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Local well-posedness and persistence properties for a model containing both Camassa-Holm and Novikov equation

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Abstract

This paper deals with the Cauchy problem for a generalized Camassa-Holm equation with high-order nonlinearities,

$$u_t - u_{xxt} + ku_x + au^m u_x = (n+2)u^n u_x u_{xx} + u^{n+1}u_{xxx},$$

where $k, a \in \mathbb{R}$ and $m, n \in \mathbb{Z}^+$. This equation is a generalization of the famous equation of Camassa-Holm and the Novikov equation. The local well-posedness of strong solutions for this equation in Sobolev space $H^s(\mathbb{R})$ with $s > \frac{3}{2}$ is obtained, and persistence properties of the strong solutions are studied. Furthermore, under appropriate hypotheses, the existence of its weak solutions in low order Sobolev space $H^s(\mathbb{R})$ with $1 < s \leq \frac{3}{2}$ is established.

Keywords: persistence properties; local well-posedness; weak solution

1 Introduction

This work is concerned with the following one-dimensional nonlinear dispersive PDE:

$$\begin{cases} u_t - u_{xxt} + ku_x + au^m u_x = (n+2)u^n u_x u_{xx} + u^{n+1}u_{xxx}, & t > 0, x \in \mathbb{R}, \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $k, a \in \mathbb{R}$ and $m, n \in \mathbb{Z}^+$.

Obviously, if $n = 0$, $m = 1$, $a = 3$, equation (1.1) becomes the Camassa-Holm equation,

$$u_t - u_{xxt} + ku_x + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0, \tag{1.2}$$

where the variable $u(t,x)$ represents the fluid velocity at time t and in the spatial direction x, and k is a nonnegative parameter related to the critical shallow water speed [1]. The Camassa-Holm equation (1.2) is also a model for the propagation of axially symmetric waves in hyperelastic rods (*cf.* [2]). It is well known that equation (1.2) has also a bi-Hamiltonian structure [3, 4] and is completely integrable (see [5, 6] and the in-depth discussion in [7, 8]). In [9], Qiao has shown that the Camassa-Holm spectral problem yields two different integrable hierarchies of nonlinear evolution equations, one is of negative order CH hierachy while the other one is of positive order CH hierarchy. Its solitary waves are smooth if $k > 0$ and peaked in the limiting case $c_0 = 0$ (*cf.* [1]). The orbital stability of the peaked solitons is proved in [10], and the stability of the smooth solitons is considered in [11]. It is worth pointing out that solutions of this type are not mere abstractions: the peakons replicate a feature that is characteristic for the waves of great height - waves of largest amplitude that are exact solutions of the governing equations for irrotational water waves (*cf.* [12–14]). The explicit interaction of the peaked solitons is given in [15] and all possible explicit single soliton solutions are shown in [16]. The Cauchy problem for the Camassa-Holm equation (1.2) has been studied extensively. It has been shown that this problem is locally well-posed for initial data $u_0 \in H^s(\mathbb{R})$ with $s > \frac{3}{2}$ [17–19]. Moreover, it has global strong solutions and also admits finite time blow-up solutions [17, 18, 20, 21]. On the other hand, it also has global weak solutions in $H^1(\mathbb{R})$ [22–25]. The advantage of the Camassa-Holm equation in comparison with the KdV equation (1.2) lies in the fact that the Camassa-Holm equation has peaked solitons and models the peculiar wave breaking phenomena [1, 21].

For $n = 0$, $m \in \mathbb{Z}^+$, $a \in \mathbb{R}$, equation (1.1) becomes a generalized Camassa-Holm equation,

$$u_t - u_{xxt} + ku_x + au^m u_x = 2u_x u_{xx} + uu_{xxx}, \tag{1.3}$$

Wazwaz [26, 27] studied the solitary wave solutions for the generalized Camassa-Holm equation (1.3) with $m = 2$, $a = 3$, and the peakon wave solutions for this equation were studied in [28–30], and the periodic blow-up solutions and limit forms for (1.3) were obtained in [31]. In [30, 32], the authors have given the traveling waves solution, peaked solitary wave solutions for (1.3).

On the other hand, taking $m = 1$, $a = 4$, $k = 0$ in (1.1) we found the Novikov equation [33]:

$$u_t - u_{xxt} + 4u^2 u_x = 3uu_x u_{xx} + u^2 u_{xxx}, \quad t > 0, x \in \mathbb{R}, \tag{1.4}$$

The Novikov equation (1.4) possesses a matrix Lax pair, many conserved densities, a bi-Hamiltonian structure as well as peakon solutions [34]. These apparently exotic waves replicate a feature that is characteristic of the waves of great height-waves of largest amplitude that are exact solutions of the governing equations for water waves, as far as the details are concerned [13, 35, 36]. The Novikov equation possesses the explicit formulas for multipeakon solutions [37]. It has been shown that the Cauchy problem for the Novikov equation is locally well-posed in the Besov spaces and in Sobolev spaces and possesses the persistence properties [38, 39]. In [40, 41], the authors showed that the data-to-solution map for equation (1.4) is not uniformly continuous on bounded subsets of H^s for $s > 3/2$. Analogous to the Camassa-Holm equation, the Novikov equation shows the blow-up phenomenon [42] and has global weak solutions [43]. Recently, Zhao and Zhou [44] discussed the symbolic analysis and exact traveling wave solutions of a modified Novikov equation, which is new in that it has a nonlinear term $u^4 u_x$ instead of $u^2 u_x$.

Other integrable CH-type equations with cubic nonlinearity have been discovered:

$$\begin{cases} m_t + (u^2 - u_x^2)m_x + 2u_x m^2 + \gamma u_x = 0, & m = u - u_{xx}, t > 0, x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.5}$$

where γ is a constant. equation (1.5) was independently proposed by Fokas [45], by Fuchssteiner [46], and Olver and Rosenau [47] as a new generalization of integrable system by using the general method of tri-Hamiltonian duality to the bi-Hamiltonian representation of the modified Korteweg-de Vries equation. Later, it was obtained by Qiao [48, 49] from the two-dimensional Euler equations, where the variables $u(t,x)$ and $m(t,x)$ represent, respectively, the velocity of the fluid and its potential density. Ivanov and Lyons [50] obtain a class of soliton solutions of the integrable hierarchy which has been put forward in a series of woks by Qiao [48, 49]. It was shown that equation (1.5) admits the Lax-pair and the Cauchy problem (1.5) may be solved by the inverse scattering transform method. The formation of singularities and the existence of peaked traveling-wave solutions for equation (1.5) was investigated in [51]. The well-posedness, blow-up mechanism, and persistence properties are given in [52]. It was also found that equation (1.5) is related to the short-pulse equation derived by Schäfer and Wayne [53].

Applying the method of pseudoparabolic regularization, Lai and Wu [54] investigated the local well-posedness and existence of weak solutions for the following generalized Camassa-Holm equation with dissipative term:

$$u_t - u_{xxt} + 2ku_x + au^m u_x = \left(\frac{n}{2}u^{n-1}u_x^2 + u^n u_{xx}\right)_x + \beta\partial_x\big[(u_x)^{2N-1}\big], \tag{1.6}$$

where $m, n, N \in \mathbb{Z}^+$, and a, k, β are constants. Hakkaev and Kirchev [55] studied the local well-posedness and orbital stability of solitary wave solution for equation (1.6) with $a = \frac{(m+2)(m+1)}{2}$, $n = m$ and $\beta = 0$.

Motivated by the results mentioned above, the goal of this paper is to establish the well-posedness of strong solutions and weak solutions for problem (1.1). First, we use Kato's theorem to obtain the existence and uniqueness of strong solutions for equation (1.1).

Theorem 1.1 *Let $u_0 \in H^s(\mathbb{R})$ with $s > 3/2$. Then there exists a maximal $T = T(\|u_0\|_{H^s(\mathbb{R})})$, and a unique solution $u(x,t)$ to the problem* (1.1) *such that*

$$u = u(\cdot, u_0) \in C\big([0,T); H^s(\mathbb{R})\big) \cap C^1\big([0,T); H^{s-1}(\mathbb{R})\big).$$

Moreover, the solution depends continuously on the initial data, i.e. the mapping

$$u_0 \to u(\cdot, u_0) : H^s(\mathbb{R}) \to C\big([0,T); H^s(\mathbb{R})\big) \cap C^1\big([0,T); H^{s-1}(\mathbb{R})\big)$$

is continuous.

In [38, 56, 57], the spatial decay rates for the strong solution to the Camassa-Holm Novikov equation were established provided that the corresponding initial datum decays at infinity. This kind of property is so-called the persistence property. Similarly, for equation (1.1), we also have the following persistence properties for the strong solution.

Theorem 1.2 *Assume that $u_0 \in C([0,T); H^s(\mathbb{R}))$ with $s > 3/2$ satisfies*

$$\big|u_0(x)\big|, \big|u_{0x}(x)\big| \sim O\big(e^{-\theta x}\big) \quad \text{as } x \uparrow \infty$$
$$\Big(\text{respectively, } \big|u_0(x)\big|, \big|u_{0x}(x)\big| \sim O\big((1+x)^{-\alpha}\big) \text{ as } x \uparrow \infty\Big)$$

for some $\theta \in (0,1)$ (*respectively*, $\alpha \geq \max\{\frac{1}{m+1}, \frac{1}{n}\}$), *then the corresponding strong solution* $u \in C([0,T); H^s(\mathbb{R}))$ *to equation* (1.1) *satisfies for some* $T > 0$

$$\begin{aligned}
&\left|u(x,t)\right|, \left|u_x(x,t)\right| \sim O\left(e^{-\theta x}\right) \quad \textit{as } x \uparrow \infty \\
&\left(\textit{respectively}, \left|u(x)\right| \sim O\left((1+x)^{-\alpha}\right) \textit{ as } x \uparrow \infty\right)
\end{aligned}$$

uniformly in the time interval $[0,T]$.

Theorem 1.3 *Assume that* $k = 0$, $m = n$, *and* $u_0 \in C([0,T); H^s(\mathbb{R}))$ *with* $s > 3/2$ *satisfies*

$$\begin{aligned}
&\left|u_0(x)\right| \sim O\left(e^{-x}\right), \qquad \left|u_{0x}(x)\right| \sim O\left(e^{-\theta x}\right) \quad \textit{as } x \uparrow \infty \\
&\left(\textit{respectively}, \left|u_0(x)\right| \sim O\left((1+x)^{-\alpha}\right), \left|u_{0x}(x)\right| \sim O\left((1+x)^{-\beta}\right) \textit{ as } x \uparrow \infty\right)
\end{aligned}$$

for some $\theta \in (1/(m+1), 1)$ (*respectively*, $\alpha \geq \frac{1}{m+1}$, $\beta \in (\frac{\alpha}{m+1}, \alpha)$), *then the corresponding strong solution* $u \in C([0,T); H^s(\mathbb{R}))$ *to equation* (1.1) *satisfies for some* $T > 0$

$$\begin{aligned}
&\left|u(x,t)\right| \sim O\left(e^{-x}\right) \quad \textit{as } x \uparrow \infty \\
&\left(\textit{respectively}, \left|u(x)\right| \sim O\left((1+x)^{-\alpha}\right) \textit{ as } x \uparrow \infty\right)
\end{aligned}$$

uniformly in the time interval $[0,T]$.

Remark 1.1 The notations mean that

$$\left|f(x)\right| \sim O\left(e^{-\theta x}\right) \quad \text{as } x \uparrow \infty \text{ if } \lim_{x \to \infty} \frac{f(x)}{e^{-\theta x}} = L.$$

Finally, we have the following theorem for the existence of a weak solution for equation (1.1).

Theorem 1.4 *Suppose that* $u_0(x) \in H^s(\mathbb{R})$ *with* $1 < s \leq \frac{3}{2}$ *and* $\|u_{0x}\|_{L^\infty(\mathbb{R})} < \infty$. *Then there exists a life span* $T > 0$ *such that problem* (1.1) *has a weak solution* $u(x,t) \in L^2([0,T], H^s(\mathbb{R}))$ *in the sense of a distribution and* $u_x \in L^\infty([0,T] \times \mathbb{R})$.

The plan of this paper is as follows. In the next section, the local well-posedness and persistence properties of strong solutions for the problem (1.1) are established, and Theorems 1.1-1.3 are proved. The existence of weak solutions for the problem (1.1) is proved in Section 3, and this proves Theorem 1.4.

2 Well-posedness and persistence properties of strong solutions

Notation The space of all infinitely differentiable functions $f(x,t)$ with compact support in $\mathbb{R} \times [0,+\infty)$ is denoted by C_0^∞. Let p be any constant with $1 \leq p < \infty$ and denote $L^p = L^p(\mathbb{R})$ to be the space of all measurable functions f such that $\|f\|_{L^p}^p = \int_{\mathbb{R}} |f(x)|^p \, dx < \infty$. The space $L^\infty = L^\infty(\mathbb{R})$ with the standard norm $\|f\|_{L^\infty} = \inf_{m(e)=0} \sup_{x \in \mathbb{R}/e} |f(x)|$. For any real number s, let $H^s = H^s(\mathbb{R})$ denote the Sobolev space with the norm defined by

$$\|f\|_{H^s} = \left(\int_{\mathbb{R}} \left(1 + |\xi|^2\right)^s \left|\hat{f}(\xi,t)\right|^2 d\xi\right)^{\frac{1}{2}} < \infty,$$

where $\hat{f}(\xi,t) = \int_{\mathbb{R}} e^{-ix\xi} f(x,t)\,dx$. Let $C([0,T];H^s(\mathbb{R}))$ denote the class of continuous functions from $[0,T]$ to $H^s(\mathbb{R})$ and $\Lambda = (1-\partial_x^2)^{\frac{1}{2}}$.

Proof of Theorem 1.1 To prove well-posedness we apply Kato's semigroup approach [58]. For this, we rewrite the Cauchy problem of equation (1.1) as follows for the transport equation:

$$\begin{cases} u_t + u^{m+1}u_x + F(u) = 0, \\ u(x,0) = u_0(x), \end{cases} \tag{2.1}$$

where $F(u) := P * E(u)$. $E(u) = 2ku_x + au^m u_x - u^{n+1}u_x + \frac{2n+1}{2}\partial_x(u^n u_x^2) + \frac{n}{2}u^{n-1}u_x^3$ and $P(x) = \frac{1}{2}e^{-|x|}$. Let $A(u) := u^{m+1}\partial_x$, $Y = H^s$, $X = H^{s-1}$ and $Q = \Lambda = (1-\partial_x^2)^{\frac{1}{2}}$. Following closely the considerations made in [17, 54, 59], we obtain the statement of Theorem 1.1. □

Proof of Theorem 1.2 We introduce the notation $M = \sup_{t\in[0,T]} \|u(t)\|_{H^s}$. The first step we will give estimates on $\|u(x,t)\|_{L^\infty}$. Integrating the both sides with respect to x variable by multiplying the first equation of (2.1) by u^{2p-1} with $p \in \mathbb{Z}^+$, we get

$$\int_{\mathbb{R}} u^{2p-1}u_t\,dx + \int_{\mathbb{R}} u^{2p-1}\bigl(u^{n+1}u_x\bigr)\,dx + \int_{\mathbb{R}} u^{2p-1}\bigl(P * E(u)\bigr)\,dx = 0. \tag{2.2}$$

Note that the estimates

$$\int_{\mathbb{R}} u^{2p-1}u_t\,dx = \frac{1}{2p}\frac{d}{dt}\bigl\|u(x,t)\bigr\|_{L^{2p}}^{2p} = \bigl\|u(x,t)\bigr\|_{L^{2p}}^{2p-1}\frac{d}{dt}\bigl\|u(x,t)\bigr\|_{L^{2p}},$$

and

$$\left|\int_{\mathbb{R}} u^{2p-1}\bigl(u^{n+1}u_x\bigr)\,dx\right| \le \bigl\|u_x(x,t)\bigr\|_{L^\infty}\bigl\|u(x,t)\bigr\|_{L^{2p}}^{2p+n}$$

are true. Moreover, using Hölder's inequality

$$\left|\int_{\mathbb{R}} u^{2p-1}\bigl(P * E(u)\bigr)\,dx\right| \le \bigl\|u(x,t)\bigr\|_{L^{2p}}^{2p-1}\bigl\|P * E(u)\bigr\|_{L^{2p}}.$$

From equation (2.2) we can obtain

$$\frac{d}{dt}\bigl\|u(x,t)\bigr\|_{L^{2p}} \le \bigl\|u_x(x,t)\bigr\|_{L^\infty}\bigl\|u(x,t)\bigr\|_{L^{2p}}^{n+1} + \bigl\|P * E(u)\bigr\|_{L^{2p}}.$$

Since $\|f\|_{L^p} \to \|f\|_{L^\infty}$ as $p \to \infty$ for any $f \in L^\infty \cap L^2$. From the above inequality we deduce that

$$\frac{d}{dt}\bigl\|u(x,t)\bigr\|_{L^\infty} \le M^{n+1}\bigl\|u(x,t)\bigr\|_{L^\infty} + \bigl\|P * E(u)\bigr\|_{L^\infty},$$

where we use

$$\bigl\|u_x(x,t)\bigr\|_{L^\infty}\bigl\|u(x,t)\bigr\|_{L^\infty}^{n} \le \bigl\|u_x(x,t)\bigr\|_{H^{\frac{1}{2}+}}\bigl\|u(x,t)\bigr\|_{H^{\frac{1}{2}+}}^{n} \le \bigl\|u(x,t)\bigr\|_{H^s}^{n+1} \le M^{n+1}.$$

Because of Gronwall's inequality, we get

$$\|u(x,t)\|_{L^\infty} \le \exp\bigl(M^{n+1}t\bigr)\left(\|u_0(x)\|_{L^\infty} + \int_0^t \bigl\|\bigl(P * E(u)\bigr)(x,\tau)\bigr\|_{L^\infty}\,d\tau\right).$$

Next, we will give estimates on $\|u_x(x,t)\|_{L^\infty}$. Differentiating (2.1) with respect to the x-variable produces the equation

$$u_{xt} + u^{n+1}u_{xx} + (n+1)u^n u_x^2 + \partial_x\bigl(P * E(u)\bigr) = 0. \tag{2.3}$$

Multiplying this equation by $(u_x)^{2p-1}$ with $p \in \mathbb{Z}^+$, integrating the result in the x-variable, and using integration by parts:

$$\begin{aligned}
&\int_{\mathbb{R}} (u_x)^{2p-1}u_{xt}\,dx = \frac{1}{2p}\frac{d}{dt}\|u_x(x,t)\|_{L^{2p}}^{2p} = \|u_x(x,t)\|_{L^{2p}}^{2p-1}\frac{d}{dt}\|u_x(x,t)\|_{L^{2p}},\\
&\left|\int_{\mathbb{R}} (u_x)^{2p-1}\bigl(u^n u_x^2\bigr)\,dx\right| \le \|u(x,t)\|_{L^\infty}^n\|u_x(x,t)\|_{L^\infty}\|u_x(x,t)\|_{L^{2p}}^{2p},\\
&\left|\int_{\mathbb{R}} (u_x)^{2p-1}\bigl(u^{n+1}u_{xx}\bigr)\,dx\right| = \left|\frac{n+1}{2p}\int_{\mathbb{R}} u^n u_x^{2p+1}\,dx\right|\\
&\qquad\qquad \le \frac{n+1}{2p}\|u(x,t)\|_{L^\infty}^n\|u_x(x,t)\|_{L^\infty}\|u_x(x,t)\|_{L^{2p}}^{2p}.
\end{aligned}$$

From the above inequalities, we also can get the following inequality:

$$\frac{d}{dt}\|u_x(x,t)\|_{L^{2p}} \le \left(n+1+\frac{n+1}{2p}\right)M^{n+1}\|u_x(x,t)\|_{L^{2p}} + \bigl\|\partial_x\bigl(P * E(u)\bigr)\bigr\|_{L^{2p}},$$

where we use $\|u_x(x,t)\|_{L^\infty}\|u(t)\|_{L^\infty}^n \le M^{n+1}$. Then passing to the limit in this inequality and using Gronwall's inequality one can obtain

$$\|u_x(x,t)\|_{L^\infty} \le \exp\bigl((n+1)M^{n+1}t\bigr)\left(\|u_{0x}(x)\|_{L^\infty} + \int_0^t \bigl\|\partial_x\bigl(P * E(u)\bigr)(x,\tau)\bigr\|_{L^\infty}\,d\tau\right).$$

We shall now repeat the arguments using the weight

$$\varphi_N(x) = \begin{cases} 1, & x \le 0,\\ e^{\theta x}, & 0 < x < N,\\ e^{\theta x}, & x \ge N,\end{cases}$$

where $N \in \mathbb{Z}$. Observe that for all N we have

$$0 \le \varphi_N'(x) \le \varphi_N(x), \quad \text{for all } x \in \mathbb{R}. \tag{2.4}$$

Using the notation $E(u)$, from (2.1) we get

$$\partial_t(u\varphi_N) + \bigl(u^{n+1}\varphi_N\bigr)u_x + \varphi_N\bigl(P * E(u)\bigr) = 0,$$

and from (2.3), we also obtain

$$\partial_t(\varphi_N\partial_x u) + u^{n+1}\varphi_N\partial_x^2 u + (n+1)u^n(\varphi_N\partial_x u)\partial_x u + \varphi_N\partial_x\bigl(P * E(u)\bigr) = 0.$$

We need to eliminate the second derivatives in the second term in the above equality. Thus, combining integration by parts and equation (2.4) we find

$$\begin{aligned}
&\left|\int_{\mathbb{R}} u^{n+1}\varphi_N \partial_x^2 u(\partial_x u\varphi_N)^{2p-1}\right| \\
&\quad = \left|\int_{\mathbb{R}} u^{n+1}(\partial_x u\varphi_N)^{2p-1}\big(\partial_x(\varphi_N\partial_x u) - \partial_x u\varphi_N'\big)\,dx\right| \\
&\quad = \left|\int_{\mathbb{R}} \frac{1}{2p}u^{n+1}\partial_x\big((\partial_x u\varphi_N)^{2p}\big) - u^{n+1}(\partial_x u\varphi_N)^{2p-1}\partial_x u\varphi_N'\,dx\right| \\
&\quad \le \big(\|u\|_{L^\infty} + \|\partial_x u\|_{L^\infty}\big)\|u\|_{L^\infty}^n\|\partial_x u\varphi_N\|_{L^{2p}}^{2p}.
\end{aligned}$$

Hence, as in the weightless case, we have

$$\begin{aligned}
&\|u\varphi_N\|_{L^\infty} + \|\partial_x u\varphi_N\|_{L^\infty} \\
&\quad \le \exp\big((n+1)M^{n+1}t\big)\big(\|u_0(x)\varphi_N\|_{L^\infty} + \|u_{0x}(x)\varphi_N\|_{L^\infty}\big) \\
&\qquad + \exp\big((n+1)M^{n+1}t\big)\int_0^t \big(\big\|\varphi_N\partial_x\big(E(u)\big)\big\|_{L^\infty} + \big\|\varphi_N\big(E(u)\big)\big\|_{L^\infty}\big)\,d\tau.
\end{aligned}$$

A simple calculation shows that there exists $C > 0$, depending only on $\theta \in (0,1)$ such that for any $N \in \mathbb{Z}^+$,

$$\varphi_N \int_{\mathbb{R}} \frac{1}{\varphi_N(y)}\,dy \le C = \frac{4}{1-\theta}.$$

Thus, we have

$$\begin{aligned}
\big|\varphi_N\big(1-\partial_x^2\big)^{-1}\big(u^{n-1}\partial_x^3 u\big)\big| &= \frac{1}{2}\left|\varphi_N\int_{\mathbb{R}} e^{-|x-y|}\big(u^{n-1}\partial_x^3 u\big)(y)\,dy\right| \\
&= \frac{1}{2}\left|\varphi_N\int_{\mathbb{R}} e^{-|x-y|}\frac{1}{\varphi_N(y)}(\varphi_N\partial_x u)\big(u^{n-1}\partial_x^2 u\big)(y)\,dy\right| \\
&\le \frac{1}{2}\left(\varphi_N\int_{\mathbb{R}} e^{-|x-y|}\frac{1}{\varphi_N(y)}\,dy\right)\|\varphi_N\partial_x u\|_{L^\infty}\big\|u^{n-1}\partial_x^2 u\big\|_{L^\infty} \\
&\le c\|\varphi_N\partial_x u\|_{L^\infty}\big\|u^{n-1}\partial_x^2 u\big\|_{L^\infty},
\end{aligned}$$

and

$$\begin{aligned}
\big|\varphi_N\big(1-\partial_x^2\big)^{-1}\partial_x\big(u^{n-1}\partial_x^3 u\big)\big| &= \frac{1}{2}\left|\varphi_N\int_{\mathbb{R}} \operatorname{sgn}(x-y)e^{-|x-y|}\big(u^{n-1}\partial_x^3 u\big)(y)\,dy\right| \\
&\le c\|\varphi_N\partial_x u\|_{L^\infty}\big\|u^{n-1}\partial_x^2 u\big\|_{L^\infty}.
\end{aligned}$$

Using the same method, we can estimate the other terms:

$$\begin{aligned}
&\big|\varphi_N\big(1-\partial_x^2\big)^{-1}\big(2ku_x + au^m u_x - u^{n+1}u_x\big)\big| \le c\big(1 + \|u\|_{L^\infty}^m + \|u\|_{L^\infty}^{n+1}\big)\|\varphi_N\partial_x u\|_{L^\infty}, \\
&\big|\varphi_N\big(1-\partial_x^2\big)^{-1}\partial_x\big(2ku_x + au^m u_x - u^{n+1}u_x\big)\big| \\
&\quad = \left|\varphi_N\big(1-\partial_x^2\big)^{-1}\partial_x^2\left(2ku + \frac{a}{m+1}u^{m+1} - \frac{1}{n+2}u^{n+2}\right)\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \left|\varphi_N\left(2ku + \frac{a}{m+1}u^{m+1} - \frac{1}{n+2}u^{n+2}\right)\right| \\
&\quad + \left|\varphi_N\left(1-\partial_x^2\right)^{-1}\left(2ku + \frac{a}{m+1}u^{m+1} - \frac{1}{n+2}u^{n+2}\right)\right| \\
&\leq c\left(1 + \|u\|_{L^\infty}^m + \|u\|_{L^\infty}^{n+1}\right)\|\varphi_N u\|_{L^\infty},
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|\varphi_N\left(1-\partial_x^2\right)^{-1}\partial_x\left(u^n u_x^2\right)\right| \leq c\|\varphi_N u\|_{L^\infty}\left\|u^{n-1}\partial_x^2 u\right\|_{L^\infty}, \\
&\left|\varphi_N\left(1-\partial_x^2\right)^{-1}\partial_x^2\left(u^n u_x^2\right)\right| \leq \left|\varphi_N u^n u_x^2\right| + \left|\varphi_N\left(1-\partial_x^2\right)^{-1}\left(u^n u_x^2\right)\right| \\
&\qquad \leq c\|\varphi_N u\|_{L^\infty}\left\|u^{n-1}\partial_x^2 u\right\|_{L^\infty}.
\end{aligned}
$$

Thus, it follows that there exists a constant $C > 0$ which depends only on M, m, n, k, a, and T, such that

$$
\begin{aligned}
&\|u\varphi_N\|_{L^\infty} + \|\partial_x u\varphi_N\|_{L^\infty} \\
&\quad \leq C\left(\|u_0\varphi_N\|_{L^\infty} + \|u_{0x}\varphi_N\|_{L^\infty}\right) \\
&\qquad + C\int_0^t \left(\left(1 + \|u\|_{L^\infty}^m + \|u\|_{L^\infty}^{n+1} + \left\|u^{n-1}\partial_x^2 u\right\|_{L^\infty}\right)\left(\|\varphi_N\partial_x u\|_{L^\infty} + \|\varphi_N u\|_{L^\infty}\right)\right) d\tau \\
&\quad \leq C\left(\|u_0\varphi_N\|_{L^\infty} + \|u_{0x}\varphi_N\|_{L^\infty}\right) + C\int_0^t \left(\|\varphi_N\partial_x u\|_{L^\infty} + \|\varphi_N u\|_{L^\infty}\right) d\tau.
\end{aligned}
$$

Hence, for any $n \in \mathbb{Z}$ and any $t \in [0, T]$ we have

$$
\begin{aligned}
\|u\varphi_N\|_{L^\infty} + \|\partial_x u\varphi_N\|_{L^\infty} &\leq C\left(\|u_0\varphi_N\|_{L^\infty} + \|u_{0x}\varphi_N\|_{L^\infty}\right) \\
&\leq C\left(\left\|u_0 \max\left(1, e^{\theta x}\right)\right\|_{L^\infty} + \left\|u_{0x}\max\left(1, e^{\theta x}\right)\right\|_{L^\infty}\right).
\end{aligned}
$$

Finally, taking the limit as N goes to infinity we find that for any $t \in [0, T]$,

$$
\left\|ue^{\theta x}\right\|_{L^\infty} + \left\|\partial_x ue^{\theta x}\right\|_{L^\infty} \leq C\left(\left\|u_0 \max\left(1, e^{\theta x}\right)\right\|_{L^\infty} + \left\|u_{0x}\max\left(1, e^{\theta x}\right)\right\|_{L^\infty}\right),
$$

which completes the proof of Theorem 1.2. □

Next, we give a simple proof for Theorem 1.3.

Proof of Theorem 1.3 We should use Theorem 1.3 to prove this theorem.

For any $t_1 \in [0, T]$, integrating equation (2.1) over the time interval $[0, t_1]$ we get

$$
u(x, t_1) - u(x, 0) + \int_0^{t_1}\left(u^{n+1}u_x\right)(x, \tau)\, d\tau + \int_0^{t_1}\left(P * E(u)\right)(x, \tau)\, d\tau = 0. \tag{2.5}
$$

From Theorem 1.2, it follows that

$$
\int_0^{t_1}\left(u^{n+1}u_x\right)(x, \tau)\, d\tau \sim O\left(e^{-(n+2)\alpha x}\right) \quad \text{as } x \uparrow \infty
$$

and so

$$\int_0^{t_1} (u^{n+1}u_x)(x,\tau)\,d\tau \sim O(e^{-x}) \quad \text{as } x \uparrow \infty.$$

We shall show that the last term in equation (2.5) is $O(e^{-x})$; thus we have

$$\int_0^{t_1} (P * E(u))(x,\tau)\,d\tau = P(x) * \int_0^{t_1} (E(u))(x,\tau)\,d\tau \doteq P(x) * \rho(x).$$

From the given condition and Theorem 1.2. we know $\rho(x) \sim O(e^{-x})$ as $x \uparrow \infty$. Since

$$P(x) * \rho(x) = \frac{1}{2}\int_{\mathbb{R}} e^{-|x-y|}\rho(y)\,dy = \frac{1}{2}e^{-x}\int_{-\infty}^{x} e^{y}\rho(y)\,dy + \frac{1}{2}e^{x}\int_{x}^{\infty} e^{-y}\rho(y)\,dy$$

we have

$$\begin{aligned} &e^{-x}\int_{-\infty}^{x} e^{y}\rho(y)\,dy = O(1)e^{-x}\int_{-\infty}^{x} e^{2y}\,dy \sim O(1)e^{-x} \sim O(e^{-x}) \quad \text{as } x \uparrow \infty, \\ &e^{x}\int_{x}^{\infty} e^{-y|}\rho(y)\,dy = O(1)e^{x}\int_{x}^{\infty} e^{-2y}\,dy \sim O(1)e^{-x} \sim O(e^{-x}) \quad \text{as } x \uparrow \infty. \end{aligned}$$

Thus

$$\int_0^{t_1} (P * E(u))(x,\tau)\,d\tau \sim O(e^{-x}) \quad \text{as } x \uparrow \infty.$$

From equation (2.5) and $|u_0(x)| \sim O(e^{-x})$ as $x \uparrow \infty$, we know

$$|u(x,t_1)| \sim O(e^{-x}) \quad \text{as } x \uparrow \infty.$$

By the arbitrariness of $t_1 \in [0,T]$, we get

$$|u(x,t)| \sim O(e^{-x}) \quad \text{as } x \uparrow \infty$$

uniformly in the time interval $[0,T]$. This completes the proof of Theorem 1.3. □

3 Existence of solution of the regularized equation

In order to prove Theorem 1.4, we consider the regularized problem for equation (1.1) in the following form:

$$\begin{cases} u_t - u_{xxt} + \epsilon u_{xxxxt} = \partial_x(-2ku - \frac{a}{m+1}u^{m+1}) + \frac{1}{n+2}\partial_x^3(u^{n+2}) \\ \qquad\qquad - \frac{2n+1}{2}\partial_x(u^n u_x^2) - \frac{n}{2}u^{n-1}u_x^3, \\ u(x,0) = u_0(x), \end{cases} \tag{3.1}$$

where $0 < \epsilon < \frac{1}{4}$, $m \geq 1$, $n \geq 1$ and a, k are constants. One can easily check that when $\epsilon = 0$, equation (3.1) is equivalent to the IVP (1.1).

Before giving the proof of Theorem 1.4, we give several lemmas.

Lemma 3.1 (See [54]) *Let p and q be real numbers such that $-p < q \leq p$. Then*

$$\|fg\|_{H^q} \leq c\|f\|_{H^p}\|g\|_{H^q}, \quad \text{if } p > \frac{1}{2},$$
$$\|fg\|_{H^{p+q-\frac{1}{2}}} \leq c\|f\|_{H^p}\|g\|_{H^q}, \quad \text{if } p < \frac{1}{2}.$$

Lemma 3.2 *Let $u_0(x) \in H^s(\mathbb{R})$ with $s > 3/2$. Then the Cauchy problem* (3.1) *has a unique solution $u(x,t) \in C([0,T]; H^s(\mathbb{R}))$ where $T > 0$ depends on $\|u_0\|_{H^s(\mathbb{R})}$. If $s \geq 2$, the solution $u(x,t) \in C([0,T]; H^s(\mathbb{R}))$ exists for all time. In particular, when $s \geq 4$, the corresponding solution is a classical globally defined solution of problem* (3.1).

Proof First, we note that, for any $0 < \epsilon < \frac{1}{4}$ and any s, the integral operator

$$\mathcal{D} = \left(1 - \partial_x^2 + \epsilon \partial_x^4\right)^{-1} : H^s \to H^{s+4}$$

defines a bounded linear operator on the indicated Sobolev spaces.

To prove the existence of a solution to the problem (3.1), we apply the operator $\mathcal{D}$ to both sides of equation (3.1) and then integrate the resulting equations with regard to t. This leads to the following equations:

$$\begin{aligned} u(x,t) = u_0(x) + \int_0^t \mathcal{D}\bigg[\partial_x\left(-2ku - \frac{a}{m+1}u^{m+1}\right) + \frac{1}{n+2}\partial_x^3\left(u^{n+2}\right) \\ - \frac{2n+1}{2}\partial_x\left(u^n u_x^2\right) - \frac{n}{2}u^{n-1}u_x^3\bigg]\,d\tau. \end{aligned} \tag{3.2}$$

Suppose that $\mathbb{A}$ is the operator in the right-hand side of equation (3.2). For fixed $t \in [0,T]$, we get

$$\begin{aligned} &\bigg\| \int_0^t \mathcal{D}\bigg[\partial_x\left(-2ku - \frac{a}{m+1}u^{m+1}\right) + \partial_x^2\left(u^{n+1}u_x\right) - \frac{2n+1}{2}\partial_x\left(u^n u_x^2\right) - \frac{n}{2}u^{n-1}u_x^3\bigg]\,d\tau \\ &\quad - \int_0^t \mathcal{D}\bigg[\partial_x\left(-2kv - \frac{a}{m+1}v^{m+1}\right) + \partial_x^3\left(v^{n+1}v_x\right) \\ &\quad - \frac{2n+1}{2}\partial_x\left(v^n v_x^2\right) - \frac{n}{2}v^{n-1}v_x^3\bigg]\,d\tau \bigg\|_{H^s} \\ &\leq CT\Big(\sup_{0\leq t\leq T}\|u-v\|_{H^s} + \sup_{0\leq t\leq T}\left\|u^{m+1} - v^{m+1}\right\|_{H^s} + \sup_{0\leq t\leq T}\left\|u^{n+2} - v^{n+2}\right\|_{H^s} \\ &\quad + \sup_{0\leq t\leq T}\left\|\mathcal{D}\partial_x\left[u^n u_x^2 - v^n v_x^2\right]\right\|_{H^s} + \sup_{0\leq t\leq T}\left\|\mathcal{D}\left[u^{n-1}u_x^3 - v^{n-1}v_x^3\right]\right\|_{H^s}\Big). \end{aligned}$$

Since H^{s_0} is an algebra for $s_0 \geq \frac{1}{2}$, we have the inequalities

$$\begin{aligned} \left\|u^{m+1} - v^{m+1}\right\|_{H^s} &= \left\|(u-v)\sum_{j=0}^{m}\left(u^{m-j}v^j\right)\right\|_{H^s} \\ &\leq C_1\left(\|u-v\|_{H^s}\sum_{j=0}^{m}\|u\|_{H^s}^{m-j}\|v\|_{H^s}^{j}\right). \end{aligned}$$

Since $s > 3/2$, by Lemma 3.1, we get

$$\begin{aligned}
&\left\|\mathcal{D}\partial_x\left[u^n u_x^2 - v^n v_x^2\right]\right\|_{H^s} \\
&\quad = \frac{1}{n+1}\left\|\mathcal{D}\partial_x\left[\partial_x\left(u^{n+1}\right)\partial_x u - \partial_x\left(v^{n+1}\right)\partial_x v\right]\right\|_{H^s} \\
&\quad \le C\left(\left\|\mathcal{D}\partial_x\left[\partial_x\left(u^{n+1}\right)\partial_x(u-v)\right]\right\|_{H^s} + \left\|\mathcal{D}\partial_x\left[\partial_x\left(u^{n+1}-v^{n+1}\right)\partial_x v\right]\right\|_{H^s}\right) \\
&\quad \le C\left(\left\|\partial_x\left(u^{n+1}\right)\partial_x(u-v)\right\|_{H^{s-2}} + \left\|\partial_x\left(u^{n+1}-v^{n+1}\right)\partial_x v\right\|_{H^{s-2}}\right) \\
&\quad \le C\left(\left\|u^{n+1}\right\|_{H^s}\|u-v\|_{H^s} + \left\|u^{n+1}-v^{n+1}\right\|_{H^s}\|v\|_{H^s}\right) \\
&\quad \le C_2\left(\|u\|_{H^s}^{n+1}\|u-v\|_{H^s} + \left(\|u-v\|_{H^s}\sum_{j=0}^{n}\|u\|_{H^s}^{n-j}\|v\|_{H^s}^{j}\right)\|v\|_{H^s}\right),
\end{aligned}$$

and

$$\begin{aligned}
&\left\|\mathcal{D}\left[u^{n-1}u_x^3 - v^{n-1}v_x^3\right]\right\|_{H^s} \\
&\quad \le \left\|\mathcal{D}\left[u^{n-1}\left(u_x^3 - v_x^3\right)\right]\right\|_{H^s} + \left\|\mathcal{D}\left[\left(u^{n-1}-v^{n-1}\right)v_x^3\right]\right\|_{H^s} \\
&\quad \le C\left[\left\|u^{n-1}\left(u_x^3 - v_x^3\right)\right\|_{H^{s-2}} + \left\|\left(u^{n-1}-v^{n-1}\right)v_x^3\right\|_{H^{s-2}}\right] \\
&\quad \le C\left[\left\|u^{n-1}\right\|_{H^{s-1}}\left\|u_x^3 - v_x^3\right\|_{H^{s-2}} + \left\|u^{n-1}-v^{n-1}\right\|_{H^{s-1}}\left\|v_x^3\right\|_{H^{s-2}}\right] \\
&\quad \le C_3\left[\|u\|_{H^s}^{n-1}\|u-v\|_{H^s}\sum_{j=0}^{2}\|u\|_{H^s}^{2-j}\|v\|_{H^s}^{j} + \left(\|u-v\|_{H^s}\sum_{j=0}^{n-2}\|u\|_{H^s}^{n-2-j}\|v\|_{H^s}^{j}\right)\|v\|_{H^s}^{3}\right],
\end{aligned}$$

where C_2, C_3 only depend on n. Suppose that both u and v are in the closed ball $B_R(0)$ of radius R about the zero function in $C([0,T];H^s(\mathbb{R}))$; by the above inequalities, we obtain

$$\|\mathbb{A}u - \mathbb{A}v\|_{C([0,T];H^s)} \le \theta\|u-v\|_{C([0,T];H^s)},$$

where $\theta = TC(R^m + R^{n+1})$ and C only depend on a, k, m, n. Choosing T sufficiently small such that $\theta < 1$, we know that $\mathbb{A}$ is a contraction. Applying the above inequality yields

$$\|\mathbb{A}u\|_{C([0,T];H^s)} \le \|u_0\|_{H^s} + \theta\|u\|_{C([0,T];H^s)}.$$

Taking T sufficiently small so that $\theta R + \|u_0\|_{H^s} < R$, we deduce that $\mathbb{A}$ maps $B_R(0)$ to itself. It follows from the contraction-mapping principle that the mapping $\mathbb{A}$ has a unique fixed point u in $B_R(0)$.

For $s \ge 2$, multiplying the first equation of the system (3.1) by $2u$, integrating with respect to x, one derives

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}}\left(u^2 + u_x^2 + \epsilon u_{xx}^2\right)dx &= \int_{\mathbb{R}} 2u\left(-2ku_x - au^m u_x + (n+2)u^n u_x u_{xx} + u^{n+1}u_{xxx}\right)dx \\
&= \int_{\mathbb{R}}\left(2(n+2)u^{n+1}u_x u_{xx} + 2u^{n+2}u_{xxx}\right)dx = 0,
\end{aligned}$$

from which we have the conservation law

$$\int_{\mathbb{R}}\left(u^2 + u_x^2 + \epsilon u_{xx}^2\right)dx = \int_{\mathbb{R}}\left(u_0^2 + u_{0x}^2 + \epsilon u_{0xx}^2\right)dx. \tag{3.3}$$

The global existence result follows from the integral from equation (3.2) and equation (3.3). □

Now we study the norms of solutions of equation (3.1) using energy estimates. First, recall the following two lemmas.

Lemma 3.3 (See [58]) *If $r > 0$, then $H^r \cap L^\infty$ is an algebra, and*

$$\|fg\|_{H^r} \le c\big(\|f\|_{L^\infty}\|g\|_{H^r} + \|g\|_{L^\infty}\|h\|_{H^r}\big),$$

here c is a constant depending only on r.

Lemma 3.4 (See [58]) *If $r > 0$, then*

$$\big\|[\Lambda^r, f]g\big\|_{L^2} \le c\big(\|\partial_x f\|_{L^\infty}\big\|\Lambda^{r-1}g\big\|_{L^2} + \big\|\Lambda^r f\big\|_{L^2}\|g\|_{L^\infty}\big),$$

where $[A,B]$ denotes the commutator of the linear operators A and B, and c is a constant depending only on r.

Theorem 3.1 *Suppose that, for some $s \ge 4$, the functions $u(t,x)$ are a solution of equation* (3.1) *corresponding to the initial data $u_0 \in H^s(\mathbb{R})$. Then the following inequality holds*:

$$\|u\|_{H^1}^2 \le c\int_{\mathbb{R}}\big(u^2 + u_x^2 + \epsilon u_{xx}^2\big)\,dx = c\int_{\mathbb{R}}\big(u_0^2 + u_{0x}^2 + \epsilon u_{0xx}^2\big)\,dx. \tag{3.4}$$

For any real number $q \in (0, s-1]$, there exists a constant c depending only on q such that

$$\begin{aligned}&\int_{\mathbb{R}}\big(\Lambda^{q+1}u\big)^2\,dx\\ &\quad\le \int_{\mathbb{R}}\big[\big(\Lambda^{q+1}u_0\big)^2 + \varepsilon\big(\Lambda^{q+1}u_{0xx}\big)^2\big]\,dx + c\int_0^t \|u\|_{H^q}^2\|u\|_{L^\infty}^{n-2}\|u_x\|_{L^\infty}^3\,d\tau\\ &\qquad + c\int_0^t \|u_x\|_{L^\infty}\big(\|u\|_{H^q}^2\big(\|u\|_{L^\infty}^{m-1} + \|u\|_{L^\infty}^n\big) + 2\|u\|_{L^\infty}^n\|u\|_{H^{q+1}}^2\big)\,d\tau.\end{aligned} \tag{3.5}$$

For $q \in [0, s-1]$, there is a constant c independent of ϵ such that

$$(1-2\epsilon)\|u_t\|_{H^q} \le c\|u\|_{H^{q+1}}\big(1 + \big(\|u\|_{L^\infty}^{m-1} + \|u\|_{L^\infty}^n\big)\|u\|_{H^1} + \|u\|_{L^\infty}^{n-1}\|u_x\|_{L^\infty}^2\big). \tag{3.6}$$

Proof Using $\|u\|_{H^1}^2 \le c\int_{\mathbb{R}}(u^2 + u_x^2)\,dx$ and (3.3) derives (3.4).

Since $\partial_x^2 = -\Lambda^2 + 1$ and the Parseval equality gives rise to

$$\int_{\mathbb{R}}\big(\Lambda^q u\big)\Lambda^q\partial_x^2 f\,dx = -\int_{\mathbb{R}}\big(\Lambda^{q+1}u\big)\Lambda^{q+1}f\,dx + \int_{\mathbb{R}}\big(\Lambda^q u\big)\Lambda^q f\,dx.$$

For any $q \in (0, s-1]$, applying $(\Lambda^q u)\Lambda^q$ to both sides of the first equation of (3.1), respectively, and integrating with regard to x again,using integration by parts, one obtains

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}\big(\big(\Lambda^q u\big)^2 + \big(\Lambda^q u_x\big)^2 + \epsilon\big(\Lambda^q u_{xx}\big)^2\big)\,dx\\ &\quad= -a\int_{\mathbb{R}}\big(\Lambda^q u\big)\Lambda^q\big(u^m u_x\big)\,dx\end{aligned}$$

$$
\begin{aligned}
&- \int_{\mathbb{R}} \left(\Lambda^{q+1} u\right) \Lambda^{q+1} \left(u^{n+1} u_x\right) dx + \frac{2n+1}{2} \int_{\mathbb{R}} \left(\Lambda^q u_x\right) \Lambda^q \left(u^n u_x^2\right) dx \\
&+ \int_{\mathbb{R}} \left(\Lambda^q u\right) \Lambda^q \left(u^{n+1} u_x\right) dx - \frac{n}{2} \int_{\mathbb{R}} \left(\Lambda^q u\right) \Lambda^q \left(u^{n-1} u_x^3\right) dx. \qquad (3.7)
\end{aligned}
$$

We will estimate the terms on the right-hand side of (3.7) separately. For the first term, by using the Cauchy-Schwartz inequality and Lemmas 3.3 and 3.4, we have

$$
\begin{aligned}
\int_{\mathbb{R}} \left(\Lambda^q u\right) \Lambda^q \left(u^m u_x\right) dx &= \int_{\mathbb{R}} \left(\Lambda^q u\right) \left[\Lambda^q \left(u^m u_x\right) - u^m \Lambda^q u_x\right] dx + \int_{\mathbb{R}} \left(\Lambda^q u\right) u^m \Lambda^q u_x \, dx \\
&\leq c \|u\|_{H^q} \left(m \|u\|_{L^\infty}^{m-1} \|u_x\|_{L^\infty} \|u\|_{H^q} + \|u\|_{L^\infty}^{m-1} \|u_x\|_{L^\infty} \|u\|_{H^q}\right) \\
&\quad + \frac{m}{2} \|u\|_{L^\infty}^{m-1} \|u_x\|_{L^\infty} \left\|\Lambda^q u\right\|_{L^2}^2 \\
&\leq c \|u\|_{L^\infty}^{m-1} \|u_x\|_{L^\infty} \|u\|_{H^q}^2. \qquad (3.8)
\end{aligned}
$$

Using the above estimate to the second term on the right-hand side of equation (3.7) yields

$$
\int_{\mathbb{R}} \left(\Lambda^{q+1} u\right) \Lambda^{q+1} \left(u^{n+1} u_x\right) dx = c \|u\|_{L^\infty}^n \|u_x\|_{L^\infty} \|u\|_{H^{q+1}}^2. \qquad (3.9)
$$

For the fourth term on the right-hand side of equation (3.7), using the Cauchy-Schwartz inequality and Lemma 3.3, we obtain

$$
\begin{aligned}
\int_{\mathbb{R}} \left(\Lambda^q u_x\right) \Lambda^q \left(u^n u_x^2\right) dx &\leq \left\|\Lambda^q u_x\right\|_{L^2} \left\|\Lambda^q \left(u^n u_x^2\right)\right\|_{L^2} \\
&\leq c \|u\|_{H^{q+1}} \left(\left\|u^n u_x\right\|_{L^\infty} \|u_x\|_{H^q} + \|u_x\|_{L^\infty} \left\|u^n u_x\right\|_{H^q}\right) \\
&\leq c \|u\|_{L^\infty}^n \|u_x\|_{L^\infty} \|u\|_{H^{q+1}}^2. \qquad (3.10)
\end{aligned}
$$

For the last term on the right-hand side of equation (3.7), using Lemma 3.3 repeatedly results in

$$
\int_{\mathbb{R}} \left(\Lambda^q u\right) \Lambda^q \left(u^{n-1} u_x^3\right) dx \leq \|u\|_{H^q} \left\|u^{n-1} u_x^3\right\|_{H^q} \leq \|u\|_{H^q}^2 \|u\|_{L^\infty}^{n-2} \|u_x\|_{L^\infty}^3. \qquad (3.11)
$$

It follows from equations (3.7)-(3.11) that there exists a constant c depending only on a, m, n, s such that

$$
\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} \left(\left(\Lambda^q u\right)^2 + \left(\Lambda^q u_x\right)^2 + \epsilon \left(\Lambda^q u_{xx}\right)^2\right) dx \\
&\quad \leq c \|u_x\|_{L^\infty} \left(\|u\|_{H^q}^2 \left(\|u\|_{L^\infty}^{m-1} + \|u\|_{L^\infty}^n\right) + 2 \|u\|_{L^\infty}^n \|u\|_{H^{q+1}}^2\right) + \|u\|_{H^q}^2 \|u\|_{L^\infty}^{n-2} \|u_x\|_{L^\infty}^3.
\end{aligned}
$$

Integrating both sides of the above inequality with respect to t results in inequality (3.5).

To estimate the norm of u_t, we apply the operator $(1-\partial_x^2)^{-1}$ to both sides of the first equation of the system (3.1) to obtain the equation

$$
\begin{aligned}
(1-\epsilon) u_t - \epsilon u_{xxt} = \left(1 - \partial_x^2\right)^{-1} \bigg[&-\epsilon u_t - \partial_x \left(2ku + \frac{a}{m+1} u^{m+1}\right) \\
&+ \frac{1}{n+2} \partial_x^3 \left(u^{n+2}\right) - \frac{2n+1}{2} \partial_x \left(u^n u_x^2\right) - \frac{n}{2} u^{n-1} u_x^3 \bigg]. \qquad (3.12)
\end{aligned}
$$

Applying $(\Lambda^q u_t)\Lambda^q$ to both sides of equation (3.12) for $q \in (0, s-1]$ gives rise to

$$\begin{aligned}
&(1-\epsilon)\int_{\mathbb{R}} (\Lambda^q u_t)^2\,dx + \epsilon\int_{\mathbb{R}} (\Lambda^q u_{xt})^2\,dx \\
&\quad= \int_{\mathbb{R}} (\Lambda^q u_t)\Lambda^{q-2}\biggl[-\epsilon u_t + \frac{1}{n+2}\partial_x^3\bigl(u^{n+2}\bigr) \\
&\qquad - \partial_x\biggl(2ku + \frac{a}{m+1}u^{m+1}\biggr) - \frac{2n+1}{2}\partial_x\bigl(u^n u_x^2\bigr) - \frac{n}{2}u^{n-1}u_x^3\biggr]\,dx.
\end{aligned} \tag{3.13}$$

For the right-hand of equation (3.13), we have

$$\int_{\mathbb{R}} (\Lambda^q u_t)\Lambda^{q-2}(-\epsilon u_t - 2ku_x)\,dx \le \epsilon\|u_t\|_{H^q}^2 + 2k\|u_t\|_{H^q}\|u\|_{H^q}, \tag{3.14}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}} (\Lambda^q u_t)\bigl(1-\partial_x^2\bigr)^{-1}\Lambda^q\partial_x\biggl(-\frac{a}{m+1}u^{m+1} - \frac{2n+1}{2}u^n u_x^2\biggr)\,dx \\
&\quad\le c\|u_t\|_{H^q}\biggl\{\int_{\mathbb{R}}\bigl(1+\xi^2\bigr)^{q-1}\biggl[\int_{\mathbb{R}}\biggl(-\frac{a}{m+1}\widehat{u^m}(\xi-\eta)\widehat{u}(\eta) \\
&\qquad - \frac{2n+1}{2}\widehat{u^n u_x}(\xi-\eta)\widehat{u_x}(\eta)\biggr)\,d\eta\biggr]^2\biggr\}^{\frac{1}{2}} \\
&\quad\le c\|u_t\|_{H^q}\|u\|_{H^1}\|u\|_{H^{q+1}}\bigl(\|u\|_{L^\infty}^{m-1} + \|u\|_{L^\infty}^n\bigr).
\end{aligned} \tag{3.15}$$

Since

$$\begin{aligned}
&\int_{\mathbb{R}} (\Lambda^q u_t)\bigl(1-\partial_x^2\bigr)^{-1}\Lambda^q\partial_x^2\bigl(u^{n+1}u_x^2\bigr)\,dx \\
&\quad= -\int_{\mathbb{R}} (\Lambda^q u_t)\Lambda^q\bigl(u^{n+1}u_x\bigr)\,dx + \int_{\mathbb{R}} (\Lambda^q u_t)\bigl(1-\partial_x^2\bigr)^{-1}\Lambda^q\bigl(u^{n+1}u_x^2\bigr)\,dx.
\end{aligned} \tag{3.16}$$

Using Lemma 3.3, $\|u^n u_x\|_{H^q} \le c\|(u^{n+1})x\|_{H^q} \le c\|u\|_{L^\infty}^n\|u\|_{H^{q+1}}$ and $\|u\|_{L^\infty} \le c\|u\|_{H^1}$, we have

$$\begin{aligned}
\int_{\mathbb{R}} (\Lambda^q u_t)\Lambda^q\bigl(u^{n+1}u_x\bigr)\,dx &\le c\|u_t\|_{H^q}\bigl\|u^{n+1}u_x\bigr\|_{H^q} \\
&\le c\|u_t\|_{H^q}\|u\|_{L^\infty}^n\|u\|_{H^{q+1}}\|u\|_{H^1},
\end{aligned} \tag{3.17}$$

and

$$\int_{\mathbb{R}} (\Lambda^q u_t)\bigl(1-\partial_x^2\bigr)^{-1}\Lambda^q\bigl(u^{n+1}u_x^2\bigr)\,dx \le c\|u_t\|_{H^q}\|u\|_{L^\infty}^n\|u\|_{H^{q+1}}\|u\|_{H^1}. \tag{3.18}$$

By the Cauchy-Schwartz inequality and Lemma 3.3, we get

$$\int_{\mathbb{R}} (\Lambda^q u_t)\bigl(1-\partial_x^2\bigr)^{-1}\Lambda^q\bigl(u^{n-1}u_x^3\bigr)\,dx \le c\|u_t\|_{H^q}\|u_x\|_{L^\infty}^2\|u\|_{L^\infty}^{n-1}\|u\|_{H^{q+1}}. \tag{3.19}$$

Substituting equations (3.14)-(3.19) into equation (3.13) yields the inequality

$$(1-2\epsilon)\|u_t\|_{H^q} \le c\|u\|_{H^{q+1}}\bigl(1 + \bigl(\|u\|_{L^\infty}^{m-1} + \|u\|_{L^\infty}^n\bigr)\|u\|_{H^1} + \|u\|_{L^\infty}^{n-1}\|u_x\|_{L^\infty}^2\bigr) \tag{3.20}$$

with a constant $c > 0$. This completes the proof of Theorem 3.1. □

For a real number s with $s > 0$, suppose that the function $u_0(x)$ is in $H^s(\mathbb{R})$, and let $u_{\epsilon 0}$ be the convolution $u_{\epsilon 0} = \phi_\epsilon u_0$ of the function $\phi_\epsilon(x) = \epsilon^{-\frac{1}{4}}\phi(\epsilon^{-\frac{1}{4}}x)$ and u_0 be such that the Fourier transform $\hat{\phi}$ of ϕ satisfies $\widehat{\phi} \in C_0^\infty$, $\widehat{\phi(\xi)} \geq 0$ and $\widehat{\phi(\xi)} = 1$ for any $\xi \in (-1, 1)$. Thus we have $u_{\epsilon 0}(x) \in C^\infty$. It follows from Theorem 3.1 that for each ϵ satisfying $0 < \epsilon < \frac{1}{4}$, the Cauchy problem

$$\begin{cases} u_t - u_{xxt} + \epsilon u_{xxxt} = \partial_x(-2ku - \frac{a}{m+1}u^{m+1}) + \frac{1}{n+2}\partial_x^3(u^{n+2}) \\ \qquad\qquad - \frac{2n+1}{2}\partial_x(u^n u_x^2) - \frac{n}{2}u^{n-1}u_x^3, \\ u(x,0) = u_{\epsilon 0}(x) \end{cases} \tag{3.21}$$

has a unique solution $u_\epsilon \in C^\infty([0, T_\epsilon), H^\infty(\mathbb{R}))$, in which T_ϵ may depend on ϵ.

For an arbitrary positive Sobolev exponent $s > 0$, we give the following lemma.

Lemma 3.5 *For $u_0 \in H^s(\mathbb{R})$ with $s > 0$ and $u_{\epsilon 0} = \phi_\epsilon \star u_0$, the following estimates hold for any ϵ with $0 < \epsilon < \frac{1}{4}$:*

$$\|u_{\epsilon 0x}\|_{L^\infty} \leq c\|u_{0x}\|_{L^\infty}, \quad \text{if } q \leq s, \tag{3.22}$$

$$\|u_{\epsilon 0}\|_{H^q} \leq c\epsilon^{\frac{s-q}{4}}, \quad \text{if } q > s, \tag{3.23}$$

$$\|u_{\epsilon 0} - u_0\|_{H^q} \leq c\epsilon^{\frac{s-q}{4}}, \quad \text{if } q \leq s, \tag{3.24}$$

$$\|u_{\epsilon 0} - u_0\|_{H^s} = o(1), \tag{3.25}$$

where c is a constant independent of ϵ.

Proof This proof is similar to that of Lemma 5 in [60] and Lemma 4.5 in [61], we omit it here. □

Remark 3.1 For $s \geq 1$, using $\|u_\epsilon\|_{L^\infty} \leq c\|u_\epsilon\|_{H^1}$, $\|u_\epsilon\|_{H^1}^2 \leq c\int_{\mathbb{R}}(u_\epsilon^2 + u_{\epsilon x}^2)\,dx$, equations (3.4), (3.22), and (3.23), we obtain

$$\begin{aligned} \|u_\epsilon\|_{L^\infty}^2 &\leq c\|u_\epsilon\|_{H^1} \leq c\int_{\mathbb{R}}\left(u_{\epsilon 0}^2 + u_{\epsilon 0x}^2 + u_{\epsilon 0xx}^2\right)dx \\ &\leq c\left(\|u_{\epsilon 0}\|_{H^1}^2 + \epsilon\|u_{\epsilon 0}\|_{H^2}^2\right) \leq c\left(c + c\epsilon \times \epsilon^{\frac{s-2}{2}}\right) \leq c_0, \end{aligned} \tag{3.26}$$

where c_0 is independent of ϵ.

Theorem 3.2 *If $u_0(x) \in H^s(\mathbb{R})$ with $s \in [1, \frac{3}{2}]$ such that $\|u_{0x}\|_{L^\infty(\mathbb{R})} < \infty$. Let $u_{\epsilon 0}$ be defined as in the system* (3.21). *Then there exist two constants, c and $T > 0$, which are independent of ϵ, such that u_ϵ of problem* (3.21) *satisfies $\|u_{\epsilon x}\|_{L^\infty(\mathbb{R})} \leq c$ for any $t \in [0, T)$.*

Proof Using the notation $u = u_\epsilon$ and differentiating equation (3.21) or equation (3.12) with respect to x give rise to

$$\begin{aligned} &(1-\epsilon)u_{xt} - \epsilon u_{xxxt} - \frac{2n+1}{2}u^n u_x^2 + \frac{1}{n+2}\partial_x^2 u^{n+2} \\ &\quad = 2ku + \frac{a}{m+1}u^{m+1} - \frac{1}{n+2}u^{n+2} \end{aligned}$$

$$
\begin{aligned}
&- \left(1-\partial_x^2\right)^{-1}\left[\epsilon u_{xt} + 2ku + \frac{a}{m+1}u^{m+1} - \frac{1}{n+2}u^{n+2}\right.\\
&\left. + \frac{2n+1}{2}u^n u_x^2 + \frac{n}{2}\partial_x\left(u^{n-1}u_x^3\right)\right].
\end{aligned}
$$

Letting $p > 0$ be an integer and multiplying the above equality by $(u_x)^{2p+1}$, then integrating the resulting equation with respect to x, and using

$$
\begin{aligned}
\frac{1}{n+2}\int_{\mathbb{R}} \partial_x^2\left(u^{n+2}\right)(u_x)^{2p+1}\,dx &= \int_{\mathbb{R}} \left((n+1)u^n u_x^2 + u^{n+1}u_{xx}\right)(u_x)^{2p+1}\,dx \\
&= (n+1)\int_{\mathbb{R}} u^n u_x^{2p+3}\,dx + \frac{1}{2p+2}\int_{\mathbb{R}} u^{n+1}\partial_x(u_x)^{2p+2}\,dx \\
&= \frac{(n+1)(2p+1)}{2p+2}\int_{\mathbb{R}} u^n u_x^{2p+3}\,dx,
\end{aligned}
$$

we find the equality

$$
\begin{aligned}
&\frac{d}{dt}\frac{1-\epsilon}{2p+2}\int_{\mathbb{R}}(u_x)^{2p+2}\,dx - \epsilon\int_{\mathbb{R}}(u_x)^{2p+1}u_{xxxt}\,dx + \frac{p-n}{2p+2}\int_{\mathbb{R}} u^n u_x^{2p+3}\,dx \\
&\quad= \int_{\mathbb{R}}(u_x)^{2p+1}\left(2ku + \frac{a}{m+1}u^{m+1} - \frac{1}{n+2}u^{n+2}\right)dx - \int_{\mathbb{R}}(u_x)^{2p+1}\left(1-\partial_x^2\right)^{-1} \\
&\qquad\cdot\left[\epsilon u_{xt} + 2ku + \frac{a}{m+1}u^{m+1} - \frac{1}{n+2}u^{n+2} + \frac{2n+1}{2}u^n u_x^2 + \frac{n}{2}\partial_x\left(u^{n-1}u_x^3\right)\right]dx. \qquad (3.27)
\end{aligned}
$$

Applying Hölder's inequality, we get

$$
\begin{aligned}
&\frac{1-\epsilon}{2p+2}\frac{d}{dt}\int_{\mathbb{R}}(u_x)^{2p+2}\,dx \\
&\quad\le \left\{\epsilon\left(\int_{\mathbb{R}}|u_{xxxt}|^{2p+2}\,dx\right)^{\frac{1}{2p+2}} + 2|k|\left(\int_{\mathbb{R}}|u|^{2p+2}\,dx\right)^{\frac{1}{2p+2}} + \left(\int_{\mathbb{R}}|G|^{2p+2}\,dx\right)^{\frac{1}{2p+2}}\right. \\
&\qquad\left. + \frac{a}{m+1}\left(\int_{\mathbb{R}}\left|u^{m+1}\right|^{2p+2}\,dx\right)^{\frac{1}{2p+2}} + \frac{1}{n+2}\left(\int_{\mathbb{R}}\left|u^{n+2}\right|^{2p+2}\,dx\right)^{\frac{1}{2p+2}}\right\} \\
&\qquad\cdot\left(\int_{\mathbb{R}}(u_x)^{2p+2}\,dx\right)^{\frac{2p+1}{2p+2}} + \frac{|p-n|}{2p+2}\|u_x\|_{L^\infty}\|u\|_{L^\infty}^n\int_{\mathbb{R}}|u_x|^{2p+2}\,dx,
\end{aligned}
$$

where $G = (1-\partial_x^2)^{-1}[\epsilon u_{xt} + 2ku + \frac{a}{m+1}u^{m+1} - \frac{1}{n+2}u^{n+2} + \frac{2n+1}{2}u^n u_x^2 + \frac{n}{2}\partial_x(u^{n-1}u_x^3)]$. Furthermore

$$
\begin{aligned}
&\frac{1-\epsilon}{2p+2}\frac{d}{dt}\left(\int_{\mathbb{R}}(u_x)^{2p+2}\,dx\right)^{\frac{1}{2p+2}} \\
&\quad\le \epsilon\left(\int_{\mathbb{R}}|u_{xxxt}|^{2p+2}\,dx\right)^{\frac{1}{2p+2}} + 2|k|\left(\int_{\mathbb{R}}|u|^{2p+2}\,dx\right)^{\frac{1}{2p+2}} \\
&\qquad + \frac{a}{m+1}\left(\int_{\mathbb{R}}\left|u^{m+1}\right|^{2p+2}\,dx\right)^{\frac{1}{2p+2}} + \frac{1}{n+2}\left(\int_{\mathbb{R}}\left|u^{n+2}\right|^{2p+2}\,dx\right)^{\frac{1}{2p+2}} \\
&\qquad + \left(\int_{\mathbb{R}}|G|^{2p+2}\,dx\right)^{\frac{1}{2p+2}} + \frac{|p-n|}{2p+2}\|u_x\|_{L^\infty}\|u\|_{L^\infty}^n\left(\int_{\mathbb{R}}(u_x)^{2p+2}\,dx\right)^{\frac{1}{2p+2}}.
\end{aligned}
$$

Since $\|f\|_{L^p} \to \|f\|_{L^\infty}$ as $p \to \infty$ for any $f \in L^\infty \cap L^2$, integrating the above inequality with respect to t and taking the limit as $p \to \infty$ result in the estimate

$$\begin{aligned}(1-\epsilon)\|u_x\|_{L^\infty} \le \|u_{0x}\|_{L^\infty} + \int_0^t \biggl[\epsilon\|u_{xxxt}\|_{L^\infty} + \frac{1}{2}\|u_x\|_{L^\infty}\|u\|_{L^\infty}^n\|u_x\|_{L^\infty}^2 \\ + c\bigl(\|u\|_{L^\infty} + \|u^{m+1}\|_{L^\infty} + \|u^{n+2}\|_{L^\infty} + \|G\|_{L^\infty}\bigr)\biggr]\,d\tau. \qquad (3.28)\end{aligned}$$

Using the algebraic property of $H^s(\mathbb{R})$ with $s > \frac{1}{2}$ and the inequality (3.26) leads to

$$\|u^{n+2}\|_{L^\infty} \le c\|u^{n+2}\|_{H^{\frac{1}{2}+}} \le c\|u^{n+2}\|_{H^1} \le c\|u\|_{H^1}^{n+2} \le c, \qquad (3.29)$$

and

$$\begin{aligned}\|G\|_{L^\infty} &\le c\bigl(\|\Lambda^{-2}u_{xt}\|_{H^{\frac{1}{2}+}} + \|\Lambda^{-2}(u^n u_x^2)\|_{H^{\frac{1}{2}+}} + \|\Lambda^{-2}\partial_x(u^{n-1}u_x^3)\|_{H^{\frac{1}{2}+}}\bigr) + c \\ &\le c\bigl(\|u_t\|_{L^2} + \|u^n u_x^2\|_{H^0} + \|u^{n-1}u_x^3\|_{H^0}\bigr) + c \\ &\le c\bigl(\|u_t\|_{L^2} + \|u^{n-1}u_x\|_{L^\infty}\|u\|_{H^1} + \|u^{n-2}u_x^2\|_{L^\infty}\|u\|_{H^1}\bigr) + c \\ &\le c\bigl(\|u_t\|_{L^2} + \|u_x\|_{L^\infty} + \|u_x\|_{L^\infty}^2\bigr) + c,\end{aligned}$$

where c is a constant independent of ϵ. Using (3.6), (3.29), and the above inequality, we get

$$\int_0^t \|G\|_{L^\infty}\,d\tau \le ct + c\int_0^t \bigl(1 + \|u_x\|_{L^\infty} + \|u_x\|_{L^\infty}^2\bigr)\,d\tau, \qquad (3.30)$$

where c is independent of ϵ. Furthermore, for any fixed $r \in (\frac{1}{2}, 1)$, there exists a constant c_r such that $\|u_{xxxt}\|_{L^\infty} \le c_r\|u_{xxxt}\|_{H^r} \le c_r\|u_t\|_{H^{r+3}}$. By (3.6) and (3.26), one has

$$\|u_{xxxt}\|_{L^\infty} \le c\|u\|_{H^{r+4}}\bigl(1 + \|u_x\|_{L^\infty}^2\bigr). \qquad (3.31)$$

Making use of the Gronwall inequality with equation (3.5), with $q = s + 3$, $u = u_\epsilon$, and equation (3.26), yields

$$\begin{aligned}\|u\|_{H^{r+4}}^2 \le \biggl(\int_{\mathbb{R}} \bigl(\Lambda^{r+4}u_0\bigr)^2 + \epsilon\bigl(\Lambda^{r+3}u_{0xx}\bigr)^2\,dx\biggr) \\ \times \exp\biggl\{c\int_0^t \bigl(\|u_x\|_{L^\infty} + \|u_x\|_{L^\infty}^2\bigr)\,d\tau\biggr\}. \qquad (3.32)\end{aligned}$$

From equations (3.22)-(3.23) and (3.31)-(3.32), we have

$$\|u_{xxxt}\|_{L^\infty} \le c\epsilon^{\frac{s-r-4}{4}}\bigl(1 + \|u_x\|_{L^\infty}^2\bigr)\exp\biggl\{c\int_0^t \bigl(\|u_x\|_{L^\infty} + \|u_x\|_{L^\infty}^2\bigr)\,d\tau\biggr\}. \qquad (3.33)$$

For $\epsilon < \frac{1}{4}$, applying equations (3.28), (3.30), and (3.33), we obtain

$$\|u_x\|_{L^\infty} \le \|u_{0x}\|_{L^\infty} + c\int_0^t \biggl\{\epsilon^{\frac{s-r}{4}}\bigl(1+\|u_x\|_{L^\infty}^2\bigr)\exp\biggl(c\int_0^\tau \bigl(\|u_x\|_{L^\infty}+\|u_x\|_{L^\infty}^2\bigr)\,d\xi\biggr) + 1 + \|u_x\|_{L^\infty} + \|u_x\|_{L^\infty}^2 + \|u_x\|_{L^\infty}^3\biggr\}\,d\tau.$$

It follows from the contraction-mapping principle that there is a $T > 0$ such that the equation

$$\|W\|_{L^\infty} = \|u_{0x}\|_{L^\infty} + c\int_0^t \biggl\{\epsilon^{\frac{s-r}{4}}\bigl(1+\|W\|_{L^\infty}^2\bigr)\exp\biggl(c\int_0^\tau \bigl(\|W\|_{L^\infty}+\|W\|_{L^\infty}^2\bigr)\,d\xi\biggr) + 1 + \|W\|_{L^\infty} + \|W\|_{L^\infty}^2 + \|W\|_{L^\infty}^3\biggr\}\,d\tau$$

has a unique solution $W \in C[0,T]$. From the above inequality, we know that the variable T only depends on c and $\|u_0^m u_{0x}\|_{L^\infty}$. Using the theorem present on p.51 in [28] or Theorem II in Section 1.1 in [62] one derives that there are constants $T > 0$ and $c > 0$ independent of ϵ such that $\|u_x\|_{L^\infty} \le W(t)$ for arbitrary $t \in [0,T]$, which leads to the conclusion of Theorem 3.2. □

Using equations (3.5)-(3.6) in Theorem 3.1 and Theorem 3.2, with the notation $u_\epsilon = u$ and with Gronwall's inequality, results in the inequalities

$$\|u_\epsilon\|_{H^q(\mathbb{R})} \le \|u_\epsilon\|_{H^{q+1}(\mathbb{R})} \le c\exp\biggl\{c\int_0^t \bigl(1+\|u_x\|_{L^\infty(\mathbb{R})}+\|u_x\|_{L^\infty(\mathbb{R})}^2\bigr)\,d\tau\biggr\} \le c,$$

and

$$\|u_{\epsilon t}\|_{H^r(\mathbb{R})} \le c\bigl(1+\|u_x\|_{L^\infty(\mathbb{R})}^2\bigr) \le c,$$

where $q \in (0,s]$, $r \in (0,s-1]$ and $t \in [0,T)$. It follows from Aubin's compactness theorem that there is a subsequence of $\{u_\epsilon\}$, denoted by $\{u_{\epsilon_n}\}$, such that $\{u_{\epsilon_n}\}$ and their temporal derivatives $\{u_{\epsilon_n t}\}$ are weakly convergent to a function $u(x,t)$ and its derivative u_t in $L^2([0,T],H^s)$ and $L^2([0,T],H^{s-1})$, respectively. Moreover, for any real number $R_1 > 0$, $\{u_{\epsilon_n}\}$ is convergent to the function u strongly in the space $L^2([0,T],H^q(-R_1,R_1))$ for $q \in (0,s]$ and $\{u_{\epsilon_n t}\}$ converges to u_t strongly in the space $L^2([0,T],H^r(-R_1,R_1))$ for $r \in (0,s-1]$. Thus, we can prove the existence of a weak solution to equation (1.1).

Proof of Theorem 1.4 From Theorem 3.2, we know that $\{u_{\epsilon_n x}\}(\epsilon_n \to 0)$ is bounded in the space L^∞. Thus, the sequences u_{ϵ_n}, $u_{\epsilon_n x}$ are weakly convergent to u, u_x in the space $L^2([0,T],H^r(-R_1,R_1))$ for any $r \in (0,s-1]$, separately. Hence, u satisfies the equation

$$-\int_0^T\int_{\mathbb{R}} u(g_t - g_{xxt})\,dx\,dt = \int_0^T\int_{\mathbb{R}}\biggl[\biggl(2ku + \frac{a}{m+1}u^{m+1} + \frac{2n+1}{2}u^n u_x^2\biggr)g_x - \frac{1}{n+2}u^{n+2}g_{xxx} - \frac{n}{2}u^{n-1}u_x^3 g\biggr]\,dx\,dt$$

with $u(x,0) = u_0(x)$ and $g \in C_0^\infty$. Since $X = L^1([0,T] \times \mathbb{R})$ is a separable Banach space and $u_{\epsilon_n x}$ is a bounded sequence in the dual space $X^* = L^\infty([0,T] \times \mathbb{R})$ of X, there exists a subsequence of $u_{\epsilon_n x}$, still denoted by $u_{\epsilon_n x}$, weakly star convergent to a function v in $L^\infty([0,T] \times \mathbb{R})$. As $u_{\epsilon_n x}$ weakly converges to u_x in $L^2([0,T] \times \mathbb{R})$, as a result $u_x = v$ almost everywhere. Thus, we obtain $u_x \in L^\infty([0,T] \times \mathbb{R})$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
This paper is the result of joint work of all authors who contributed equally to the final version of this paper. All authors read and approved the final manuscript.

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