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A single exponential BKM type estimate for the 3D incompressible ideal MHD equations

Jianli Liu[1], Fenglun Wei[1] and Kejia Pan[2,3*]

*Correspondence: pankejia@hotmail.com
[2]School of Mathematics and Statistics, Central South University, Changsha, 410083, P.R. China
[3]Key Laboratory of Metallogenic Prediction of Nonferrous Metals, Ministry of Education, Central South University, Changsha, 410083, P.R. China
Full list of author information is available at the end of the article

Abstract
In this paper, we give a Beale-Kato-Majda type criterion of strong solutions to the incompressible ideal MHD equations. Instead of double exponential estimates, we get a single exponential bound on $\|(u,h)\|_{H^s}$ $(s>\frac{5}{2})$. It can be applied to a system of an ideal viscoelastic flow.

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1 Introduction

In this paper, we will get the Beale-Kato-Majda type criterion for the breakdown of smooth solutions to the incompressible ideal MHD equations in $\mathbb{R}^3$ as follows:

$$\begin{cases} u_t + u\cdot\nabla u + \nabla(p+\frac{1}{2}|h|^2) = h\cdot\nabla h, \\ h_t + u\cdot\nabla h = h\cdot\nabla u, \\ \nabla\cdot u = 0, \qquad \nabla\cdot h = 0, \\ t=0: \quad u=u_0, \qquad h=h_0, \end{cases} \tag{1}$$

where $x\in\mathbb{R}^3$, $t\geq 0$, u is the flow velocity, h is the magnetic field, p is the pressure, while u_0 and h_0 are, respectively, the given initial velocity and initial magnetic field satisfying $\nabla\cdot u_0 = 0$, $\nabla\cdot h_0 = 0$.

Using the standard energy method [1], it is well known that for $(u_0,h_0)\in H^s(\mathbb{R}^3)$, $s\geq 3$, there exists a $T>0$ such that the Cauchy problem (1) has a unique smooth solution (u,h) on $[0,T]$ satisfying

$$\bigl(u(t,x),h(t,x)\bigr)\in C\bigl([0,T];H^s\bigr)\cap C^1\bigl([0,T];H^{s-1}\bigr). \tag{2}$$

Recently, Caflisch *et al.* [2] extended the well-known result of Beale *et al.* [3] to the 3D ideal MHD equations. More precisely, they showed that if the smooth solution (u,h) satisfies the following condition:

$$\int_0^T \|\nabla\times u\|_{L^\infty}\,dt<\infty \quad\text{and}\quad \int_0^T \|\nabla\times h\|_{L^\infty}\,dt<\infty, \tag{3}$$

then the solution (u,h) can be extended beyond $t=T$, namely, for some $T^\star>T$, $(u,h)\in C([0,T^\star);H^s(\mathbb{R}^3))\cap C^1([0,T^\star);H^{s-1}(\mathbb{R}^3))$. Many authors also considered the blow-up criterion of the ideal MHD equations in other spaces; see [4–6] and references therein. More recently, for the following incompressible Euler equations:

$$\begin{cases} u_t + u \cdot \nabla u + \nabla p = 0, \\ \nabla \cdot u = 0, \\ t = 0: \quad u = u_0, \end{cases} \tag{4}$$

with $\nabla \cdot u_0 = 0$, Chen and Pavlovic [7] showed that if the solution u to (4) satisfies

$$\int_0^T \bigl(\iota_\gamma(\tau)\bigr)^{-\frac{5}{2}}\,d\tau < \infty, \tag{5}$$

where $\iota_\gamma(t) = \min\{L, (\frac{\|\omega(t)\|_{C^\gamma}}{\|u_0\|_{L^2}})^{-\frac{2}{2\gamma+5}}\}$, $\omega = \nabla \times u$ and $\|\omega\|_{C^\gamma} = \sup_{|x-y|<L} \frac{|\omega(x)-\omega(y)|}{|x-y|^\gamma}$, then the solution u can be extended beyond $t = T$. The quantity $\iota_\gamma(t)$ was introduced by Constantin in [8] (see also the work of Constantin *et al.* [9]). For the blow-up criterion of incompressible Euler equations, we refer to [7, 10] and references therein.

2 Main results

In this short note, we develop these ideas further and establish an analogous blow-up criterion for solutions of the 3D ideal MHD equations (1). More precisely, we can get the following theorem.

Theorem 2.1 *Let* (u,h) *be a solution to* (1) *in the class* (2) *for* $s = \frac{5}{2} + \gamma$. *Assume that*

$$\int_0^T \bigl(l_\gamma(\tau)\bigr)^{-\frac{5}{2}}\,d\tau < \infty, \tag{6}$$

where $l_\gamma(t) = \min\{L, (\frac{\|\omega(t)\|_{C^\gamma}}{\|u_0\|_{L^2}})^{-\frac{2}{2\gamma+5}}, (\frac{\|\Omega(t)\|_{C^\gamma}}{\|h_0\|_{L^2}})^{-\frac{2}{2\gamma+5}}\}$, $\omega = \nabla \times u$, $\Omega = \nabla \times h$ *and the definition of* C^γ *as above. Then there exists a finite positive constant* $C_\gamma = O(\gamma^{-1})$ *independent of* (u,h) *and* t *such that*

$$\bigl\|(u,h)\bigr\|_{H^s} \le \bigl\|(u_0,h_0)\bigr\|_{H^s} \exp\biggl\{C_\gamma \bigl\|(u_0,h_0)\bigr\|_{L^2} \int_0^t \bigl(l_\gamma(\tau)\bigr)^{-\frac{5}{2}}\,d\tau\biggr\} \tag{7}$$

holds for $0 \le t \le T$.

Remark 2.1 Using a similar method, we also can get the blow-up criterion result about ideal viscoelastic flow

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = \sum_{k=1}^3 (F_k \cdot \nabla) F_k, \\ \partial_t F_k + u \cdot \nabla F_k = (F_k \cdot \nabla) u, \\ \nabla \cdot u = 0, \qquad \nabla \cdot F_k = 0, \\ t = 0: \quad u = u_0(x), \qquad F_k = F_{k0}(x) \end{cases} \tag{8}$$

with $\nabla \cdot u_0 = 0$, $\nabla \cdot F_{k0} = 0$.

Theorem 2.2 *Let (u,F) be a solution to* (8) *in the class* (2) *for $s = \frac{5}{2} + \gamma$. Assume that $L_\gamma(t)$ is defined as above, and that*

$$\int_0^T \big(L_\gamma(\tau)\big)^{-\frac{5}{2}}\,d\tau < \infty, \tag{9}$$

where $L_\gamma(t) = \min\{L, (\frac{\|(\nabla\times u)(t)\|_{C^\gamma}}{\|u_0\|_{L^2}})^{-\frac{2}{2\gamma+5}}, (\frac{\|(\nabla\times F)(t)\|_{C^\gamma}}{\|F_0\|_{L^2}})^{-\frac{2}{2\gamma+5}}\}$. Then there exists a finite positive constant $C_\gamma = O(\gamma^{-1})$ independent of (u,F) and t such that

$$\big\|(u,F)\big\|_{H^s} \le \big\|(u_0,F_0)\big\|_{H^s}\exp\left\{C_\gamma\big\|(u_0,F_0)\big\|_{L^2}\int_0^t \big(L_\gamma(\tau)\big)^{-\frac{5}{2}}\,d\tau\right\} \tag{10}$$

holds for $0 \le t \le T$.

This system arises in the Oldroyd model for an ideal viscoelastic flow, *i.e.* a viscoelastic fluid whose elastic properties dominate its behavior. Here $F = F(x,t) \in \mathbb{R}^{3\times 3}$ represents the local deformation gradient of the fluid. The blow-up criterion of the ideal viscoelastic system can be found in [11] and references therein.

3 Proof of Theorem 2.1

For the proof of our main result, firstly we give some properties about the gradient of velocity. Recall that the full gradient of the velocity, ∇u, can be decomposed into symmetric and antisymmetric parts,

$$\nabla u = Du^+ + Du^-, \tag{11}$$

where

$$Du^\pm = \frac{1}{2}\big(\nabla u \pm \nabla u^T\big), \tag{12}$$

Du^+ is called the deformation tensor.

In the following lemmas, we recall some important properties of Du^+ and Du^- without proof [7, 8].

Lemma 3.1 *For both the symmetric and the antisymmetric parts Du^+, Du^- of ∇u, the L^2 bound*

$$\big\|Du^\pm\big\|_{L^2} \le C\|\omega\|_{L^2} \tag{13}$$

holds.

The antisymmetric part Du^- satisfies

$$Du^- v = \frac{1}{2}\omega \wedge v \tag{14}$$

for any vector $v \in \mathbb{R}^3$. The vorticity ω satisfies the identity

$$\omega(x) = \frac{1}{4}P.V.\int \sigma(\hat{y})\omega(x+y)\frac{dy}{|y|^3}, \tag{15}$$

(*'P.V.' denotes principal value*) *where* $\sigma(\hat{y}) = 3\hat{y} \otimes \hat{y} - 1$, *with* $\hat{y} = \frac{y}{|y|}$. *Notably,*

$$\int_{S^2} \sigma(\hat{y})\, d\mu_{S^2}(y) = 0, \tag{16}$$

where $d\mu_{S^2}$ *denotes the standard measure on the sphere* S^2.

The matrix components of the symmetric part have the form

$$Du_{ij}^{+} = \sum_{k} T_{ij}^{k}(\omega_k) = \sum_{k} \Gamma_{ij}^{k} * \omega_k, \tag{17}$$

where ω_l *are the vector components of* ω, *and where the integral kernels* Γ_{ij}^{k} *have the properties*

$$\Gamma_{ij}^{k}(y) = \sigma_{ij}^{k}(\hat{y})|y|^{-3}, \tag{18}$$

$$\|\sigma_{ij}^{k}\|_{C^1(S^2)} \le C, \tag{19}$$

$$\int_{S^2} \sigma_{ij}^{k}(\hat{y})\, d\mu_{S^2}(y) = 0. \tag{20}$$

Thus, in particular, T_{ij}^{k} *is a Calderon-Zygmund operator, for every* $i, j, k \in \{1, 2, 3\}$.

We can also give the following useful lemma to provide an upper bound of singular integral operator for the incompressible Euler equations in [7].

Lemma 3.2 *For* $L > 0$ *fixed, and* $\gamma > 0$, *let* $\iota_\gamma(t)$ *be defined as above. Moreover, let* ω_k ($k = 1, 2, 3$) *denote the components of the vorticity vector* $\omega(t)$. *Then any singular integral operator*

$$T\omega_k(x) = \frac{1}{4\pi} P.V. \int \sigma_T(\hat{y})\omega_k(x+y)\frac{dy}{|y|^3}, \tag{21}$$

with

$$\int_{S^2} \sigma_T(\hat{y})\, d\mu_{S^2}(y) = 0, \qquad \|\sigma_T\|_{C^1(S^2)} < C, \tag{22}$$

satisfies

$$\|T\omega_k\|_{L^\infty} \le C\|u_0\|_{L^2}\iota_\gamma^{\frac{5}{2}}(t) \tag{23}$$

for $k \in \{1, 2, 3\}$ *and the constant C independent of u and t.*

Now we are ready to give a proof of Theorem 2.1, which is based on combining an energy estimate for ideal MHD equations with the estimate of $(\|\nabla u\|_{L^\infty} + \|\nabla h\|_{L^\infty})$.

For $s > \frac{5}{2}$, we recall the definitions of the homogeneous and inhomogeneous Besov norms for $1 \le p, q \le \infty$,

$$\|f\|_{\dot{B}^s_{p,q}} = \left(\sum_{j\in\mathbb{Z}} 2^{jqs}\|f_j\|^q_{L^p}\right)^{\frac{1}{q}} \tag{24}$$

and

$$\|f\|_{B^s_{p,q}} = \|f\|_{L^p} + \left(\sum_{j\in\mathbb{Z}} 2^{jqs}\|f_j\|^q_{L^p}\right)^{\frac{1}{q}}, \tag{25}$$

where $f_j = P_j f$ is the Paley-Littlewood projection of f of scale j. We take the $B^s_{2,2}$ Besov norm of $u(t)$ and $h(t)$; then

$$\begin{aligned}&\frac{d}{dt}\left(\|u(t)\|^2_{B^s_{2,2}} + \|h(t)\|^2_{B^s_{2,2}}\right)\\&\quad\le C\left(\|\nabla u\|_\infty + \|\nabla h\|_{L^\infty}\right)\left(\|u(\cdot,t)\|^2_{B^s_{2,2}} + \|h(\cdot,t)\|^2_{B^s_{2,2}}\right).\end{aligned} \tag{26}$$

Therefore,

$$\begin{aligned}&\frac{d}{dt}\left(\|u(t)\|_{B^s_{2,2}} + \|h(t)\|_{B^s_{2,2}}\right)\\&\quad\le C\left(\|\nabla u\|_\infty + \|\nabla h\|_{L^\infty}\right)\left(\|u(\cdot,t)\|_{B^s_{2,2}} + \|h(\cdot,t)\|_{B^s_{2,2}}\right).\end{aligned} \tag{27}$$

However, applying the results of Lemma 3.1 and Lemma 3.2 to u and h, and by the definition of $l_\gamma(t)$, we obtain

$$\begin{aligned}\|\nabla u\|_{L^\infty} + \|\nabla h\|_{L^\infty} &\le \|Du^+\|_{L^\infty} + \|Du^-\|_{L^\infty} + \|Dh^+\|_{L^\infty} + \|Dh^-\|_{L^\infty}\\&\le C_\gamma\left(\|u_0\|_{L^2} + \|h_0\|_{L^2}\right)\left(l_\gamma(t)\right)^{-\frac{5}{2}}.\end{aligned} \tag{28}$$

Therefore, we get

$$\begin{aligned}&\|u(t)\|_{H^s} + \|h(t)\|_{H^s}\\&\quad\simeq \|u(t)\|_{B^s_{2,2}} + \|h(t)\|_{B^s_{2,2}}\\&\quad\le \left(\|u_0\|_{B^s_{2,2}} + \|h_0\|_{B^s_{2,2}}\right)\exp\left\{C\left(\|u_0\|_{L^2} + \|h_0\|_{L^2}\right)\int_0^t l_\gamma(s)^{-\frac{5}{2}}\,ds\right\}\\&\quad\simeq \left(\|u_0\|_{H^s} + \|h_0\|_{H^s}\right)\exp\left\{C\left(\|u_0\|_{L^2} + \|h_0\|_{L^2}\right)\int_0^t l_\gamma(s)^{-\frac{5}{2}}\,ds\right\}\end{aligned}$$

for $s \ge \frac{5}{2} + \gamma$. Thus we complete the proof of Theorem 2.1.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the work was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Shanghai University, Shanghai, 200444, P.R. China. [2]School of Mathematics and Statistics, Central South University, Changsha, 410083, P.R. China. [3]Key Laboratory of Metallogenic Prediction of Nonferrous Metals, Ministry of Education, Central South University, Changsha, 410083, P.R. China.

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