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Existence of fast homoclinic orbits for a class of second-order non-autonomous problems

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Abstract
By applying the mountain pass theorem and the symmetric mountain pass theorem in critical point theory, the existence and multiplicity of fast homoclinic solutions are obtained for the following second-order non-autonomous problem: $\ddot{u}(t)+q(t)\dot{u}(t)-a(t)|u(t)|^{p-2}u(t)+\nabla W(t,u(t))=0$, where $p\geq 2$, $t\in\mathbb{R}$, $u\in\mathbb{R}^N$, $a\in C(\mathbb{R},\mathbb{R})$, $W\in C^1(\mathbb{R}\times\mathbb{R}^N,\mathbb{R})$ are not periodic in t and $q:\mathbb{R}\to\mathbb{R}$ is a continuous function and $Q(t)=\int_0^s q(s)\,ds$ with $\lim_{|t|\to+\infty}Q(t)=+\infty$.

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Keywords: fast homoclinic solutions; variational methods; critical point

1 Introduction

Consider fast homoclinic solutions of the following problem:

$$\ddot{u}(t)+q(t)\dot{u}(t)-a(t)\big|u(t)\big|^{p-2}u(t)+\nabla W\big(t,u(t)\big)=0,\quad t\in\mathbb{R},\tag{1.1}$$

where $p\geq 2$, $t\in\mathbb{R}$, $u\in\mathbb{R}^N$, $a\in C(\mathbb{R},\mathbb{R})$, $W\in C^1(\mathbb{R}\times\mathbb{R}^N,\mathbb{R})$ are not periodic in t, and $q:\mathbb{R}\to\mathbb{R}$ is a continuous function and $Q(t)=\int_0^s q(s)\,ds$ with

$$\lim_{|t|\to+\infty}Q(t)=+\infty.\tag{1.2}$$

When $q(t)\equiv 0$, problem (1.1) reduces to the following special second-order Hamiltonian system:

$$\ddot{u}(t)-a(t)\big|u(t)\big|^{p-2}u(t)+\nabla W\big(t,u(t)\big)=0,\quad t\in\mathbb{R}.\tag{1.3}$$

When $p=0$, problem (1.1) reduces to the following second-order damped vibration problem:

$$\ddot{u}(t)+q(t)\dot{u}(t)-a(t)u(t)+\nabla W\big(t,u(t)\big)=0,\quad t\in\mathbb{R}.\tag{1.4}$$

If we take $p=2$ and $q(t)\equiv 0$, then problem (1.1) reduces to the following second-order Hamiltonian system:

$$\ddot{u}(t)-a(t)u(t)+\nabla W\big(t,u(t)\big)=0,\quad t\in\mathbb{R}.\tag{1.5}$$

The existence of homoclinic orbits plays an important role in the study of the behavior of dynamical systems. If a system has transversely intersected homoclinic orbits, then it must be chaotic. If it has smoothly connected homoclinic orbits, then it cannot stand the perturbation, and its perturbed system probably produces chaotic phenomena. The first work about homoclinic orbits was done by Poincaré [1].

Recently, the existence and multiplicity of homoclinic solutions and periodic solutions for Hamiltonian systems have been extensively studied by critical point theory. For example, see [2–30] and references therein. In [6, 16, 17], the authors considered homoclinic solutions for the special Hamiltonian system (1.3) in weighted Sobolev space. Later, Shi *et al.* [31] obtained some results for system (1.3) with a p-Laplacian, which improved and generalized the results in [6, 16, 17]. However, there is little research as regards the existence of homoclinic solutions for damped vibration problems (1.4) when $q(t) \not\equiv 0$. In 2008, Wu and Zhou [32] obtained some results for damped vibration problems (1.4) with some boundary value conditions by variational methods. Zhang and Yuan [33, 34] studied the existence of homoclinic solutions for (1.4) when $q(t) \equiv c$ is a constant. Later, Chen *et al.* [35] investigated fast homoclinic solutions for (1.4) and obtained some new results under more relaxed assumptions on $W(t,x)$, which resolved some open problems in [33]. Zhang [36] obtained infinitely many solutions for a class of general second-order damped vibration systems by using the variational methods. Zhang [37] investigated subharmonic solutions for a class of second-order impulsive systems with damped term by using the mountain pass theorem.

Motivated by [21, 23, 32–34, 38–42], we will establish some new results for (1.1) in weighted Sobolev space. In order to introduce the concept of fast homoclinic solutions for problem (1.1), we first state some properties of the weighted Sobolev space E on which the certain variational functional associated with (1.1) is defined and the fast homoclinic solutions are the critical points of the certain functional.

Let

$$X = \left\{ u \in H^{1,2}(\mathbb{R}, \mathbb{R}^N) \,\Big|\, \int_{\mathbb{R}} e^{Q(t)} \left[|\dot{u}(t)|^2 + |u(t)|^2 \right] dt < +\infty \right\},$$

where $Q(t)$ is defined in (1.2) and for $u, v \in X$, let

$$\langle u, v \rangle = \int_{\mathbb{R}} e^{Q(t)} \left[\left(\dot{u}(t), \dot{v}(t) \right) + \left(u(t), v(t) \right) \right] dt.$$

Then X is a Hilbert space with the norm given by

$$\|u\| = \left(\int_{\mathbb{R}} e^{Q(t)} \left[|\dot{u}(t)|^2 + |u(t)|^2 \right] dt \right)^{1/2}.$$

It is obvious that

$$X \subset L^2\left(e^{Q(t)}\right)$$

with the embedding being continuous. Here $L^p(e^{Q(t)})$ $(2 \leq p < +\infty)$ denotes the Banach spaces of functions on $\mathbb{R}$ with values in $\mathbb{R}^N$ under the norm

$$\|u\|_p = \left\{ \int_{\mathbb{R}} e^{Q(t)} |u(t)|^p \, dt \right\}^{1/p}.$$

If σ is a positive, continuous function on $\mathbb{R}$ and $1 < s < +\infty$, let

$$L_\sigma^s\left(e^{Q(t)}\right) = \left\{u \in L_{\mathrm{loc}}^1\left(e^{Q(t)}\right) \middle| \int_{\mathbb{R}} \sigma(t)e^{Q(t)}\left|u(t)\right|^s dt < +\infty\right\}.$$

L_σ^s equipped with the norm

$$\|u\|_{s,\sigma} = \left(\int_{\mathbb{R}} \sigma(t)e^{Q(t)}\left|u(t)\right|^s dt\right)^{1/s}$$

is a reflexive Banach space.

Set $E = X \cap L_a^p(e^{Q(t)})$, where a is the function given in condition (A). Then E with its standard norm $\|\cdot\|$ is a reflexive Banach space. Similar to [33, 35], we have the following definition of fast homoclinic solutions.

Definition 1.1 If (1.2) holds, a solution of (1.1) is called a fast homoclinic solution if $u \in E$.

The functional φ corresponding to (1.1) on E is given by

$$\varphi(u) = \int_{\mathbb{R}} e^{Q(t)}\left[\frac{1}{2}\left|\dot{u}(t)\right|^2 + \frac{a(t)}{p}\left|u(t)\right|^p - W\left(t, u(t)\right)\right] dt, \quad u \in E. \tag{1.6}$$

Clearly, it follows from (W1) or (W1)′ that $\varphi : E \to \mathbb{R}$. By Theorem 2.1 of [43], we can deduce that the map

$$u \to a(t)e^{Q(t)}\left|u(t)\right|^{p-2}u(t)$$

is continuous from $L_a^p(e^{Q(t)})$ in the dual space $L_{a^{-1/(p-1)}}^{p_1}(e^{Q(t)})$, where $p_1 = \frac{p}{p-1}$. As the embeddings $E \subset X \subset L^\gamma(e^{Q(t)})$ for all $\gamma \geq 2$ are continuous, if (A) and (W1) or (W1)′ hold, then $\varphi \in C^1(E, \mathbb{R})$ and one can easily check that

$$\begin{aligned}\left\langle \varphi'(u), v\right\rangle = \int_{\mathbb{R}} e^{Q(t)}\Big[&\left(\dot{u}(t), \dot{v}(t)\right) + a(t)\left|u(t)\right|^{p-2}\left(u(t), v(t)\right) \\ &- \left(\nabla W\left(t, u(t)\right), v(t)\right)\Big] dt, \quad u \in E.\end{aligned} \tag{1.7}$$

Furthermore, the critical points of φ in E are classical solutions of (1.1) with $u(\pm\infty) = 0$.

Now, we state our main results.

Theorem 1.1 *Suppose that a, q, and W satisfy* (1.2) *and the following conditions*:

(A) *Let $p > 2$, $a(t)$ is a continuous, positive function on $\mathbb{R}$ such that for all $t \in \mathbb{R}$*

$$a(t) \geq \alpha|t|^\beta, \quad \alpha > 0, \beta > (p-2)/2.$$

(W1) $W(t,x) = W_1(t,x) - W_2(t,x)$, $W_1, W_2 \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$, *and there exists a constant $R > 0$ such that*

$$\frac{1}{a(t)}\left|\nabla W(t,x)\right| = o\left(|x|^{p-1}\right) \quad \text{as } x \to 0$$

uniformly in $t \in (-\infty, -R] \cup [R, +\infty)$.

(W2) *There is a constant* $\mu > p$ *such that*

$$0 < \mu W_1(t,x) \le \big(\nabla W_1(t,x), x\big), \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N \backslash \{0\}.$$

(W3) $W_2(t,0) = 0$ *and there exists a constant* $\varrho \in (p, \mu)$ *such that*

$$W_2(t,x) \ge 0, \quad \big(\nabla W_2(t,x), x\big) \le \varrho W_2(t,x), \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N.$$

Then problem (1.1) *has at least one nontrivial fast homoclinic solution.*

Theorem 1.2 *Suppose that* a, q, *and* W *satisfy* (1.2), (A), (W2), *and the following conditions*:

(W1)′ $W(t,x) = W_1(t,x) - W_2(t,x)$, $W_1, W_2 \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$, *and*

$$\frac{1}{a(t)} \big|\nabla W(t,x)\big| = o\big(|x|^{p-1}\big) \quad \text{as } x \to 0$$

uniformly in $t \in \mathbb{R}$.

(W3)′ $W_2(t,0) = 0$ *and there exists a constant* $\varrho \in (p, \mu)$ *such that*

$$\big(\nabla W_2(t,x), x\big) \le \varrho W_2(t,x), \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N.$$

Then problem (1.1) *has at least one nontrivial fast homoclinic solution.*

Theorem 1.3 *Suppose that* a, q, *and* W *satisfy* (1.2), (A), (W1)-(W3), *and the following assumption*:

(W4) $W(t,-x) = W(t,x)$, $\forall (t,x) \in \mathbb{R} \times \mathbb{R}^N$.

Then problem (1.1) *has an unbounded sequence of fast homoclinic solutions.*

Theorem 1.4 *Suppose that* a, q, *and* W *satisfy* (1.2), (A), (W1)′, (W2), (W3)′, *and* (W4). *Then problem* (1.1) *has an unbounded sequence of fast homoclinic solutions.*

Remark 1.1 It is easy to see that our results hold true even if $p = 2$. To the best of our knowledge, similar results for problem (1.1) cannot be seen in the literature, from this point, our results are new. As pointed out in [17], condition (A) can be replaced by more general assumption: $a(t) \to +\infty$ as $|t| \to +\infty$.

The rest of this paper is organized as follows: in Section 2, some preliminaries are presented. In Section 3, we give the proofs of our results. In Section 4, some examples are given to illustrate our results.

2 Preliminaries

Let E and $\|\cdot\|$ be given in Section 1, by a similar argument in [41], we have the following important lemma.

Lemma 2.1 *For any* $u \in E$,

$$\|u\|_\infty \le \frac{1}{\sqrt{2e_0}} \|u\| = \frac{1}{\sqrt{2e_0}} \left\{ \int_{\mathbb{R}} e^{Q(s)} \big[\big|\dot{u}(s)\big|^2 + \big|u(s)\big|^2 \big] \, ds \right\}^{1/2}, \tag{2.1}$$

$$\begin{aligned}|u(t)| &\le \left\{\int_t^{+\infty} e^{-Q(s)}e^{Q(s)}\left[|\dot{u}(s)|^2 + |u(s)|^2\right] ds\right\}^{1/2} \\ &\le \frac{1}{\sqrt[4]{e_0}}\left\{\int_t^{+\infty} e^{Q(s)}\left[|\dot{u}(s)|^2 + |u(s)|^2\right] ds\right\}^{1/2}\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}|u(t)| &\le \left\{\int_{-\infty}^{t} e^{-Q(s)}e^{Q(s)}\left[|\dot{u}(s)|^2 + |u(s)|^2\right] ds\right\}^{1/2} \\ &\le \frac{1}{\sqrt[4]{e_0}}\left\{\int_{-\infty}^{t} e^{Q(s)}\left[|\dot{u}(s)|^2 + |u(s)|^2\right] ds\right\}^{1/2},\end{aligned} \tag{2.3}$$

where $\|u\|_\infty = \operatorname{ess\,sup}_{t\in\mathbb{R}} |u(t)|$, $e_0 = e^{\min\{Q(t):t\in\mathbb{R}\}}$.

The following lemma is an improvement result of [16].

Lemma 2.2 *If a satisfies assumption* (A), *then*

$$\text{the embedding } L_a^p\left(e^{Q(t)}\right) \subset L^2\left(e^{Q(t)}\right) \text{ is continuous.} \tag{2.4}$$

Moreover, there exists a Hilbert space Z such that

$$\text{the embeddings } L_a^p\left(e^{Q(t)}\right) \subset Z \subset L^2\left(e^{Q(t)}\right) \text{ are continuous;} \tag{2.5}$$

$$\text{the embedding } X \cap Z \subset L^2\left(e^{Q(t)}\right) \text{ is compact.} \tag{2.6}$$

Proof Let $\theta = p/(p-2)$, $\theta' = p/2$, we have

$$\begin{aligned}\|u\|_2^2 &= \int_{\mathbb{R}} e^{Q(t)}|u(t)|^2\,dt \\ &= \int_{\mathbb{R}} a^{-1/\theta'}a^{1/\theta'}e^{Q(t)/\theta'}e^{Q(t)/\theta}|u(t)|^2\,dt \\ &\le \left(\int_{\mathbb{R}} a^{-\theta/\theta'}e^{Q(t)}\,dt\right)^{1/\theta}\left(\int_{\mathbb{R}} ae^{Q(t)}|u(t)|^{2\theta'}\,dt\right)^{1/\theta'} \\ &= a_1\left(\int_{\mathbb{R}} ae^{Q(t)}|u(t)|^p\,dt\right)^{2/p} \\ &= a_1\|u\|_{p,a}^2,\end{aligned}$$

where from (A) and (1.2), $a_1 = (\int_{\mathbb{R}} a^{-2/(p-2)}e^{Q(t)}\,dt)^{(p-2)/p} < +\infty$. Then (2.4) holds.

By (A), there exists a continuous positive function ρ such that $\rho(t) \to +\infty$ as $|t| \to +\infty$ and

$$a_2 = \left(\int_{\mathbb{R}} \rho^\theta a^{-\theta/\theta'}e^{Q(t)}\,dt\right)^{1/\theta} < +\infty.$$

Since

$$\begin{aligned}\|u\|_{2,\rho}^2 &= \int_{\mathbb{R}} \rho e^{Q(t)}|u(t)|^2\,dt \\ &= \int_{\mathbb{R}} \rho a^{-1/\theta'}a^{1/\theta'}e^{Q(t)/\theta'}e^{Q(t)/\theta}|u(t)|^2\,dt\end{aligned}$$

$$\begin{aligned}&\le \left(\int_{\mathbb{R}} \rho^\theta a^{-\theta/\theta'} e^{Q(t)}\,dt\right)^{1/\theta} \left(\int_{\mathbb{R}} a e^{Q(t)} \left|u(t)\right|^p\,dt\right)^{1/\theta'}\\ &= a_2\|u\|_{p,a}^2,\end{aligned}$$

(2.5) holds by taking $Z = L_\rho^2(e^{Q(t)})$.

Finally, as $X \cap Z$ is the weighted Sobolev space $\Gamma^{1,2}(\mathbb{R}, \rho, 1)$, it follows from [43] that (2.6) holds. □

The following two lemmas are the mountain pass theorem and the symmetric mountain pass theorem, which are useful in the proofs of our theorems.

Lemma 2.3 [44] *Let E be a real Banach space and $I \in C^1(E, \mathbb{R})$ satisfying (PS)-condition. Suppose $I(0) = 0$ and:*

(i) *There exist constants $\rho, \alpha > 0$ such that $I_{\partial B_\rho(0)} \ge \alpha$.*

(ii) *There exists an $e \in E \backslash \bar{B}_\rho(0)$ such that $I(e) \le 0$.*

Then I possesses a critical value $c \ge \alpha$ which can be characterized as

$$c = \inf_{h\in\Phi} \max_{s\in[0,1]} I\big(h(s)\big),$$

where $\Phi = \{h \in C([0,1], E) | h(0) = 0, h(1) = e\}$ and $B_\rho(0)$ is an open ball in E of radius ρ centered at 0.

Lemma 2.4 [44] *Let E be a real Banach space and $I \in C^1(E, \mathbb{R})$ with I even. Assume that $I(0) = 0$ and I satisfies (PS)-condition, assumption* (i) *of Lemma* 2.3 *and the following condition:*

(iii) *For each finite dimensional subspace $E' \subset E$, there is $r = r(E') > 0$ such that $I(u) \le 0$ for $u \in E' \backslash B_r(0)$, $B_r(0)$ is an open ball in E of radius r centered at 0.*

Then I possesses an unbounded sequence of critical values.

Lemma 2.5 *Assume that* (W2) *and* (W3) *or* (W3)$'$ *hold. Then for every $(t, x) \in \mathbb{R} \times \mathbb{R}^N$,*

(i) *$s^{-\mu} W_1(t, sx)$ is nondecreasing on $(0, +\infty)$;*

(ii) *$s^{-\varrho} W_2(t, sx)$ is nonincreasing on $(0, +\infty)$.*

The proof of Lemma 2.5 is routine and we omit it.

3 Proofs of theorems

Proof of Theorem 1.1 Step 1. The functional φ satisfies the (PS)-condition. Let $\{u_n\} \subset E$ satisfying $\varphi(u_n)$ is bounded and $\varphi'(u_n) \to 0$ as $n \to \infty$. Then there exists a constant $C_1 > 0$ such that

$$\left|\varphi(u_n)\right| \le C_1, \qquad \left\|\varphi'(u_n)\right\|_{E^*} \le \mu C_1. \tag{3.1}$$

From (1.6), (1.7), (3.1), (W2), and (W3), we have

$$\begin{aligned}2C_1 + 2C_1\|u_n\| &\ge 2\varphi(u_n) - \frac{2}{\mu}\big\langle\varphi'(u_n), u_n\big\rangle\\ &= \frac{\mu-2}{\mu}\|\dot{u}_n\|_2^2 + 2\int_{\mathbb{R}} e^{Q(t)}\left[W_2\big(t, u_n(t)\big) - \frac{1}{\mu}\big(\nabla W_2\big(t, u_n(t)\big), u_n(t)\big)\right]dt\end{aligned}$$

$$
\begin{aligned}
&\quad - 2\int_{\mathbb{R}} e^{Q(t)}\left[W_1\big(t,u_n(t)\big) - \frac{1}{\mu}\big(\nabla W_1\big(t,u_n(t)\big),u_n(t)\big)\right]dt \\
&\quad + \left(\frac{2}{p} - \frac{2}{\mu}\right)\int_{\mathbb{R}} a(t)e^{Q(t)}\big|u_n(t)\big|^p\,dt \\
&\geq \frac{\mu-2}{\mu}\|\dot{u}_n\|_2^2 + \left(\frac{2}{p} - \frac{2}{\mu}\right)\|u_n\|_{p,a}^p.
\end{aligned}
$$

It follows from Lemma 2.2, $\mu > p > 2$, and the above inequalities that there exists a constant $C_2 > 0$ such that

$$
\|u_n\| \leq C_2, \quad n \in \mathbb{N}. \tag{3.2}
$$

Now we prove that $u_n \to u_0$ in E. Passing to a subsequence if necessary, it can be assumed that $u_n \rightharpoonup u_0$ in E. Since $Q(t) \to \infty$ as $|t| \to \infty$, we can choose $T > R$ such that

$$
Q(t) \geq \ln\left(\frac{C_2^2}{\xi^2}\right) \quad \text{for } |t| \geq T. \tag{3.3}
$$

It follows from (2.2), (3.2), and (3.3) that

$$
\begin{aligned}
\big|u_n(t)\big|^2 &\leq \int_t^{+\infty} e^{-Q(s)}e^{Q(s)}\big[\big|\dot{u}_n(s)\big|^2 + \big|u_n(s)\big|^2\big]\,ds \\
&\leq \frac{\xi^2}{C_2^2}\|u_n\|^2 \leq \xi^2 \quad \text{for } t \geq T \text{ and } n \in \mathbb{N}.
\end{aligned} \tag{3.4}
$$

Similarly, by (2.3), (3.2), and (3.3), we have

$$
\big|u_n(t)\big|^2 \leq \xi^2 \quad \text{for } t \leq -T \text{ and } n \in \mathbb{N}. \tag{3.5}
$$

Since $u_n \rightharpoonup u_0$ in E, it is easy to verify that $u_n(t)$ converges to $u_0(t)$ pointwise for all $t \in \mathbb{R}$. Hence, it follows from (3.4) and (3.5) that

$$
\big|u_0(t)\big| \leq \xi \quad \text{for } t \in (-\infty,-T] \cup [T,+\infty). \tag{3.6}
$$

Since $e^{Q(t)} \geq e_0 > 0$ on $[-T,T] = J$, the operator defined by $S : E \to X(J) : u \to u|_J$ is a linear continuous map. So $u_n \rightharpoonup u_0$ in $X(J)$. The Sobolev theorem implies that $u_n \to u_0$ uniformly on J, so there is $n_0 \in \mathbb{N}$ such that

$$
\int_{-T}^{T} e^{Q(t)}\big|\nabla W\big(t,u_n(t)\big) - \nabla W\big(t,u_0(t)\big)\big|\big|u_n(t) - u_0(t)\big|\,dt < \varepsilon \quad \text{for } n \geq n_0. \tag{3.7}
$$

For any given number $\varepsilon > 0$, by (W1), we can choose $\xi > 0$ such that

$$
\big|\nabla W(t,x)\big| \leq \varepsilon a(t)|x|^{p-1} \quad \text{for } |t| \geq R \text{ and } |x| \leq \xi. \tag{3.8}
$$

From (3.8), we have

$$
\begin{aligned}
&e^{Q(t)}\big|\nabla W\big(t,u_n(t)\big) - \nabla W\big(t,u_0(t)\big)\big|^2 \\
&\quad \leq e^{Q(t)}\big[\varepsilon a(t)\big(\big|u_n(t)\big|^{p-1} + \big|u_0(t)\big|^{p-1}\big)\big]^2
\end{aligned}
$$

$$\begin{aligned}
&\le e^{Q(t)}\big[\varepsilon 2^{p-1}a(t)\big|u_n(t)-u_0(t)\big|^{p-1}+\varepsilon\big(1+2^{p-1}\big)a(t)\big|u_0(t)\big|^{p-1}\big]^2\\
&\le 2^{2p}\varepsilon^2a^2(t)e^{Q(t)}\big|u_n(t)-u_0(t)\big|^{2(p-1)}+(2\varepsilon)^2\big(1+2^{p-1}\big)^2a^2(t)e^{Q(t)}\big|u_0(t)\big|^{2(p-1)}\\
&:= g_n(t). \qquad (3.9)
\end{aligned}$$

Moreover, since $a(t)$ is a positive continuous function on $\mathbb{R}$ and $u_n(t)$ converges to $u_0(t)$ pointwise for all $t\in\mathbb{R}$, it follows from (3.9) that

$$\lim_{n\to\infty}g_n(t)=(2\varepsilon)^2\big(1+2^{p-1}\big)^2a^2(t)e^{Q(t)}\big|u_0(t)\big|^{2(p-1)}:=g(t)\quad\text{for a.e. } t\in\mathbb{R}$$

and

$$\begin{aligned}
\lim_{n\to\infty}\int_{\mathbb{R}\setminus(-T,T)}g_n(t)\,dt &= \lim_{n\to\infty}\int_{\mathbb{R}\setminus(-T,T)}e^{Q(t)}\big[2^{2p}\varepsilon^2a^2(t)\big|u_n(t)-u_0(t)\big|^{2(p-1)}\\
&\quad+(2\varepsilon)^2\big(1+2^{p-1}\big)^2a^2(t)\big|u_0(t)\big|^{2(p-1)}\big]\,dt\\
&=2^{2p}(\varepsilon)^2\lim_{n\to\infty}\int_{\mathbb{R}\setminus(-T,T)}a^2(t)e^{Q(t)}\big|u_n(t)-u_0(t)\big|^{2(p-1)}\,dt\\
&\quad+(2\varepsilon)^2\big(1+2^{p-1}\big)^2\int_{\mathbb{R}\setminus(-T,T)}a^2(t)e^{Q(t)}\big|u_0(t)\big|^{2(p-1)}\,dt\\
&=(2\varepsilon)^2\big(1+2^{p-1}\big)^2\int_{\mathbb{R}\setminus(-T,T)}a^2(t)e^{Q(t)}\big|u_0(t)\big|^{2(p-1)}\,dt\\
&=\int_{\mathbb{R}}g(t)\,dt<+\infty.
\end{aligned}$$

From Lebesgue's dominated convergence theorem, (3.4), (3.5), (3.6), (3.9), and the above inequalities, we have

$$\lim_{n\to\infty}\int_{\mathbb{R}\setminus(-T,T)}e^{Q(t)}\big|\nabla W\big(t,u_n(t)\big)-\nabla W\big(t,u_0(t)\big)\big|^2\,dt=0. \qquad (3.10)$$

From Lemma 2.2, we have $u_n\to u_0$ in $L^2(e^{Q(t)})$. Hence, by (3.10),

$$\begin{aligned}
&\int_{\mathbb{R}\setminus(-T,T)}e^{Q(t)}\big|\nabla W\big(t,u_n(t)\big)-\nabla W\big(t,u_0(t)\big)\big|\big|u_n(t)-u_0(t)\big|\,dt\\
&\quad\le\left(\int_{\mathbb{R}\setminus(-T,T)}e^{Q(t)}\big|\nabla W\big(t,u_n(t)\big)-\nabla W\big(t,u_0(t)\big)\big|^2\,dt\right)^{1/2}\\
&\qquad\times\left(\int_{\mathbb{R}\setminus(-T,T)}e^{Q(t)}\big|u_n(t)-u_0(t)\big|^2\,dt\right)^{1/2}
\end{aligned}$$

tends to 0 as $n\to+\infty$, which together with (3.7) shows that

$$\int_{\mathbb{R}}e^{Q(t)}\big|\nabla W\big(t,u_n(t)\big)-\nabla W\big(t,u_0(t)\big)\big|\big|u_n(t)-u_0(t)\big|\,dt\to0\quad\text{as } n\to\infty. \qquad (3.11)$$

From (1.7), we have

$$\begin{aligned}
0&\leftarrow\big\langle\varphi'(u_n)-\varphi'(u_0),u_n-u_0\big\rangle\\
&=\|\dot u_n-\dot u_0\|_2^2+\int_{\mathbb{R}}a(t)e^{Q(t)}\big(\big|u_n(t)\big|^{p-2}u_n(t)-\big|u_0(t)\big|^{p-2}u_0(t)\big)\big(u_n(t)-u_0(t)\big)\,dt
\end{aligned}$$

$$
\begin{aligned}
&-\int_{\mathbb{R}} e^{Q(t)}\big(\nabla W\big(t,u_n(t)\big)-\nabla W\big(t,u_0(t)\big),u_n(t)-u_0(t)\big)\,dt \\
\geq\ & \|\dot{u}_n-\dot{u}_0\|_2^2 + C_3\int_{\mathbb{R}} a(t)e^{Q(t)}\big(\big|u_n(t)-u_0(t)\big|^p\big)\,dt \\
&-\int_{\mathbb{R}} e^{Q(t)}\big(\nabla W\big(t,u_n(t)\big)-\nabla W\big(t,u_0(t)\big),u_n(t)-u_0(t)\big)\,dt, \quad n\to\infty,
\end{aligned}
\tag{3.12}
$$

where C_3 is a positive constant. It follows from (3.11) and (3.12) that

$$
\|\dot{u}_n\|_2 \to \|\dot{u}_0\|_2 \quad \text{as } n\to\infty \tag{3.13}
$$

and

$$
\int_{\mathbb{R}} a(t)e^{Q(t)}\big|u_n(t)\big|^p\,dt \to \int_{\mathbb{R}} a(t)e^{Q(t)}\big|u_0(t)\big|^p\,dt \quad \text{as } n\to\infty. \tag{3.14}
$$

Hence, $u_n \to u_0$ in E by (3.13) and (3.14). This shows that φ satisfies (PS)-condition.

Step 2. From (W1), there exists $\delta\in(0,1)$ such that

$$
\big|\nabla W(t,x)\big| \leq \frac{1}{2}a(t)|x|^{p-1} \quad \text{for } |t|\geq R, |x|\leq\delta. \tag{3.15}
$$

By (3.15) and $W(t,0)=0$, we have

$$
\big|W(t,x)\big| \leq \frac{1}{2p}a(t)|x|^{p} \quad \text{for } |t|\geq R, |x|\leq\delta. \tag{3.16}
$$

Let

$$
C_4 = \sup\left\{\frac{W_1(t,x)}{a(t)}\,\Big|\, t\in[-R,R], x\in\mathbb{R}, |x|=1\right\}. \tag{3.17}
$$

Set $\sigma=\min\{1/(2pC_4+1)^{1/(\mu-p)},\delta\}$ and $\|u\|=\sqrt{2e_0}\sigma:=\rho$, it follows from Lemma 2.1 that $|u(t)|\leq\sigma\leq\delta<1$ for $t\in\mathbb{R}$. From Lemma 2.5(i) and (3.17), we have

$$
\begin{aligned}
\int_{-R}^{R} e^{Q(t)}W_1\big(t,u(t)\big)\,dt &\leq \int_{\{t\in[-R,R]:u(t)\neq 0\}} e^{Q(t)}W_1\left(t,\frac{u(t)}{|u(t)|}\right)\big|u(t)\big|^\mu\,dt \\
&\leq C_4\int_{-R}^{R} a(t)e^{Q(t)}\big|u(t)\big|^\mu\,dt \\
&\leq C_4\sigma^{\mu-p}\int_{-R}^{R} a(t)e^{Q(t)}\big|u(t)\big|^p\,dt \\
&\leq \frac{1}{2p}\int_{-R}^{R} a(t)e^{Q(t)}\big|u(t)\big|^p\,dt.
\end{aligned}
\tag{3.18}
$$

By (W3), (3.16), and (3.18), we have

$$
\begin{aligned}
\varphi(u) &= \frac{1}{2}\int_{\mathbb{R}} e^{Q(t)}\big|\dot{u}(t)\big|^2\,dt + \int_{\mathbb{R}}\frac{a(t)}{p}e^{Q(t)}\big|u(t)\big|^p\,dt - \int_{\mathbb{R}} e^{Q(t)}W\big(t,u(t)\big)\,dt \\
&= \frac{1}{2}\|\dot{u}\|_2^2 + \frac{1}{p}\|u\|_{p,a}^p - \int_{\mathbb{R}\setminus(-R,R)} e^{Q(t)}W\big(t,u(t)\big)\,dt - \int_{-R}^{R} e^{Q(t)}W\big(t,u(t)\big)\,dt
\end{aligned}
$$

$$\begin{aligned}
&\geq \frac{1}{2}\|\dot{u}\|_2^2 + \frac{1}{p}\|u\|_{p,a}^p - \int_{\mathbb{R}\setminus(-R,R)} e^{Q(t)} W\big(t,u(t)\big)\,dt - \int_{-R}^{R} e^{Q(t)} W_1\big(t,u(t)\big)\,dt \\
&\geq \frac{1}{2}\|\dot{u}\|_2^2 + \frac{1}{p}\|u\|_{p,a}^p - \frac{1}{2p}\int_{\mathbb{R}\setminus(-R,R)} a(t)e^{Q(t)}\big|u(t)\big|^p\,dt \\
&\quad - \frac{1}{2p}\int_{-R}^{R} a(t)e^{Q(t)}\big|u(t)\big|^p\,dt \\
&= \frac{1}{2}\|\dot{u}\|_2^2 + \frac{1}{2p}\|u\|_{p,a}^p.
\end{aligned}$$

Therefore, we can choose a constant $\alpha > 0$ depending on ρ such that $\varphi(u) \geq \alpha$ for any $u \in E$ with $\|u\| = \rho$.

Step 3. From Lemma 2.5(ii) and (2.1), we have for any $u \in E$

$$\begin{aligned}
&\int_{-3}^{3} e^{Q(t)} W_2\big(t,u(t)\big)\,dt \\
&\quad = \int_{\{t\in[-3,3]:|u(t)|>1\}} e^{Q(t)} W_2\big(t,u(t)\big)\,dt + \int_{\{t\in[-3,3]:|u(t)|\leq 1\}} e^{Q(t)} W_2\big(t,u(t)\big)\,dt \\
&\quad \leq \int_{\{t\in[-3,3]:|u(t)|>1\}} e^{Q(t)} W_2\left(t,\frac{u(t)}{|u(t)|}\right)\big|u(t)\big|^{\varrho}\,dt + \int_{-3}^{3} e^{Q(t)} \max_{|x|\leq 1} W_2(t,x)\,dt \\
&\quad \leq \|u\|_\infty^{\varrho}\int_{-3}^{3} e^{Q(t)} \max_{|x|=1} W_2(t,x)\,dt + \int_{-3}^{3} e^{Q(t)} \max_{|x|\leq 1} W_2(t,x)\,dt \\
&\quad \leq \left(\frac{1}{\sqrt{2e_0}}\right)^{\varrho}\|u\|^{\varrho}\int_{-3}^{3} e^{Q(t)} \max_{|x|=1} W_2(t,x)\,dt + \int_{-3}^{3} e^{Q(t)} \max_{|x|\leq 1} W_2(t,x)\,dt \\
&\quad = C_5\|u\|^{\varrho} + C_6,
\end{aligned} \tag{3.19}$$

where $C_5 = (\frac{1}{\sqrt{2e_0}})^{\varrho}\int_{-3}^{3} e^{Q(t)}\max_{|x|=1} W_2(t,x)\,dt$, $C_6 = \int_{-3}^{3} e^{Q(t)}\max_{|x|\leq 1} W_2(t,x)\,dt$. Take $\omega \in E$ such that

$$\big|\omega(t)\big| = \begin{cases} 1 & \text{for } |t| \leq 1, \\ 0 & \text{for } |t| \geq 3, \end{cases} \tag{3.20}$$

and $|\omega(t)| \leq 1$ for $|t| \in (1,3]$. For $s > 1$, from Lemma 2.5(i) and (3.20), we get

$$\int_{-1}^{1} e^{Q(t)} W_1\big(t,s\omega(t)\big)\,dt \geq s^{\mu}\int_{-1}^{1} e^{Q(t)} W_1\big(t,\omega(t)\big)\,dt = C_7 s^{\mu}, \tag{3.21}$$

where $C_7 = \int_{-1}^{1} e^{Q(t)} W_1(t,\omega(t))\,dt > 0$. From (W3), (1.6), (3.19), (3.20), and (3.21), we get for $s > 1$

$$\begin{aligned}
\varphi(s\omega) &= \frac{s^2}{2}\|\dot{\omega}\|_2^2 + \frac{s^p}{p}\|\omega\|_{p,a}^p + \int_{\mathbb{R}} e^{Q(t)}\big[W_2\big(t,s\omega(t)\big) - W_1\big(t,s\omega(t)\big)\big]\,dt \\
&\leq \frac{s^2}{2}\|\dot{\omega}\|_2^2 + \frac{s^p}{p}\|\omega\|_{p,a}^p + \int_{-3}^{3} e^{Q(t)} W_2\big(t,s\omega(t)\big)\,dt - \int_{-1}^{1} e^{Q(t)} W_1\big(t,s\omega(t)\big)\,dt \\
&\leq \frac{s^2}{2}\|\dot{\omega}\|_2^2 + \frac{s^p}{p}\|\omega\|_{p,a}^p + C_5 s^{\varrho}\|\omega\|^{\varrho} + C_6 - C_7 s^{\mu}.
\end{aligned} \tag{3.22}$$

Since $\mu > \varrho > p$ and $C_7 > 0$, it follows from (3.22) that there exists $s_1 > 1$ such that $\|s_1\omega\| > \rho$ and $\varphi(s_1\omega) < 0$. Set $e = s_1\omega(t)$, then $e \in E$, $\|e\| = \|s_1\omega\| > \rho$, and $\varphi(e) = \varphi(s_1\omega) < 0$. It is easy to see that $\varphi(0) = 0$. By Lemma 2.3, φ has a critical value $c > \alpha$ given by

$$c = \inf_{g\in\Phi} \max_{s\in[0,1]} \varphi\big(g(s)\big), \tag{3.23}$$

where

$$\Phi = \big\{g \in C\big([0,1],E\big) : g(0) = 0, g(1) = e\big\}.$$

Hence, there exists $u^* \in E$ such that

$$\begin{aligned} &\varphi\big(u^*\big) = c, \\ &\varphi'\big(u^*\big) = 0. \end{aligned}$$

The function u^* is the desired solution of problem (1.1). Since $c > 0$, u^* is a nontrivial fast homoclinic solution. The proof is complete. □

Proof of Theorem 1.2 In the proof of Theorem 1.1, the condition $W_2(t,x) \geq 0$ in (W3) is only used in the proofs of (3.2) and Step 2. Therefore, we only need to prove that (3.2) and Step 2 still hold if we use (W1)′ and (W3)′ instead of (W1) and (W3). We first prove that (3.2) holds. From (W2), (W3)′, (1.6), (1.7), and (3.1), we have

$$\begin{aligned} 2C_1 + \frac{2C_1\mu}{\varrho}\|u_n\| &\geq 2\varphi(u_n) - \frac{2}{\varrho}\big\langle\varphi'(u_n), u_n\big\rangle \\ &= \frac{\varrho-2}{\varrho}\|\dot{u}_n\|_2^2 \\ &\quad + 2\int_{\mathbb{R}} e^{Q(t)}\bigg[W_2\big(t,u_n(t)\big) - \frac{1}{\varrho}\big(\nabla W_2\big(t,u_n(t)\big), u_n(t)\big)\bigg]\,dt \\ &\quad - 2\int_{\mathbb{R}} e^{Q(t)}\bigg[W_1\big(t,u_n(t)\big) - \frac{1}{\varrho}\big(\nabla W_1\big(t,u_n(t)\big), u_n(t)\big)\bigg]\,dt \\ &\quad + 2\bigg(\frac{1}{p} - \frac{1}{\varrho}\bigg)\int_{\mathbb{R}} a(t)e^{Q(t)}\big|u_n(t)\big|^p\,dt \\ &\geq \frac{\varrho-2}{\varrho}\|\dot{u}_n\|_2^2 + 2\bigg(\frac{1}{p} - \frac{1}{\varrho}\bigg)\|u_n\|_{p,a}^p, \end{aligned}$$

which implies that there exists a constant $C_3 > 0$ such that (3.2) holds. Next, we prove that Step 2 still holds. From (W1)′, there exists $\delta \in (0,1)$ such that

$$\big|\nabla W(t,x)\big| \leq \frac{1}{2}a(t)|x|^{p-1} \quad \text{for } t \in \mathbb{R}, |x| \leq \delta. \tag{3.24}$$

By (3.24) and $W(t,0) = 0$, we have

$$\big|W(t,x)\big| \leq \frac{1}{2p}a(t)|x|^p \quad \text{for } t \in \mathbb{R}, |x| \leq \delta. \tag{3.25}$$

Let $\|u\| = \sqrt{2e_0}\delta := \rho$, it follows from Lemma 2.1 that $|u(t)| \le \delta$. It follows from (1.6) and (3.25) that

$$\begin{aligned}\varphi(u) &= \frac{1}{2}\int_{\mathbb{R}} e^{Q(t)}\left|\dot{u}(t)\right|^2 dt + \int_{\mathbb{R}} \frac{a(t)e^{Q(t)}}{p}\left|u(t)\right|^p dt - \int_{\mathbb{R}} e^{Q(t)} W\bigl(t, u(t)\bigr)\,dt \\ &\ge \frac{1}{2}\|\dot{u}\|_2^2 + \frac{1}{p}\|u\|_{p,a}^p - \int_{\mathbb{R}} \frac{1}{2p} a(t)e^{Q(t)}\left|u(t)\right|^p dt \\ &= \frac{1}{2}\|\dot{u}\|_2^2 + \frac{1}{2p}\|u\|_{p,a}^p.\end{aligned}$$

Therefore, we can choose a constant $\alpha > 0$ depending on ρ such that $\varphi(u) \ge \alpha$ for any $u \in E$ with $\|u\| = \rho$. The proof of Theorem 1.2 is complete. □

Proof of Theorem 1.3 Condition (W4) shows that φ is even. In view of the proof of Theorem 1.1, we know that $\varphi \in C^1(E, \mathbb{R})$ and satisfies (PS)-condition and assumption (i) of Lemma 2.3. Now, we prove that (iii) of Lemma 2.4. Let E' be a finite dimensional subspace of E. Since all norms of a finite dimensional space are equivalent, there exists $d > 0$ such that

$$\|u\| \le d\|u\|_\infty. \tag{3.26}$$

Assume that $\dim E' = m$ and $\{u_1, u_2, \ldots, u_m\}$ is a basis of E' such that

$$\|u_i\| = d, \quad i = 1, 2, \ldots, m. \tag{3.27}$$

For any $u \in E'$, there exists $\lambda_i \in \mathbb{R}$, $i = 1, 2, \ldots, m$ such that

$$u(t) = \sum_{i=1}^m \lambda_i u_i(t) \quad \text{for } t \in \mathbb{R}. \tag{3.28}$$

Let

$$\|u\|_* = \sum_{i=1}^m |\lambda_i|\|u_i\|. \tag{3.29}$$

It is easy to see that $\|\cdot\|_*$ is a norm of E'. Hence, there exists a constant $d' > 0$ such that $d'\|u\|_* \le \|u\|$. Since $u_i \in E$, by Lemma 2.1, we can choose $R_1 > R$ such that

$$\left|u_i(t)\right| < \frac{d'\delta}{1+m}, \quad |t| > R_1, i = 1, 2, \ldots, m, \tag{3.30}$$

where δ is given in (3.25). Let

$$\Theta = \left\{\sum_{i=1}^m \lambda_i u_i(t) : \lambda_i \in \mathbb{R}, i = 1, 2, \ldots, m; \sum_{i=1}^m |\lambda_i| = 1\right\} = \bigl\{u \in E' : \|u\|_* = d\bigr\}. \tag{3.31}$$

Hence, for $u \in \Theta$, let $t_0 = t_0(u) \in \mathbb{R}$ such that

$$\left|u(t_0)\right| = \|u\|_\infty. \tag{3.32}$$

Then by (3.26)-(3.29), (3.31), and (3.32), we have

$$\begin{aligned} d = \|u\|_* &\le \frac{\|u\|}{d'} \le \frac{d}{d'}\|u\|_\infty = \frac{d}{d'}\big|u(t_0)\big| \\ &= \frac{d}{d'}\left|\sum_{i=1}^m \lambda_i u_i(t_0)\right| \le d\sum_{i=1}^m |\lambda_i|\big|u_i(t_0)\big|, \quad u \in \Theta. \end{aligned} \tag{3.33}$$

This shows that $|u(t_0)| \ge d'$ and there exists $i_0 \in \{1,2,\dots,m\}$ such that $|u_{i_0}(t_0)| \ge d'/m$, which together with (3.30), implies that $|t_0| \le R_1$. Let $R_2 = R_1 + 1$ and

$$\gamma = \min\left\{e^{Q(t)}W_1(t,x) : -R_2 \le t \le R_2, \frac{d'}{\sqrt{2}} \le |x| \le \frac{d}{\sqrt{2e_0}}\right\}. \tag{3.34}$$

Since $W_1(t,x) > 0$ for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^N\backslash\{0\}$, and $W_1 \in C^1(\mathbb{R}\times\mathbb{R}^N,\mathbb{R})$, it follows that $\gamma > 0$. For any $u \in E$, from Lemma 2.1 and Lemma 2.5(i), we have

$$\begin{aligned} &\int_{-R_2}^{R_2} e^{Q(t)}W_2\big(t,u(t)\big)\,dt \\ &\quad= \int_{\{t\in[-R_2,R_2]:|u(t)|>1\}} e^{Q(t)}W_2\big(t,u(t)\big)\,dt \\ &\qquad+ \int_{\{t\in[-R_2,R_2]:|u(t)|\le 1\}} e^{Q(t)}W_2\big(t,u(t)\big)\,dt \\ &\quad\le \int_{\{t\in[-R_2,R_2]:|u(t)|>1\}} e^{Q(t)}W_2\left(t,\frac{u(t)}{|u(t)|}\right)\big|u(t)\big|^\varrho\,dt \\ &\qquad+ \int_{-R_2}^{R_2} e^{Q(t)}\max_{|x|\le 1}W_2(t,x)\,dt \\ &\quad\le \|u\|_\infty^\varrho \int_{-R_2}^{R_2} e^{Q(t)}\max_{|x|=1}W_2(t,x)\,dt + \int_{-R_2}^{R_2} e^{Q(t)}\max_{|x|\le 1}W_2(t,x)\,dt \\ &\quad\le \left(\frac{1}{\sqrt{2e_0}}\right)^\varrho \|u\|^\varrho \int_{-R_2}^{R_2} e^{Q(t)}\max_{|x|=1}W_2(t,x)\,dt + \int_{-R_2}^{R_2} e^{Q(t)}\max_{|x|\le 1}W_2(t,x)\,dt \\ &\quad= C_8\|u\|^\varrho + C_9, \end{aligned} \tag{3.35}$$

where $C_8 = (\frac{1}{\sqrt{2e_0}})^\varrho \int_{-R_2}^{R_2} e^{Q(t)}\max_{|x|=1}W_2(t,x)\,dt$, $C_9 = \int_{-R_2}^{R_2} e^{Q(t)}\max_{|x|\le 1}W_2(t,x)\,dt$. Since $\dot{u}_i \in L^2(e^{Q(t)})$, $i = 1,2,\dots,m$, it follows that there exists $\varepsilon \in (0, ((d')^2e_0)/(32m^2d^2))$ such that

$$\begin{aligned} \int_{t+\varepsilon}^{t-\varepsilon} \big|\dot{u}_i(s)\big|\,ds &= \int_{t+\varepsilon}^{t-\varepsilon} e^{-\frac{Q(s)}{2}}e^{\frac{Q(s)}{2}}\big|\dot{u}_i(s)\big|\,ds \\ &\le \frac{1}{\sqrt{e_0}}\int_{t+\varepsilon}^{t-\varepsilon} e^{\frac{Q(s)}{2}}\big|\dot{u}_i(s)\big|\,ds \\ &\le \frac{1}{\sqrt{e_0}}(2\varepsilon)^{1/2}\left(\int_{t+\varepsilon}^{t-\varepsilon} e^{Q(s)}\big|\dot{u}_i(s)\big|^2\,ds\right)^{1/2} \\ &\le \left(\frac{2\varepsilon}{e_0}\right)^{1/2}\|\dot{u}_i\|_2 \\ &\le \frac{d'}{4m} \quad \text{for } t \in \mathbb{R}, i = 1,2,\dots,m. \end{aligned} \tag{3.36}$$

Then for $u \in \Theta$ with $|u(t_0)| = \|u\|_\infty$ and $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$, it follows from (3.28), (3.31), (3.32), (3.33), and (3.36) that

$$\begin{aligned} |u(t)|^2 &= |u(t_0)|^2 + 2\int_{t_0}^{t} \big(\dot{u}(s), u(s)\big)\,ds \\ &\geq |u(t_0)|^2 - 2\int_{t_0-\varepsilon}^{t_0+\varepsilon} |u(s)||\dot{u}(s)|\,ds \\ &\geq |u(t_0)|^2 - 2|u(t_0)|\int_{t_0-\varepsilon}^{t_0+\varepsilon} |\dot{u}(s)|\,ds \\ &\geq |u(t_0)|^2 - 2|u(t_0)|\sum_{i=1}^{m}|\lambda_i|\int_{t_0-\varepsilon}^{t_0+\varepsilon} |\dot{u}_i(s)|\,ds \\ &\geq \frac{(d')^2}{2}. \end{aligned} \tag{3.37}$$

On the other hand, since $\|u\| \leq d$ for $u \in \Theta$, then

$$|u(t)| \leq \|u\|_\infty \leq \frac{d}{\sqrt{2e_0}}, \quad t \in \mathbb{R}, u \in \Theta. \tag{3.38}$$

Therefore, from (3.34), (3.37), and (3.38), we have

$$\int_{-R_2}^{R_2} e^{Q(t)} W_1\big(t, u(t)\big)\,dt \geq \int_{t_0-\varepsilon}^{t_0+\varepsilon} e^{Q(t)} W_1\big(t, u(t)\big)\,dt \geq 2\varepsilon\gamma \quad \text{for } u \in \Theta. \tag{3.39}$$

By (3.30) and (3.31), we have

$$|u(t)| \leq \sum_{i=1}^{m} |\lambda_i||u_i(t)| \leq \delta \quad \text{for } |t| \geq R_1, u \in \Theta. \tag{3.40}$$

By (1.6), (3.16), (3.35), (3.39), (3.40), and Lemma 2.5, we have for $u \in \Theta$ and $r > 1$

$$\begin{aligned} \varphi(ru) &= \frac{r^2}{2}\|\dot{u}\|_2^2 + \frac{r^p}{p}\|u\|_{p,a}^p + \int_{\mathbb{R}} e^{Q(t)}\big[W_2\big(t, ru(t)\big) - W_1\big(t, ru(t)\big)\big]\,dt \\ &\leq \frac{r^2}{2}\|\dot{u}\|_2^2 + \frac{r^p}{p}\|u\|_{p,a}^p + r^\varrho \int_{\mathbb{R}} e^{Q(t)} W_2\big(t, u(t)\big)\,dt - r^\mu \int_{\mathbb{R}} e^{Q(t)} W_1\big(t, u(t)\big)\,dt \\ &= \frac{r^2}{2}\|\dot{u}\|_2^2 + \frac{r^p}{p}\|u\|_{p,a}^p + r^\varrho \int_{\mathbb{R}\setminus(-R_2,R_2)} e^{Q(t)} W_2\big(t, u(t)\big)\,dt \\ &\quad - r^\mu \int_{\mathbb{R}\setminus(-R_2,R_2)} e^{Q(t)} W_1\big(t, u(t)\big)\,dt + r^\varrho \int_{-R_2}^{R_2} e^{Q(t)} W_2\big(t, u(t)\big)\,dt \\ &\quad - r^\mu \int_{-R_2}^{R_2} e^{Q(t)} W_1\big(t, u(t)\big)\,dt \\ &\leq \frac{r^2}{2}\|\dot{u}\|_2^2 + \frac{r^p}{p}\|u\|_{p,a}^p - r^\varrho \int_{\mathbb{R}\setminus(-R_2,R_2)} e^{Q(t)} W\big(t, u(t)\big)\,dt \\ &\quad - r^\mu \int_{-R_2}^{R_2} e^{Q(t)} W_1\big(t, u(t)\big)\,dt + r^\varrho \int_{-R_2}^{R_2} e^{Q(t)} W_2\big(t, u(t)\big)\,dt \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{r^2}{2}\|\dot{u}\|_2^2 + \frac{r^p}{p}\|u\|_{p,a}^p + \frac{r^\varrho}{2p}\int_{\mathbb{R}\setminus(-R_2,R_2)} a(t)e^{Q(t)}\left|u(t)\right|^p\,dt \\
&\quad + r^\varrho\left(C_8\|u\|^\varrho + C_9\right) - 2\varepsilon\gamma r^\mu \\
&\leq \frac{r^2}{2}\|\dot{u}\|_2^2 + \frac{r^p}{p}\|u\|_{p,a}^p + \frac{r^\varrho}{2p}\|u\|_{p,a}^p + r^\varrho\left(C_8\|u\|^\varrho + C_9\right) - 2\varepsilon\gamma r^\mu \\
&\leq \frac{r^2}{2}d^2 + \frac{r^p}{p}d^p + \frac{r^\varrho}{2p}d^p + C_8(rd)^\varrho + C_9 r^\varrho - 2\varepsilon\gamma r^\mu.
\end{aligned}
\tag{3.41}
$$

Since $\mu > \varrho > p > 2$, we deduce that there exists $r_0 = r_0(d, d', C_8, C_9, R_1, R_2, \varepsilon, \gamma) = r_0(E') > 1$ such that

$$\varphi(ru) < 0 \quad \text{for } u \in \Theta \text{ and } r \geq r_0.$$

It follows that

$$\varphi(u) < 0 \quad \text{for } u \in E' \text{ and } \|u\| \geq dr_0,$$

which shows that (iii) of Lemma 2.4 holds. By Lemma 2.4, φ possesses an unbounded sequence $\{c_n\}_{n=1}^{\infty}$ of critical values with $c_n = \varphi(u_n)$, where u_n is such that $\varphi'(u_n) = 0$ for $n = 1, 2, \ldots$. If $\{\|u_n\|\}$ is bounded, then there exists $C_{10} > 0$ such that

$$\|u_n\| \leq C_{10} \quad \text{for } n \in \mathbb{N}. \tag{3.42}$$

In a similar fashion to the proof of (3.4) and (3.5), for the given δ in (3.16), there exists $R_3 > R$ such that

$$\left|u_n(t)\right| \leq \delta \quad \text{for } |t| \geq R_3, n \in \mathbb{N}. \tag{3.43}$$

Hence, by (1.6), (2.1), (3.16), (3.42), and (3.43), we have

$$
\begin{aligned}
&\frac{1}{2}\|\dot{u}_n\|_2^2 + \frac{1}{p}\|u_n\|_{p,a}^p \\
&\quad = c_n + \int_{\mathbb{R}} e^{Q(t)} W\left(t, u_n(t)\right)\,dt \\
&\quad = c_n + \int_{\mathbb{R}\setminus[-R_3,R_3]} e^{Q(t)} W\left(t, u_n(t)\right)\,dt + \int_{-R_3}^{R_3} e^{Q(t)} W\left(t, u_n(t)\right)\,dt \\
&\quad \geq c_n - \frac{1}{2p}\int_{\mathbb{R}\setminus[-R_3,R_3]} a(t)e^{Q(t)}\left|u_n(t)\right|^p\,dt - \int_{-R_3}^{R_3} e^{Q(t)}\left|W\left(t, u_n(t)\right)\right|\,dt \\
&\quad \geq c_n - \frac{1}{2p}\|u_n\|_{p,a}^p - \int_{-R_3}^{R_3} e^{Q(t)} \max_{|x|\leq\sqrt{2e_0}C_{10}}\left|W(t,x)\right|\,dt,
\end{aligned}
$$

which, together with (3.42), implies that

$$c_n \leq \frac{1}{2}\|\dot{u}_n\|_2^2 + \frac{3}{2p}\|u_n\|_{p,a}^p + \int_{-R_3}^{R_3} \max_{|x|\leq\sqrt{2e_0}C_{10}} e^{Q(t)}\left|W(t,x)\right|\,dt < +\infty.$$

This contradicts the fact that $\{c_n\}_{n=1}^{\infty}$ is unbounded, and so $\{\|u_n\|\}$ is unbounded. The proof is complete. □

Proof of Theorem 1.4 In view of the proofs of Theorem 1.2 and Theorem 1.3, the conclusion of Theorem 1.4 holds. The proof is complete. □

4 Examples

Example 4.1 Consider the following system:

$$\ddot{u}(t) + t\dot{u}(t) - a(t)\left|u(t)\right|^{1/2}u(t) + \nabla W\big(t, u(t)\big) = 0, \quad \text{a.e. } t \in \mathbb{R}, \tag{4.1}$$

where $q(t) = t$, $p = 5/2$, $t \in \mathbb{R}$, $u \in \mathbb{R}^N$, $a \in C(\mathbb{R}, (0,\infty))$, and a satisfies (A). Let

$$W(t,x) = a(t)\left(\sum_{i=1}^{m} a_i|x|^{\mu_i} - \sum_{j=1}^{n} b_j|x|^{\varrho_j}\right),$$

where $\mu_1 > \mu_2 > \cdots > \mu_m > \varrho_1 > \varrho_2 > \cdots > \varrho_n > 5/2$, $a_i, b_j > 0$, $i = 1, \dots, m$, $j = 1, \dots, n$. Let

$$W_1(t,x) = a(t)\sum_{i=1}^{m} a_i|x|^{\mu_i}, \qquad W_2(t,x) = a(t)\sum_{j=1}^{n} b_j|x|^{\varrho_j}.$$

Then it is easy to check that all the conditions of Theorem 1.3 are satisfied with $\mu = \mu_m$ and $\varrho = \varrho_1$. Hence, problem (4.1) has an unbounded sequence of fast homoclinic solutions.

Example 4.2 Consider the following system:

$$\ddot{u}(t) + \big(t + t^3\big)\dot{u}(t) - a(t)\left|u(t)\right|^{4}u(t) + \nabla W\big(t, u(t)\big) = 0, \quad \text{a.e. } t \in \mathbb{R}, \tag{4.2}$$

where $q(t) = t + t^3$, $p = 6$, $t \in \mathbb{R}$, $u \in \mathbb{R}^N$, $a \in C(\mathbb{R}, (0,\infty))$, and a satisfies (A). Let

$$W(t,x) = a(t)\big[a_1|x|^{\mu_1} + a_2|x|^{\mu_2} - b_1(\cos t)|x|^{\varrho_1} - b_2|x|^{\varrho_2}\big],$$

where $\mu_1 > \mu_2 > \varrho_1 > \varrho_2 > 6$, $a_1, a_2 > 0$, $b_1, b_2 > 0$. Let

$$W_1(t,x) = a(t)\big(a_1|x|^{\mu_1} + a_2|x|^{\mu_2}\big), \qquad W_2(t,x) = a(t)\big[b_1(\cos t)|x|^{\varrho_1} + b_2|x|^{\varrho_2}\big].$$

Then it is easy to check that all the conditions of Theorem 1.4 are satisfied with $\mu = \mu_2$ and $\varrho = \varrho_1$. Hence, by Theorem 1.4, problem (4.2) has an unbounded sequence of fast homoclinic solutions.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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