s}}^2\biggl(\frac{1}{2}\theta^kR\biggr)^{n+2(1-\frac{n}{s})}+2^{m+1}\alpha_nC_s^m\|g\|_{H^{1,s}}^m\biggl(\frac{1}{2}\theta^kR\biggr)^{n+m(1-\frac{n}{s})},\end{aligned}$$

and hence, finally

$$I(y,\rho) \le C_{11} I\big(y, \theta^k R/2\big)$$

for $C_{11} = \max\{2^{2m-3}, 8C_s^2\theta^{-n}, 2^{m+1}C_s^m\theta^{-n}\}$. We combine these estimates with (4.22) and (4.23):

$$\begin{aligned} I(y,\rho) &\le C_{11}\theta^{2\sigma k} I(y,R/2) \le C_{10}C_{11}\theta^{-2\sigma}\left(\frac{2\rho}{R}\right)^{2\sigma} I(x_0,R) \\ &\le C_{10}C_{11}\left(\frac{2}{\theta}\right)^{2\sigma}\left(\frac{\rho}{R}\right)^{2\sigma} I(x_0,R), \end{aligned}$$

and more particularly

$$\inf_{\mu\in R^N}\int_{B_\rho^+(y)} |u-\mu|^2\,dx \le C_{12} I(x_0,R)\left(\frac{\rho}{R}\right)^{2\sigma} \tag{4.24}$$

for C_{12} given by $C_{12} = C_{10}C_{11}(\frac{2}{\theta})^{2\sigma}$. We recall that this estimate is valid for all $y \in D$ and ρ with $D_\rho(y) \subset D_{R/2}(x_0)$, and we assume only the smallness condition (4.19) on $I(x_0,R)$. This yields after replacing R by $6R$ the boundary estimate required to apply Lemma 2.2.

Similarly, one can get the analogous interior estimate as (4.24). Applying Lemma 2.2, we can conclude the desired Hölder continuity. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
SC participated in design of the study and drafted the manuscript. ZT participated in conceiving of the study and the amendment of the paper. All authors read and approved the final manuscript.

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