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Existence of an unbounded branch of the set of solutions for Neumann problems involving the $p(x)$-Laplacian

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Abstract

We are concerned with the following nonlinear problem:
$-\operatorname{div}(w(x)|\nabla u|^{p(x)-2}\nabla u)+|u|^{p(x)-2}u=\mu g(x)|u|^{p(x)-2}u+f(\lambda,x,u,\nabla u)$ in Ω, $\frac{\partial u}{\partial n}=0$ on $\partial\Omega$, which is subject to a Neumann boundary condition, provided that μ is not an eigenvalue of the $p(x)$-Laplacian. The aim of this paper is to study the structure of the set of solutions for the degenerate $p(x)$-Laplacian Neumann problems by applying a bifurcation result for nonlinear operator equations.

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Keywords: $p(x)$-Laplacian; weighted variable exponent Lebesgue-Sobolev spaces; Neumann boundary condition; eigenvalue

1 Introduction

In recent years, there has been much interest in studying differential equations and variational problems involving $p(x)$-growth conditions since they can model physical phenomena which arise in the study of elastic mechanics, electro-rheological fluid dynamics and image processing, *etc.* We refer the readers to [1–5] and references therein. In the case of $p(x)$ a constant, called the p-Laplacian, there are a lot of papers, for instance, [6–14] and references therein.

In the present paper, we are concerned with the existence of an unbounded branch of the set of solutions for the $p(x)$-Laplacian problem with degeneracy subject to the Neumann boundary condition

$$\begin{cases} -\operatorname{div}(w(x)|\nabla u|^{p(x)-2}\nabla u)+|u|^{p(x)-2}u=\mu g(x)|u|^{p(x)-2}u+f(\lambda,x,u,\nabla u) & \text{in } \Omega,\\ \frac{\partial u}{\partial n}=0 & \text{on } \partial\Omega, \end{cases} \tag{B}$$

when μ is not an eigenvalue of the divergence form

$$\begin{cases} -\operatorname{div}(w(x)|\nabla u|^{p(x)-2}\nabla u)+|u|^{p(x)-2}u=\mu g(x)|u|^{p(x)-2}u & \text{in } \Omega,\\ \frac{\partial u}{\partial n}=0 & \text{on } \partial\Omega, \end{cases} \tag{E}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with the Lipschitz boundary $\partial\Omega$, $\frac{\partial u}{\partial n}$ denotes the outer normal derivative of u with respect to $\partial\Omega$, the variable exponent $p:\overline{\Omega}\to(1,\infty)$ is a continuous function, $g \in L^\infty(\Omega)$, w is a weighted function in Ω and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies a Carathéodory condition.

Since the inceptive study of bifurcation theory by Krasnoselskii [15], Rabinowitz [16] claimed that the bifurcation occurring in the Krasnoselskii theorem is actually a global phenomenon. As regards the p-Laplacian and generalized operators, the nonlinear eigenvalue and bifurcation problems have been widely studied by many researchers in various approaches in the spirit of Rabinowitz [16]; see also [6–9, 13, 17].

The authors in [6, 7] obtained the bifurcation phenomenon for the nonlinear Dirichlet problem which bifurcates from the first eigenvalue of the p-Laplacian. As in [6, 7], Khalil and Ouanan [18] got the result for the nonlinear Neumann problem of the form

$$-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = \lambda m(x)|u|^{p-2}u + f(\lambda, x, u) \quad \text{in } \Omega, \tag{A}$$

which is based on the fact [19] that the first eigenvalue of the p-Laplacian is simple and isolated under suitable conditions on m.

While many researchers considered global branches bifurcating from the first eigenvalue of the p-Laplacian, Väth [20] came at it from another viewpoint to establish the existence of a global branch of solutions for the p-Laplacian with Dirichlet boundary condition by applying nonlinear spectral theory for homogeneous operators. From this point of view, for the case that $p(x)$ is a constant function, the existence of a global branch of solutions for the problem (B) was attained in [11] (for generalization to equations involving nonhomogeneous operators, see also [12]) when μ is not eigenvalue of (E).

Compared to the p-Laplacian equation, an analysis for the $p(x)$-Laplacian equation has to be carried out more carefully because it has complicated nonlinearities (it is nonhomogeneous) and includes a weighted function. As mentioned before, the fact that the principal eigenvalue for nonlinear eigenvalue problems related to the p-Laplacian under either Dirichlet boundary condition or Neumann boundary condition is isolated plays a key role in obtaining the bifurcation result from the principal eigenvalue of the p-Laplacian. However, unlike the p-Laplacian case, under some conditions on $p(x)$, the first eigenvalue for the $p(x)$-Laplacian Neumann problems is not isolated (see [21]), that is, the infimum of all eigenvalues of the problem might be zero (see [22] for Dirichlet boundary condition). Thus we cannot investigate the existence of global branches bifurcating from the principal eigenvalue of the $p(x)$-Laplacian. For this reason, the global behavior of solutions for nonlinear problems involving the $p(x)$-Laplacian had been considered in [23]. To the best of our knowledge, there are no papers concerned with the bifurcation theory for the $p(x)$-Laplacian Neumann problems with weighted functions.

This paper is organized as follows. We first state some basic results for the weighted variable exponent Lebesgue-Sobolev spaces which were given in [23]. Next we give some properties of the corresponding integral operators. Finally we show the existence of a global bifurcation for a Neumann problem involving the $p(x)$-Laplacian by using a bifurcation result in an abstract setting.

2 Preliminaries

In this section, we state some elementary properties for the (weighted) variable exponent Lebesgue-Sobolev spaces which will be used in the next sections. The basic properties of the variable exponent Lebesgue-Sobolev spaces, that is, when $w(x) \equiv 1$ can be found from [24].

To make a self-contained paper, we recall some definitions and basic properties of the weighted variable exponent Lebesgue spaces $L^{p(x)}(w,\Omega)$ and the weighted variable exponent Lebesgue-Sobolev spaces $W^{1,p(x)}(w,\Omega)$.

Set

$$C_+(\overline{\Omega}) = \Big\{h \in C(\overline{\Omega}) : \min_{x\in\overline{\Omega}} h(x) > 1\Big\}.$$

For any $h \in C_+(\overline{\Omega})$ we define

$$h_+ = \sup_{x\in\Omega} h(x) \quad \text{and} \quad h_- = \inf_{x\in\Omega} h(x).$$

Let w is a measurable positive and a.e. finite function in Ω. For any $p \in C_+(\overline{\Omega})$, we introduce the weighted variable exponent Lebesgue space

$$L^{p(x)}(w,\Omega) := \Big\{u : u \text{ is a measurable real-valued function}, \int_\Omega w(x)\big|u(x)\big|^{p(x)}\,dx < \infty\Big\},$$

endowed with the Luxemburg norm

$$\|u\|_{L^{p(x)}(w,\Omega)} = \inf\Big\{\lambda > 0 : \int_\Omega w(x)\left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \le 1\Big\}.$$

The weighted variable exponent Sobolev space $X := W^{1,p(x)}(w,\Omega)$ is defined by

$$X = \big\{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(w,\Omega)\big\},$$

where the norm is

$$\|u\|_X = \|u\|_{L^{p(x)}(\Omega)} + \|\nabla u\|_{L^{p(x)}(w,\Omega)}. \tag{2.1}$$

It is significant that smooth functions are not dense in $W^{1,p(x)}(\Omega)$ without additional assumptions on the exponent $p(x)$. This feature was observed by Zhikov [25] in connection with the Lavrentiev phenomenon. However, if the exponent $p(x)$ is log-Hölder continuous, *i.e.*, there is a constant C such that

$$\big|p(x) - p(y)\big| \le \frac{C}{-\log|x-y|} \tag{2.2}$$

for every $x, y \in \Omega$ with $|x-y| \le 1/2$, then smooth functions are dense in variable exponent Sobolev spaces and there is no confusion in defining the Sobolev space with zero boundary values, $W_0^{1,p(x)}(\Omega)$, as the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_{W^{1,p(x)}(\Omega)}$ (see [26]).

Lemma 2.1 ([24]) *The space $L^{p(x)}(\Omega)$ is a separable, uniformly convex Banach space, and its conjugate space is $L^{p'(x)}(\Omega)$ where $1/p(x) + 1/p'(x) = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have*

$$\left|\int_\Omega uv\,dx\right| \le \left(\frac{1}{p_-} + \frac{1}{(p')_-}\right)\|u\|_{L^{p(x)}(\Omega)}\|v\|_{L^{p'(x)}(\Omega)} \le 2\|u\|_{L^{p(x)}(\Omega)}\|v\|_{L^{p'(x)}(\Omega)}.$$

Lemma 2.2 ([23]) *Denote*

$$\rho(u) = \int_\Omega w(x)|u|^{p(x)}\,dx, \quad \textit{for all } u \in L^{p(x)}(w,\Omega).$$

Then

(1) $\rho(u) > 1$ $(= 1; < 1)$ *if and only if* $\|u\|_{L^{p(x)}(w,\Omega)} > 1$ $(= 1; < 1)$, *respectively*;
(2) *if* $\|u\|_{L^{p(x)}(w,\Omega)} > 1$, *then* $\|u\|^{p_-}_{L^{p(x)}(w,\Omega)} \le \rho(u) \le \|u\|^{p_+}_{L^{p(x)}(w,\Omega)}$;
(3) *if* $\|u\|_{L^{p(x)}(w,\Omega)} < 1$, *then* $\|u\|^{p_+}_{L^{p(x)}(w,\Omega)} \le \rho(u) \le \|u\|^{p_-}_{L^{p(x)}(w,\Omega)}$.

Lemma 2.3 ([27]) *Let* $q \in L^\infty(\Omega)$ *be such that* $1 \le p(x)q(x) \le \infty$ *for almost all* $x \in \Omega$. *If* $u \in L^{q(x)}(\Omega)$ *with* $u \neq 0$, *then*

(1) *if* $\|u\|_{L^{p(x)q(x)}(w,\Omega)} > 1$, *then* $\|u\|^{q_-}_{L^{p(x)q(x)}(w,\Omega)} \le \||u|^{q(x)}\|_{L^{p(x)}(w,\Omega)} \le \|u\|^{q_+}_{L^{p(x)q(x)}(w,\Omega)}$;
(2) *if* $\|u\|_{L^{p(x)q(x)}(w,\Omega)} < 1$, *then* $\|u\|^{q_+}_{L^{p(x)q(x)}(w,\Omega)} \le \||u|^{q(x)}\|_{L^{p(x)}(w,\Omega)} \le \|u\|^{q_-}_{L^{p(x)q(x)}(w,\Omega)}$.

We assume that w is a measurable positive and a.e. finite function in Ω satisfying that
(w1) $w \in L^1_{\mathrm{loc}}(\Omega)$ and $w^{-1/(p(x)-1)} \in L^1_{\mathrm{loc}}(\Omega)$;
(w2) $w^{-s(x)} \in L^1(\Omega)$ with $s(x) \in (\frac{N}{p(x)}, \infty) \cap [\frac{1}{p(x)-1}, \infty)$.
The reasons that we assume (w1) and (w2) can be found in [23].

Lemma 2.4 ([23]) *Let* $p \in C_+(\overline{\Omega})$ *and* (w1) *hold. Then* X *is a reflexive and separable Banach space.*

For $p, s \in C_+(\overline{\Omega})$, let us denote

$$p_s(x) := \frac{p(x)s(x)}{1 + s(x)} < p(x),$$

where $s(x)$ is given in (w2) and

$$p_s^*(x) := \begin{cases} \frac{p(x)s(x)N}{(s(x)+1)N - p(x)s(x)} & \text{if } N > p_s(x), \\ +\infty & \text{if } N \le p_s(x), \end{cases} \tag{2.3}$$

for almost all $x \in \Omega$.

We shall frequently make use of the following (compact) imbedding theorem for the weighted variable exponent Lebesgue-Sobolev space in the next sections.

Lemma 2.5 ([23]) *Let* $\Omega \subset \mathbb{R}^N$ *be an open, bounded set with Lipschitz boundary and* $p \in C_+(\overline{\Omega})$ *with* $1 < p_- \le p_+ < \infty$ *satisfy the log-Hölder continuity condition* (2.2). *If assumptions* (w1) *and* (w2) *hold and* $r \in L^\infty(\Omega)$ *with* $r_- > 1$ *satisfies* $1 < r(x) \le p_s^*(x)$ *for all* $x \in \Omega$, *then we have*

$$X \hookrightarrow L^{r(x)}(\Omega)$$

and the imbedding is compact if $\inf_{x\in\Omega}(p_s^*(x) - r(x)) > 0$.

3 Properties of the integral operators

In this section, we give the definitions and some properties of the integral operators corresponding to the problem (B), by applying the basic properties of the spaces $L^{p(x)}(w,\Omega)$ and X which are given in the previous section.

Throughout this paper, let $p \in C_+(\overline{\Omega})$ satisfy the log-Hölder continuity condition (2.2). We define an operator $J\colon X \to X^*$ by

$$\langle J(u),\varphi\rangle = \int_\Omega \langle w(x)\big|\nabla u(x)\big|^{p(x)-2}\nabla u(x), \nabla\varphi(x)\rangle\,dx + \int_\Omega \big|u(x)\big|^{p(x)-2}u(x)\varphi(x)\,dx, \tag{3.1}$$

for any $\varphi \in X$ where $\langle\cdot,\cdot\rangle$ denotes the pairing of X and its dual X^* and the Euclidean scalar product on $\mathbb{R}^N$, respectively.

The following estimate, which can be found in [17], plays a key role in obtaining the homeomorphism of the operator J.

Lemma 3.1 *For any $u, v \in \mathbb{R}^N$, the following inequalities hold*:

$$\langle |u|^{p-2}u - |v|^{p-2}v, u - v\rangle \geq \begin{cases} (p-1)(|u|+|v|)^{p-2}|u-v|^2 & \text{if } 1 < p < 2 \text{ and } (u,v) \neq (0,0), \\ 4^{1-p}|u-v|^p & \text{if } p \geq 2. \end{cases}$$

From Lemma 3.1, we can obtain the following topological result, which will be needed in the main result. Compared to the case of $p(x)$ being constant (see [11]), the following result is hard to prove because it has complicated nonlinearities.

Theorem 3.2 *Let* (w1) *and* (w2) *be satisfied. The operator $J : X \to X^*$ is homeomorphism onto X^* with a bounded inverse.*

Proof Let $\Psi_1 : X \to L^{p'(x)}(\Omega)$ and $\Psi_2 : X \to L^{p'(x)}(\Omega,\mathbb{R}^N)$ be operators defined by

$$\Psi_1(u)(x) := \big|u(x)\big|^{p(x)-2}u(x) \quad \text{and} \quad \Psi_2(u)(x) := w^{\frac{1}{p'(x)}}(x)\big|\nabla u(x)\big|^{p(x)-2}\nabla u(x).$$

Then the operators Ψ_1, Ψ_2 are bounded and continuous. In fact, for any $u \in X$, let $u_n \to u$ in X as $n \to \infty$. Then there exist a subsequence (u_{n_k}) and functions v, w_j in $L^{p(x)}(w,\Omega)$ for $j = i,\dots,N$ such that $u_{n_k}(x) \to u(x)$ as $k \to \infty$, $|u_{n_k}(x)| \leq v(x)$ and $|(\partial u_{n_k}/\partial x_j)(x)| \leq w_j(x)$ for all $k \in \mathbb{N}$ and for almost all $x \in \Omega$. Without loss of generality, we assume that $\|\Psi_i(u_{n_k}) - \Psi_i(u)\|_{L^{p'(x)}(\Omega)} < 1$ for $i = 1, 2$. Then we have

$$\big\|\Psi_1(u_{n_k}) - \Psi_1(u)\big\|^{(p')_+}_{L^{p'(x)}(\Omega)} \leq \int_\Omega \Big|\big|u_{n_k}(x)\big|^{p(x)-2}u_{n_k}(x) - \big|u(x)\big|^{p(x)-2}u(x)\Big|^{p'(x)}\,dx \tag{3.2}$$

and

$$\begin{aligned} &\big\|\Psi_2(u_{n_k}) - \Psi_2(u)\big\|^{(p')_+}_{L^{p'(x)}(\Omega,\mathbb{R}^N)} \\ &\quad\leq \int_\Omega \Big|w^{\frac{1}{p'(x)}}\big|\nabla u_{n_k}(x)\big|^{p(x)-2}\nabla u_{n_k}(x) - w^{\frac{1}{p'(x)}}\big|\nabla u(x)\big|^{p(x)-2}\nabla u(x)\Big|^{p'(x)}\,dx, \end{aligned} \tag{3.3}$$

and the integrands at the right-hand sides in (3.2) and (3.3) are dominated by some integrable functions. Since $u_{n_k} \to u$ in X as $k \to \infty$, we can deduce that $|u_{n_k}(x)|^{p(x)-2}u_{n_k}(x) \to$

$|u(x)|^{p(x)-2}u(x)$ and $w^{\frac{1}{p'(x)}}(x)|\nabla u_{n_k}(x)|^{p(x)-2}\nabla u_{n_k}(x) \to w^{\frac{1}{p'(x)}}(x)|\nabla u(x)|^{p(x)-2}\nabla u(x)$ as $k \to \infty$ for almost all $x \in \Omega$. Therefore, the Lebesgue dominated convergence theorem tells us that $\Psi_1(u_{n_k}) \to \Psi_1(u)$ in $L^{p'(x)}(\Omega)$ and $\Psi_2(u_{n_k}) \to \Psi_2(u)$ in $L^{p'(x)}(\Omega, \mathbb{R}^N)$ as $k \to \infty$, that is, Ψ_1, Ψ_2 are continuous on X. Also it is easy to show that these operators are bounded on X.

Using the continuity for the operators Ψ_1 and Ψ_2 on X, we finally show that J is continuous on X. From Hölder's inequality, we have

$$\begin{aligned}
&\bigl|\langle J(u_n) - J(u), \varphi\rangle\bigr| \\
&\quad = \left|\int_\Omega \bigl(|u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u\bigr)\varphi\,dx\right| \\
&\qquad + \left|\int_\Omega \bigl\langle w^{\frac{1}{p'(x)}}|\nabla u_n|^{p(x)-2}\nabla u_n - w^{\frac{1}{p'(x)}}|\nabla u|^{p(x)-2}\nabla u, \nabla\varphi\bigr\rangle\,dx\right| \\
&\quad \le 2\bigl\| |u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u\bigr\|_{L^{p'(x)}(\Omega)}\|\varphi\|_{L^{p(x)}(\Omega)} \\
&\qquad + 2\bigl\| w^{\frac{1}{p'}}|\nabla u_n|^{p(x)-2}\nabla u_n - w^{\frac{1}{p'(x)}}|\nabla u|^{p(x)-2}\nabla u\bigr\|_{L^{p'(x)}(\Omega,\mathbb{R}^N)}\|\nabla\varphi\|_{L^{p(x)}(\Omega)}
\end{aligned}$$

for all $\varphi \in X$. Hence we get

$$\begin{aligned}
\bigl\|J(u_n) - J(u)\bigr\|_{X^*} &= \sup_{\|\varphi\|_X \le 1}\bigl|\langle J(u_n) - J(u), \varphi\rangle\bigr| \\
&\le 2\Bigl\{\bigl\| |u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u\bigr\|_{L^{p'(x)}(\Omega)} \\
&\qquad + \bigl\| w^{\frac{1}{p'}}|\nabla u_n|^{p(x)-2}\nabla u_n - w^{\frac{1}{p'(x)}}|\nabla u|^{p(x)-2}\nabla u\bigr\|_{L^{p'(x)}(\Omega,\mathbb{R}^N)}\Bigr\},
\end{aligned} \tag{3.4}$$

and the right-hand side in (3.4) converges to zero as $n \to 0$. Therefore the operator J is continuous on X.

For any u in X with $\|u\|_X > 1$, it follows that

$$\langle J(u), u\rangle \ge C\|u\|_X^{p_-}$$

for some positive constant C. Thus we get

$$\frac{\langle J(u), u\rangle}{\|u\|_X} \to \infty$$

as $\|u\|_X \to \infty$ and therefore the operator J is coercive on X.

Denote

$$\Omega_1 = \bigl\{x \in \Omega : 1 < p(x) < 2\bigr\}, \qquad \Omega_2 = \bigl\{x \in \Omega : p(x) \ge 2\bigr\}.$$

Set

$$p_0 = \inf_{x\in\Omega_1} p(x), \qquad p_1 = \sup_{x\in\Omega_1} p(x)$$

and

$$p_2 = \inf_{x\in\Omega_2} p(x), \qquad p_3 = \sup_{x\in\Omega_2} p(x).$$

(Of course, if both the sets Ω_1 and Ω_2 are nonempty, then $p_1 = p_2 = 2$ by the continuity of $p(x)$.) It is clear that

$$\begin{aligned}\langle J(u) - J(v), u - v\rangle &= \int_\Omega \langle w|\nabla u|^{p(x)-2}\nabla u - w|\nabla v|^{p(x)-2}\nabla v, \nabla u - \nabla v\rangle \,dx \\ &\quad + \int_\Omega \bigl(|u|^{p(x)-2}u - |v|^{p(x)-2}v\bigr)(u - v)\,dx \\ &= \int_{\Omega_1} \langle w|\nabla u|^{p(x)-2}\nabla u - w|\nabla v|^{p(x)-2}\nabla v, \nabla u - \nabla v\rangle \,dx \\ &\quad + \int_{\Omega_1} \bigl(|u|^{p(x)-2}u - |v|^{p(x)-2}v\bigr)(u - v)\,dx \\ &\quad + \int_{\Omega_2} \langle w|\nabla u|^{p(x)-2}\nabla u - w|\nabla v|^{p(x)-2}\nabla v, \nabla u - \nabla v\rangle \,dx \\ &\quad + \int_{\Omega_2} \bigl(|u|^{p(x)-2}u - |v|^{p(x)-2}v\bigr)(u - v)\,dx. \end{aligned} \tag{3.5}$$

By using Lemma 3.1 and (3.5), we find that J is strictly monotone on X. The Browder-Minty theorem hence implies that the inverse operator $J^{-1} : X^* \to X$ exists and is bounded; see Theorem 26.A in [28].

Next we will show that J^{-1} is continuous on X^*. Assume that u and v are any elements in X with $\|u - v\|_X < 1$. According to Lemma 3.1, we have

$$\langle|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v, \nabla u - \nabla v\rangle \ge C_1|\nabla u - \nabla v|^{p(x)}$$

and

$$\langle|u|^{p(x)-2}u - |v|^{p(x)-2}v, u - v\rangle \ge C_2|u - v|^{p(x)}$$

for almost all $x \in \Omega_2$ and for some positive constants C_1 and C_2. Integrating the above inequalities over Ω and using Lemma 2.2, we assert that

$$\begin{aligned}\langle J(u) - J(v), u - v\rangle &= \int_\Omega \langle w|\nabla u|^{p(x)-2}\nabla u - w|\nabla v|^{p(x)-2}\nabla v, \nabla u - \nabla v\rangle \,dx \\ &\quad + \int_\Omega \bigl(|u|^{p(x)-2}u - |v|^{p(x)-2}v\bigr)(u - v)\,dx \\ &\ge C_3\|\nabla u - \nabla v\|^{p_3}_{L^{p(x)}(\Omega_2)} + C_4\|u - v\|^{p_3}_{L^{p(x)}(\Omega_2)} \\ &\ge C_5\bigl(\|\nabla u - \nabla v\|_{L^{p(x)}(\Omega_2)} + \|u - v\|_{L^{p(x)}(\Omega_2)}\bigr)^{p_3} \end{aligned} \tag{3.6}$$

for some positive constants C_3, C_4, and C_5. For almost all $x \in \Omega_1$, the following inequalities hold:

$$m_1^{p(x)-2}|\nabla u - \nabla v|^2 \le |\nabla u - \nabla v|^{p(x)} \tag{3.7}$$

and

$$m_2^{p(x)-2}|u - v|^2 \le |u - v|^{p(x)}, \tag{3.8}$$

where we put $\Omega_0 := \{x \in \Omega_1 : (u(x), v(x)) \neq (0,0)\}$ and use the shortcuts

$$m_1(x) = |\nabla u(x)| + |\nabla v(x)| \quad \text{and} \quad m_2(x) = |u(x)| + |v(x)|.$$

Hence using Lemma 3.1, we assert that

$$\begin{aligned}
&\langle J(u) - J(v), u - v\rangle \\
&\quad = \int_\Omega \langle w|\nabla u|^{p(x)-2}\nabla u - w|\nabla v|^{p(x)-2}\nabla v, \nabla u - \nabla v\rangle \, dx \\
&\qquad + \int_\Omega \big(|u|^{p(x)-2}u - |v|^{p(x)-2}v\big)(u - v)\, dx \\
&\quad \geq C_6 \int_{\Omega_0} m_1^{p(x)-2}|\nabla u - \nabla v|^2 + m_2^{p(x)-2}|u - v|^2 \, dx
\end{aligned}$$

for some positive constant C_6. From Hölder's and Minkowski's inequalities, and the inequality

$$a^{\frac{1}{q'}} r^{\frac{1}{q}} + b^{\frac{1}{q'}} s^{\frac{1}{q}} \geq (a+b)^{\frac{1}{q'}} (r+s)^{\frac{1}{q}} \tag{3.9}$$

for any positive numbers a, b, r, and s, it follows that

$$\begin{aligned}
&\int_{\Omega_0} |\nabla u - \nabla v|^{p(x)}\, dx + \int_{\Omega_0} |u - v|^{p(x)}\, dx \\
&\quad = \int_{\Omega_0} m_1^{\frac{p(x)(2-p(x))}{2}} \big(m_1^{\frac{p(x)(p(x)-2)}{2}} |\nabla u - \nabla v|^{p(x)}\big)\, dx \\
&\qquad + \int_{\Omega_0} m_2^{\frac{p(x)(2-p(x))}{2}} \big(m_2^{\frac{p(x)(p(x)-2)}{2}} |u - v|^{p(x)}\big)\, dx \\
&\quad \leq 2\big\| m_1^{\frac{p(x)(2-p(x))}{2}} \big\|_{L^{\frac{2}{2-p(x)}}(\Omega_1)} \big\| m_1^{\frac{p(x)(p(x)-2)}{2}} |\nabla u - \nabla v|^{p(x)} \big\|_{L^{\frac{2}{p(x)}}(\Omega_1)} \\
&\qquad + 2\big\| m_2^{\frac{p(x)(2-p(x))}{2}} \big\|_{L^{\frac{2}{2-p(x)}}(\Omega_1)} \big\| m_2^{\frac{p(x)(p(x)-2)}{2}} |u - v|^{p(x)} \big\|_{L^{\frac{2}{p(x)}}(\Omega_1)}.
\end{aligned} \tag{3.10}$$

Applying Lemma 2.3 and Minkowski's inequality,

$$\begin{aligned}
\big\| m_1^{\frac{p(x)(2-p(x))}{2}} \big\|_{L^{\frac{2}{2-p(x)}}(\Omega_1)} &\leq \|m_1\|^\alpha_{L^{p(x)}(\Omega_1)} \\
&\leq \big\| |\nabla u| + |\nabla v| \big\|^\alpha_{L^{p(x)}(\Omega_1)} \\
&\leq \big(\|\nabla u\|_{L^{p(x)}(\Omega)} + \|\nabla v\|_{L^{p(x)}(\Omega)}\big)^\alpha
\end{aligned}$$

for any $u, v \in X$ where α is either $p_1(2-p_0)/2$ or $p_0(2-p_1)/2$. In a similar way,

$$\big\| m_2^{\frac{p(x)(2-p(x))}{2}} \big\|_{L^{\frac{2}{2-p(x)}}(\Omega_1)} \leq \big(\|u\|_{L^{p(x)}(\Omega)} + \|v\|_{L^{p(x)}(\Omega)}\big)^\beta$$

for any $u,v \in X$ where β is either $p_1(2-p_0)/2$ or $p_0(2-p_1)/2$. It follows from (3.7)-(3.10) and Lemma 2.2 that

$$
\begin{aligned}
&\big(\|\nabla u-\nabla v\|_{L^{p(x)}(\Omega_1)} + \|u-v\|_{L^{p(x)}(\Omega_1)}\big)^{p_1} \\
&\quad\le 2^{p_1}\left(\int_{\Omega_0} |\nabla u-\nabla v|^{p(x)}\,dx + \int_{\Omega_0} |u-v|^{p(x)}\,dx\right) \\
&\quad\le 2^{p_1}\left(2\big(\|\nabla u\|_{L^{p(x)}(\Omega)} + \|\nabla v\|_{L^{p(x)}(\Omega)}\big)^{\alpha}\left(\int_{\Omega_0} m_1^{p(x)-2}|\nabla u-\nabla v|^2\,dx\right)^{\frac{p_0}{2}}\right. \\
&\qquad\left. + 2\big(\|u\|_{L^{p(x)}(\Omega)} + \|v\|_{L^{p(x)}(\Omega)}\big)^{\beta}\left(\int_{\Omega_0} m_2^{p(x)-2}|u-v|^2\,dx\right)^{\frac{p_0}{2}}\right) \\
&\quad\le 2^{p_1}\Big(\big(\|\nabla u\|_{L^{p(x)}(\Omega)} + \|\nabla v\|_{L^{p(x)}(\Omega)}\big)^{\frac{2\alpha}{2-p_0}} + \big(\|u\|_{L^{p(x)}(\Omega)} + \|v\|_{L^{p(x)}(\Omega)}\big)^{\frac{2\beta}{2-p_0}}\Big)^{\frac{2-p_0}{2}} \\
&\qquad\times\left(\int_{\Omega_0} m_1^{p(x)-2}|\nabla u-\nabla v|^2\,dx + \int_{\Omega_0} m_2^{p(x)-2}|u-v|^2\,dx\right)^{\frac{p_0}{2}} \\
&\quad\le 2^{p_1}\Big(\big(\|u\|_X + \|v\|_X\big)^{\frac{2\alpha}{2-p_0}} + \big(\|u\|_X + \|v\|_X\big)^{\frac{2\beta}{2-p_0}}\Big)^{\frac{2-p_0}{2}} \\
&\qquad\times\left(\int_{\Omega_0} m_1^{p(x)-2}|\nabla u-\nabla v|^2\,dx + \int_{\Omega_0} m_2^{p(x)-2}|u-v|^2\,dx\right)^{\frac{p_0}{2}} \\
&\quad\le C_7\big(\|u\|_X + \|v\|_X\big)^{\gamma}\times\left(\int_{\Omega_0} m_1^{p(x)-2}|\nabla u-\nabla v|^2\,dx + \int_{\Omega_0} m_2^{p(x)-2}|u-v|^2\,dx\right)^{\frac{p_0}{2}},
\end{aligned}
$$

where γ is either $p_1(2-p_0)/2$ or $p_0(2-p_1)/2$ and C_7 is positive constant. So

$$
\begin{aligned}
&\big\langle J(u)-J(v),u-v\big\rangle \\
&\quad\ge C_8\big(\|u\|_X + \|v\|_X\big)^{\frac{-2\gamma}{p_0}}\big(\|\nabla u-\nabla v\|_{L^{p(x)}(\Omega_1)} + \|u-v\|_{L^{p(x)}(\Omega_1)}\big)^{\frac{2p_1}{p_0}}
\end{aligned}
\tag{3.11}
$$

for some positive constant C_8. Consequently, it follows from (3.6) and (3.11) that

$$
\begin{aligned}
&\big\langle J(u)-J(v),u-v\big\rangle \\
&\quad\ge C_9\big(\|u\|_X + \|v\|_X\big)^{\frac{-2\gamma}{p_0}}\big(\|\nabla u-\nabla v\|_{L^{p(x)}(\Omega_1)} + \|u-v\|_{L^{p(x)}(\Omega_1)}\big)^{\frac{2p_1}{p_0}} \\
&\qquad + C_9\big(\|\nabla u-\nabla v\|_{L^{p(x)}(\Omega_2)} + \|u-v\|_{L^{p(x)}(\Omega_2)}\big)^{p_3} \\
&\quad\ge C_{10}\min\Big\{C_9\big(\|u\|_X + \|v\|_X\big)^{\frac{-2\gamma}{p_0}}, C_9\Big\}\big(\|\nabla u-\nabla v\|_{L^{p(x)}(\Omega)} + \|u-v\|_{L^{p(x)}(\Omega)}\big)^{\delta} \\
&\quad= C_{10}\min\Big\{C_9\big(\|u\|_X + \|v\|_X\big)^{\frac{-2\gamma}{p_0}}, C_9\Big\}\|u-v\|_X^{\delta}
\end{aligned}
\tag{3.12}
$$

for some positive constants C_9 and C_{10} where $\delta = \max\{2p_1/p_0, p_3\}$. For each $h \in X^*$, let (h_n) be any sequence in X^* that converges to h in X^*. Set $u_n = J^{-1}(h_n)$ and $u = J^{-1}(h)$ with $\|u_n - u\|_X < 1$. We obtain from (3.12)

$$
\|u_n - u\|_X \le C_{10}^{-\frac{1}{\delta}}\min\Big\{C_9\big(\|u_n\|_X + \|u\|_X\big)^{\frac{-2\gamma}{p_0}}, C_9\Big\}^{-\frac{1}{\delta}}\big\|J(u_n)-J(u)\big\|_{X^*}^{\frac{1}{\delta}}.
$$

Since $\{u_n : n \in \mathbb{N}\}$ is bounded in X and $J(u_n) \to J(u)$ in X^* as $n \to \infty$, it follows that (u_n) converges to u in X. Thus, J^{-1} is continuous at each $h \in X^*$. This completes the proof. □

From now on we deal with the properties for the superposition operator induced by the function f in (B). We assume that the variable exponents are subject to the following restrictions:

$$\begin{cases} q(x) \in (\frac{p(x)s(x)N}{p(x)s(x)N-s(x)N-N+p(x)s(x)}, \infty) & \text{if } N > p_s(x), \\ q(x) \in (1,\infty) \text{ arbitrary} & \text{if } N \le p_s(x) \end{cases}$$

for almost all $x \in \Omega$. Assume that:

(F1) $f : \mathbb{R} \times \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfies the Carathéodory condition in the sense that $f(\lambda, \cdot, u, v)$ is measurable for all $(\lambda, u, v) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N$ and $f(\cdot, x, \cdot, \cdot)$ is continuous for almost all $x \in \Omega$.

(F2) For each bounded interval $I \subset \mathbb{R}$, there are a function $a_I \in L^{q(x)}(\Omega)$ and a nonnegative constant b_I such that

$$\left|f(\lambda, x, u, v)\right| \le a_I(x) + b_I\left(|u|^{\frac{p(x)}{q(x)}} + |v|^{\frac{p_s(x)}{q(x)}}\right)$$

for almost all $x \in \Omega$ and all $(\lambda, u, v) \in I \times \mathbb{R} \times \mathbb{R}^N$.

(F3) f satisfies the following inequality:

$$\left|f(\lambda, x, u, v) - f(\lambda_0, x, u, v)\right| \le C(\lambda, \lambda_0)\left(a_{\lambda,\lambda_0}(x) + |u|^{\frac{p(x)}{q(x)}} + |v|^{\frac{p_s(x)}{q(x)}}\right),$$

where $\|a_{\lambda,\lambda_0}\|_{L^{q(x)}(\Omega)} \le 1$ and $\lim_{\lambda \to \lambda_0} C(\lambda, \lambda_0) = 0$ for each $\lambda_0 \in \mathbb{R}$.

(F4) There exist a function $a \in L^{p'(x)}(\Omega)$ and a locally bounded function $b : [0,\infty) \to \mathbb{R}$ with $\lim_{r\to\infty} b(r)/r = 0$ such that

$$\left|f(0, x, u, v)\right| \le a(x) + \left[b\left(|u| + |v|\right)\right]^{\frac{(p_- -1)s_-}{s_- +1}}$$

for almost all $x \in \Omega$ and all $(u, v) \in \mathbb{R} \times \mathbb{R}^N$.

Under assumptions (F1) and (F2), we can define an operator $F : \mathbb{R} \times X \to X^*$ by

$$\left\langle F(\lambda, u), \varphi\right\rangle = \int_\Omega f\left(\lambda, x, u(x), \nabla u(x)\right)\varphi(x)\,dx \tag{3.13}$$

and an operator $G : X \to X^*$ by

$$\left\langle G(u), \varphi\right\rangle = \int_\Omega g(x)\left|u(x)\right|^{p(x)-2}u(x)\varphi(x)\,dx \tag{3.14}$$

for any $\varphi \in X$.

For our aim, we need some properties of the operators F and G. In contrast with [23], we give a direct proofs for the continuity and compactness of F and G without using a continuity result on superposition operators.

Theorem 3.3 *If* (w1), (w2), *and* (F1)-(F3) *hold, then the operator* $F : \mathbb{R} \times X \to X^*$ *is continuous and compact. Also the operator* $G : X \to X^*$ *is continuous and compact.*

Proof Let $\Psi : \mathbb{R} \times X \to L^{q(x)}(\Omega)$ be an operator defined by

$$\Psi(\lambda, u)(x) := f\big(\lambda, x, u(x), \nabla u(x)\big).$$

Then for fixed $\lambda \in \mathbb{R}$, the operator $\Psi(\lambda, \cdot) : X \to L^{q(x)}(\Omega)$ is bounded and continuous. In fact, for any $u \in X$, let $u_n \to u$ in X as $n \to \infty$. Then there exist a subsequence (u_{n_k}) and functions v, w_j in $L^{q(x)}(\Omega)$ for $j = i, \ldots, N$ such that $u_{n_k}(x) \to u(x)$ and $\nabla u_{n_k}(x) \to \nabla u(x)$ as $k \to \infty$, and $|u_{n_k}(x)| \leq v(x)$ and $|(\partial u_{n_k}/\partial x_j)(x)| \leq w_j(x)$ for all $k \in \mathbb{N}$ and for almost all $x \in \Omega$. Suppose that we can choose $K \in \mathbb{N}$ such that $k \geq K$ implies that $\|\Psi(\lambda, u_{n_k}) - \Psi(\lambda, u)\|_{L^{q(x)}(\Omega)} \leq 1$. For $k \geq K$, we have

$$\big\|\Psi(\lambda, u_{n_k}) - \Psi(\lambda, u)\big\|^{q_+}_{L^{q(x)}(\Omega)} \leq \int_\Omega \big|f\big(\lambda, x, u_{n_k}(x), \nabla u_{n_k}(x)\big) - f\big(\lambda, x, u(x), \nabla u(x)\big)\big|^{q(x)}\, dx$$

and (F2) implies that the integrand at the right-hand side is dominated by an integrable function. Since the function f satisfies a Carathéodory condition, we obtain $f(\lambda, x, u_{n_k}(x), \nabla u_{n_k}(x)) \to f(\lambda, x, u(x), \nabla u(x))$ as $k \to \infty$ for almost all $x \in \Omega$. Therefore, the Lebesgue dominated convergence theorem tells us that $\Psi(\lambda, u_{n_k}) \to \Psi(\lambda, u)$ in $L^{q(x)}(\Omega)$ as $k \to \infty$. We conclude that $\Psi(\lambda, u_n) \to \Psi(\lambda, u)$ in $L^{q(x)}(\Omega)$ as $n \to \infty$ and thus $\Psi(\lambda, \cdot)$ is continuous on X. The boundedness of $\Psi(\lambda, \cdot)$ follows from (F2), Minkowski's inequality, and the imbedding $X \hookrightarrow W^{1,p_s(x)}(w, \Omega)$ continuously (see Theorem 2.11 in [23]) as follows:

$$\begin{aligned}
\big\|\Psi(\lambda, u)\big\|_{L^{q(x)}(\Omega)} &\leq 1 + \|a_I\|_{L^{q(x)}(\Omega)} + \Big\||u|^{\frac{p(x)}{q(x)}} + |\nabla u|^{\frac{p_s(x)}{q(x)}}\Big\|_{L^{q(x)}(\Omega)} \\
&\leq 1 + \|a_I\|_{L^{q(x)}(\Omega)} + \Big\||u|^{\frac{p(x)}{q(x)}}\Big\|_{L^{q(x)}(\Omega)} + \Big\||\nabla u|^{\frac{p_s(x)}{q(x)}}\Big\|_{L^{q(x)}(\Omega)} \\
&\leq 3 + \|a_I\|_{L^{q(x)}(\Omega)} + \|u\|^{\frac{p_+}{q_-}}_{L^{p(x)}(\Omega)} + \|\nabla u\|^{\frac{(p_s)_+}{q_-}}_{L^{p_s(x)}(\Omega)} \\
&\leq 3 + \|a_I\|_{L^{q(x)}(\Omega)} + \|u\|^{\frac{p_+}{q_-}}_X + d\|u\|^{\frac{(p_s)_+}{q_-}}_X
\end{aligned} \tag{3.15}$$

for all $u \in X$ and for some positive constant d.

Minkowski's inequality and (3.12) imply in view of (F3) that

$$\begin{aligned}
\big\|\Psi(\lambda, u) - \Psi(\lambda_0, u)\big\|_{L^{q(x)}(\Omega)} &\leq C(\lambda, \lambda_0)\big(3 + \|a_{\lambda,\lambda_0}\|_{L^{q(x)}(\Omega)} + \|u\|^{\frac{p_+}{q_-}}_X + d\|u\|^{\frac{(p_s)_+}{q_-}}_X\big) \\
&\leq C(\lambda, \lambda_0)\big(4 + \|u\|^{\frac{p_+}{q_-}}_X + d\|u\|^{\frac{(p_s)_+}{q_-}}_X\big)
\end{aligned}$$

for all $\lambda, \lambda_0 \in \mathbb{R}$ and for all $u \in X$. This shows that for any bounded subset $B \subseteq X$, the family $\{\Psi(\cdot, u) : u \in B\}$ is equicontinuous at each $\lambda_0 \in \mathbb{R}$. Hence it follows from the continuity of $\Psi(\lambda_0, \cdot)$ that Ψ is continuous on $\mathbb{R} \times X$, on observing the following relation:

$$\big\|\Psi(\lambda, u) - \Psi(\lambda_0, v)\big\|_{L^{q(x)}(\Omega)} \leq \big\|\Psi(\lambda, u) - \Psi(\lambda_0, u)\big\|_{L^{q(x)}(\Omega)} + \big\|\Psi(\lambda_0, u) - \Psi(\lambda_0, v)\big\|_{L^{q(x)}(\Omega)}.$$

Moreover, Ψ is bounded. Indeed, if $B \subseteq X$ and $\Lambda_0 \subseteq \mathbb{R}$ are bounded, we have to verify that $\Psi(\Lambda_0 \times B)$ is bounded. We may assume that Λ_0 is compact. By the equicontinuity and the compactness of Λ_0, we can find finitely many numbers $\lambda_1, \dots, \lambda_m \in \Lambda_0$ such that for every $\lambda \in \Lambda_0$ there is an integer $k \in \{1, \dots, m\}$ with

$$\left\| \Psi(\lambda, u) - \Psi(\lambda_k, u) \right\|_{L^{q(x)}(\Omega)} \le 1 \quad \text{for all } u \in B.$$

Since $\Psi(\{\lambda_k\} \times B)$ is bounded for each $k \in \{1, \dots, m\}$, Minkowski's inequality hence implies that $\Psi(\Lambda_0 \times B)$ is bounded.

Recall that the embedding $I : X \hookrightarrow L^{q'(x)}(\Omega)$ is continuous and compact (see *e.g.* [8]) and so the adjoint operator $I^* : L^{q(x)}(\Omega) \to X^*$ given by

$$\left(I^* v\right)(u) := \int_\Omega vu \, dx$$

is also compact. As F can be written as a composition of I^* with Ψ, we conclude that F is continuous and compact on $\mathbb{R} \times X$. The operator G is continuous and compact because G can be regarded as a special case of F. This completes the proof. □

The analog of the following result can be found in [23]. However, our growth condition described in assumption (F4) is slightly different from that of [23].

Lemma 3.4 *Let assumptions* (w1), (w2), (F1) *and* (F4) *be fulfilled*. *Then the operator* $F(0, \cdot) : X \to X^*$ *has the following property*:

$$\lim_{\|u\|_X \to \infty} \frac{\|F(0, u)\|_{X^*}}{\|u\|_X^{p_- - 1}} = 0.$$

Proof Let $0 < \varepsilon < 1$. Choose a positive constant R such that $|b(r)| \le \varepsilon r$ for all $r \ge R$. Since b is locally bounded, there is a nonnegative constant C_R such that $|b(r)| \le C_R$ for all $r \in [0, R]$. Let $u \in X$ with $\|u\|_X > 1$. Set $\Omega_R = \{x \in \Omega : |u(x)| + |\nabla u(x)| \le R\}$. Without loss of generality, we may suppose that

$$\int_\Omega b\left(|u| + |\nabla u|\right)^{(p_s)_-} dx > 1 \quad \text{and} \quad \int_\Omega |u|^{(p_s)_-} + |\nabla u|^{(p_s)_-} dx > 1.$$

By assumption (F4), Lemma 2.5 and the continuous imbedding $X \hookrightarrow W^{1,p_s(x)}(\Omega) \hookrightarrow W^{1,(p_s)_-}(\Omega)$, we obtain that

$$\begin{aligned}
&\left\| f\left(0, x, u(x), \nabla u(x)\right) \right\|_{L^{p'(x)}(\Omega)} \\
&\quad \le \left\| a + b\left(|u| + |\nabla u|\right)^{\frac{(p_- - 1)s_-}{s_- + 1}} \right\|_{L^{p'(x)}(\Omega)} \\
&\quad \le \|a\|_{L^{p'(x)}(\Omega)} + \left\| b\left(|u| + |\nabla u|\right)^{\frac{(p_- - 1)s_-}{s_- + 1}} \right\|_{L^{(p')_+}(\Omega)} \\
&\quad \le \|a\|_{L^{p'(x)}(\Omega)} + \left(\int_\Omega \left| b\left(\left|u(x)\right| + \left|\nabla u(x)\right|\right)\right|^{(p_s)_-} dx \right)^{\frac{1}{(p')_+}} \\
&\quad \le \|a\|_{L^{p'(x)}(\Omega)} + \left(\int_{\Omega_R} (C_R)^{(p_s)_-} dx \right)^{\frac{p_- - 1}{p_-}} \\
&\qquad + \left(\int_{\Omega \setminus \Omega_R} \varepsilon^{(p_s)_-} \left(\left|u(x)\right| + \left|\nabla u(x)\right|\right)^{(p_s)_-} dx \right)^{\frac{p_- - 1}{p_-}}
\end{aligned}$$

$$\begin{aligned}
&\leq \|a\|_{L^{p'(x)}(\Omega)} + \left(C_R^{(p_s)_-} \operatorname{meas}(\Omega_R)\right)^{\frac{p_- - 1}{p_-}} \\
&\quad + 2^{\frac{(p_- - 1)s_-}{1+s_-}} \varepsilon^{\frac{(p_- - 1)(p_s)_-}{p_-}} \left(\int_\Omega |u(x)|^{(p_s)_-} + |\nabla u(x)|^{(p_s)_-} \, dx\right)^{\frac{p_- - 1}{p_-}} \\
&\leq \|a\|_{L^{p'(x)}(\Omega)} + \left(C_R^{(p_s)_-} \operatorname{meas}(\Omega_R)\right)^{\frac{p_- - 1}{p_-}} + 2^{\frac{(p_- - 1)s_-}{1+s_-}} \varepsilon^{\frac{(p_- - 1)(p_s)_-}{p_-}} c_1 \|u\|_{W^{1,(p_s)_-}(\Omega)}^{\frac{(p_- - 1)s_-}{1+s_-}} \\
&\leq \|a\|_{L^{p'(x)}(\Omega)} + \left(C_R^{(p_s)_-} \operatorname{meas}(\Omega_R)\right)^{\frac{p_- - 1}{p_-}} + 2^{\frac{(p_- - 1)s_-}{1+s_-}} \varepsilon^{\frac{(p_- - 1)(p_s)_-}{p_-}} c_1 d_1 \|u\|_X^{\frac{(p_- - 1)s_-}{1+s_-}} \\
&\leq \|a\|_{L^{p'(x)}(\Omega)} + \left(C_R^{(p_s)_-} \operatorname{meas}(\Omega_R)\right)^{\frac{p_- - 1}{p_-}} + 2^{\frac{(p_- - 1)s_-}{1+s_-}} \varepsilon^{\frac{(p_- - 1)(p_s)_-}{p_-}} c_1 d_1 \|u\|_X^{p_- - 1},
\end{aligned}$$

where c_1 and d_1 are positive constants. It follows from Hölder's inequality that

$$\begin{aligned}
\left|\langle F(0,u), \varphi\rangle\right| &= \left|\int_\Omega f\left(0, x, u(x), \nabla u(x)\right)\varphi(x)\, dx\right| \\
&\leq 2\left\|f\left(0, x, u(x), \nabla u(x)\right)\right\|_{L^{p'(x)}(\Omega)} \|\varphi\|_{L^{p(x)}(\Omega)} \\
&\leq 2d_2\Big(\|a\|_{L^{p'(x)}(\Omega)} + \left(C_R^{(p_s)_-} \operatorname{meas}(\Omega_R)\right)^{\frac{p_- - 1}{p_-}} \\
&\quad + 2^{\frac{(p_- - 1)s_-}{1+s_-}} \varepsilon^{\frac{(p_- - 1)(p_s)_-}{p_-}} c_1 d_1 \|u\|_X^{p_- - 1}\Big)\|\varphi\|_X
\end{aligned}$$

for all $u, \varphi \in X$ with $\|u\|_X > 1$, where d_2 is a positive constant. Consequently, we get

$$\lim_{\|u\|_X \to \infty} \frac{\|F(0,u)\|_{X^*}}{\|u\|_X^{p_- - 1}} = 0. \qquad \square$$

Recall that a real number μ is called an *eigenvalue of* (E) if the equation

$$J(u) = \mu G(u)$$

has a solution u_0 in X which is different from the origin.

The following lemma is a consequence about nonlinear spectral theory and its proof can be found in [23]. For the case that $p(x)$ is a constant, this assertion has been obtained by using the Furi-Martelli-Vignoli spectrum; see Theorem 4 of [29] or Lemma 27 of [20].

Lemma 3.5 *Suppose that assumptions* (w1) *and* (w2) *are fulfilled. If* μ *is not an eigenvalue of* (E), *then we have*

$$\liminf_{\|u\|_X \to \infty} \frac{\|J(u) - \mu G(u)\|_{X^*}}{\|u\|_X^{p_- - 1}} > 0. \tag{3.16}$$

4 Bifurcation result

In this section, we are ready to prove the main result. We give the definition of weak solutions for our problem.

Definition 4.1 A weak solution of (B) is a pair (λ, u) in $\mathbb{R} \times X$ such that

$$J(u) - \mu G(u) = F(\lambda, u) \quad \text{in } X^*,$$

where J, F and G are defined by (3.1), (3.13) and (3.14), respectively.

The following result, taken from Theorem 2.2 of [20], is a key tool to obtain our bifurcation result.

Lemma 4.2 *Let X be a Banach space and Y be a normed space. Suppose that $J : X \to Y$ is a homeomorphism and $G : X \to Y$ is a continuous and compact operator such that the composition $J^{-1} \circ (-G)$ is odd. Let $F : \mathbb{R} \times X \to Y$ be a continuous and compact operator. If the set*

$$\bigcup_{t\in[0,1]} \{u \in X : J(u) + G(u) = tF(0,u)\}$$

is bounded, then the set

$$\{(\lambda,u) \in \mathbb{R} \times X : J(u) + G(u) = F(\lambda,u)\}$$

has an unbounded connected set $C \subseteq (\mathbb{R} \setminus \{0\}) \times X$ such that $\overline{C}$ intersects $\{0\} \times X$.

Finally we establish the existence of an unbounded branch of the set of solutions for Neumann problem (B) thereby using Lemma 4.2.

Theorem 4.3 *Let conditions* (w1), (w2), *and* (F1)-(F4) *be satisfied. If μ is not an eigenvalue of* (E), *then there is an unbounded connected set $C \subseteq (\mathbb{R} \setminus \{0\}) \times X$ such that every point (λ,u) in C is a weak solution of the above problem* (B) *and $\overline{C}$ intersects* $\{0\} \times X$.

Proof By Theorem 3.2 and Lemma 3.3, $J : X \to X^*$ is a homeomorphism, the operators G and F are continuous and compact, and $J^{-1} \circ (\mu G)$ is odd. Since μ is not an eigenvalue of (E), we get by Lemma 3.5

$$\liminf_{\|u\|_X\to\infty} \frac{\|J(u) - \mu G(u)\|_{X^*}}{\|u\|_X^{p_- -1}} > 0.$$

This together with Lemma 3.4 implies that for some $\beta > 0$, there is a positive constant $R > 1$ such that

$$\left\|J(u) - \mu G(u)\right\|_{X^*} > \beta\|u\|_X^{p_- -1} > \left\|F(0,u)\right\|_{X^*} \geq \left\|tF(0,u)\right\|_{X^*}$$

for all $u \in X$ with $\|u\|_X \geq R$ and for all $t \in [0,1]$. Therefore, the set

$$\bigcup_{t\in[0,1]} \{u \in X : J(u) - \mu G(u) = tF(0,u)\}$$

is bounded. By Lemma 4.2, the set

$$\{(\lambda,u) \in \mathbb{R} \times X : J(u) - \mu G(u) = F(\lambda,u)\}$$

contains an unbounded connected set C which $\overline{C}$ intersects $\{0\} \times X$. This completes the proof. □

In particular the following example illustrates an application of our bifurcation result.

Example 4.4 Suppose that assumptions (w1) and (w2) are fulfilled and $g \in L^\infty(\Omega)$. If μ is not an eigenvalue of (E), then there is an unbounded connected set C such that every point (λ, u) in C is a weak solution of the following nonlinear problem:

$$\begin{cases} -\operatorname{div}(w(x)|\nabla u|^{p(x)-2}\nabla u) + |u|^{p(x)-2}u = \mu g(x)|u|^{p(x)-2}u + \lambda(a(x) + |u|^{\frac{p(x)}{q(x)}-1}u) & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $a \in L^{q(x)}(\Omega)$ and the conjugate function of $q(x)$ is strictly less than $p_s^*(x)$.

Proof Let $f(\lambda, x, u, \nabla u) = \lambda(a(x) + |u|^{p(x)/q(x)-1}u)$. Then it is clear that f satisfies conditions (F1)-(F4). Therefore, the conclusion follows from Theorem 4.3. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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