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On the spectral investigation of the scattering problem for some version of one-dimensional Schrödinger equation with turning point

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Abstract
In this paper we introduce and investigate the eigenvalues and the normalizing numbers as well as the scattering function for some version of the one-dimensional Schrödinger equation with turning point on the half line.
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1 Introduction

The solution of many problems of mathematical physics are reduced to the spectral investigation of a differential operator. The differential operator is called regular if its domain is finite and its coefficients are continuous, otherwise it is called a singular differential operator. The Sturm-Liouville theory occupies a central position in the spectral theory of regular operator. During the development of quantum mechanics there was an increase in the interest of spectral theory of singular operators, on which we will restrict our attention. The first basic role in the development of the spectral theory of singular operators dates back to Titchmarsh [1]. He gave a new approach in the spectral theory of singular differential operator of the second order by using contour integration. Also Levitan [2] gave a new method, he obtained the eigenfunction expansion in an infinite interval by taking the limit of a regular case. In the last 35 or so years, due to the needs of mathematical physics, in particular, quantum mechanics, the question of solving various spectral problems with explosive factor has appeared in the study of geophysics and electromagnetic fields; see [3, 4]. The spectral theory of differential operators with explosive factor is studied by Tikhonov [5], Gasymov [6]. For earlier results on various aspects of solvability theory of boundary value problems and spectral theory in the half line case, the situation closely related to the principal topic of this paper, we refer, for instance, to [7–10]. Notice that the paper [11] presented an approximate construction of the Jost function for some Sturm-Liouville boundary value problem in the case $\rho(x) = 1$ by means of the collocation method. In the present paper we introduce and investigate the eigenvalues and the normalizing numbers as well as the scattering function for some version of the one-dimensional Schrödinger equation with turning point on the half line as in (1.1), (1.2). In [12, 13], and [14] the weight functions introduced are considered as applications of the discontinuous wave speed problem on a non-homogeneous medium as in our case, while the introduction of the weight function $\rho(x)$ which is given by (1.3) as $\pm$ signs causes an excess of analytical difficulties. In [15] the author studied the spectral property in a finite interval, while in the present work we consider the half line which gives rise both to a continuous and a discrete spectrum; the latter is treated by the scattering function. In [16] the author considered the weight function of the form

$$\rho(x) = \begin{cases} \alpha^2; & \operatorname{Im}\alpha \neq 0, 0 \leq x \leq a < \pi, \\ 1; & a < x \leq \pi, \end{cases}$$

and the spectra were both continuous and discrete as in our problem. We must notice that the result of this paper is a starting point in calculating the regularized trace formula and solving the inverse scattering problem, which will be investigated later on.

Consider the initial value problem

$$-y'' + q(x)y = \mu\rho(x)y, \quad 0 \leq x < \infty, \tag{1.1}$$

$$y'(0) - hy(0) = 0, \quad h > 0, \tag{1.2}$$

where

$$\rho(x) = \begin{cases} -1; & 0 \leq x \leq 1, \\ 1; & 1 < x < \infty, \end{cases} \tag{1.3}$$

$q(x)$ is a finite real valued function which satisfies

$$\int_0^\infty (1+x)\left|q(x)\right| dx < \infty,$$

and μ is a complex spectral parameter. To study the eigenvalues of (1.1)-(1.2), we first consider the case when $q(x) \equiv 0$ and $h = 0$.

For $q(x) \equiv 0$ and $h = 0$ problem (1.1)-(1.2) takes the form

$$-y'' = \mu\rho(x)y, \quad 0 \leq x \leq \infty, \tag{1.4}$$

$$y'(0) = 0, \tag{1.5}$$

$$\mu = \lambda^2. \tag{1.6}$$

From now on we consider $\operatorname{Im}\lambda \geq 0$ because according to (1.6) μ covers all the complex plane. Denote by $\varphi_o(x,\lambda)$ the solution of (1.4) with the initial conditions $\varphi_o(0,\lambda) = 1$, $\varphi_o'(0,\lambda) = 0$. According to (1.3), (1.4) is equivalent to the two equations

$$\begin{aligned} -y'' &= -\lambda^2 y, \quad 0 \leq x < 1, \\ -y'' &= \lambda^2 y, \quad 1 < x \leq \infty. \end{aligned} \tag{1.7}$$

It is easy to see that

$$\varphi_o(x,\lambda) = \begin{cases} \cosh\lambda x; & 0 \leq x < 1, \\ a_o(\lambda)e^{i\lambda x} + b_o(\lambda)e^{-i\lambda x}; & 1 < x < \infty, \end{cases} \tag{1.8}$$

where $a_o(\lambda)$, $b_o(\lambda)$ are calculated from the requirements $\varphi_o(1-0,\lambda) = \varphi_o(1+0,\lambda)$ and $\varphi_o'(1-0,\lambda) = \varphi_o'(1+0,\lambda)$, so that (1.8) takes the form

$$\varphi_o(x,\lambda) = \begin{cases} \cosh\lambda x; & 0 \le x < 1, \\ \frac{e^{-i\lambda}}{2}(\cosh\lambda - i\sinh\lambda)e^{i\lambda x} + \frac{e^{i\lambda}}{2}(\cosh\lambda + i\sinh\lambda)e^{-i\lambda x}; & 1 < x < \infty. \end{cases} \tag{1.9}$$

For $\operatorname{Im}\lambda = 0$, the function $\varphi_o(x,\lambda)$ does not belong to $L_2(0,\infty)$ also, for $\operatorname{Im}\lambda > 0$, $e^{i\lambda x} \to 0$ as $x \to \infty$ whereas $e^{-i\lambda x} \to \infty$ as $x \to \infty$, so that it is convenient to consider

$$e^{i\lambda}(\cosh\lambda + i\sinh\lambda) = 0 \tag{1.10}$$

as the equation of the eigenvalues $\mu_o = \lambda_o^2$.

From this we have $\lambda_o = (n + \frac{1}{4})\pi i$, $n = 0, 1, \pm1, \pm2, \dots$ or

$$\mu_n^o = -\left(n + \frac{1}{4}\right)^2 \pi^2, \quad n = 1, 2, \dots. \tag{1.11}$$

Together with the solution $\varphi_o(x,\lambda)$ of (1.4) we introduce the second solution $f_o(x,\lambda)$, which is known as the Jost solution. This solution is defined by the condition

$$f_o(x,\lambda) \approx e^{i\lambda x}, \quad x \to \infty. \tag{1.12}$$

With the aid of (1.7), we have

$$f_o(x,\lambda) = \begin{cases} c_o(\lambda)\cosh\lambda x + d_o(\lambda)\sinh\lambda x, & 0 \le x < 1, \\ e^{i\lambda x}, & 1 < x < \infty, \end{cases}$$

where the coefficients $c_o(\lambda)$, $d_o(\lambda)$ are calculated from the requirements $f_o(1-0,\lambda) = f_o(1+0,\lambda)$ and $f_o'(1-0,\lambda) = f_o'(1+0,\lambda)$, and the solution becomes

$$f_o(x,\lambda) = \begin{cases} e^{i\lambda}(\cosh\lambda - i\sinh\lambda)\cosh\lambda x \\ \quad + e^{i\lambda}(i\cosh\lambda - \sinh\lambda)\sinh\lambda x, & 0 \le x < 1, \\ e^{i\lambda x}, & 1 < x < \infty. \end{cases} \tag{1.13}$$

It should be noted, here, that the equation of the eigenvalues can be obtained, also, from the condition that the solution $f_o(x,\lambda) \in L_2(0,\infty;\rho)$; this condition implies that $f_o'(0,\lambda) = 0$, which is the same as (1.10).

Now for $q(x) \neq 0$, $h \neq 0$ we denote by $f(x,\lambda)$ the solution of (1.1) which satisfies the condition

$$f(x,\lambda) \approx e^{i\lambda x}, \quad x \to \infty, -\infty < \lambda < \infty.$$

For $x > 1$, (1.1) takes the form $-y'' + q(x)y = \lambda^2 y$, and in the following, we study its solution and the related spectrum. From [4] this solution has the following representation:

$$f(x,\lambda) = e^{i\lambda x} + \int_x^\infty K(x,t)e^{i\lambda t}\,dt, \quad 1 < x < \infty, \tag{1.14}$$

where $\operatorname{Im}\lambda \ge 0$, $K(x,x) = \frac{1}{2}\int_0^x q(t)\,dt$, $1 < x < \infty$.

For $0 \leq x \leq 1$, the solution $f(x,\lambda)$ has the form

$$f(x,\lambda) = \begin{cases} a(\lambda)\varphi(x,\lambda) + b(\lambda)\theta(x,\lambda), & 0 \leq x \leq 1, \\ e^{i\lambda x} + \int_x^\infty K(x,t)e^{i\lambda t}\,dt, & 1 < x < \infty, \end{cases} \tag{1.15}$$

where $\varphi(x,\lambda)$, $\theta(x,\lambda)$ is the fundamental system of solutions of (1.1) subject to the initial conditions

$$\begin{aligned} &\varphi(0,\lambda) = 1, \qquad \varphi'(0,\lambda) = h, \\ &\theta(0,\lambda) = 0, \qquad \theta'(0,\lambda) = 1, \end{aligned} \tag{1.16}$$

where the coefficients $a(\lambda)$, $b(\lambda)$ are calculated from the requirements $f(1-0,\lambda) = f(1+0,\lambda)$, $f'(1-0,\lambda) = f'(1+0,\lambda)$, from which

$$\begin{aligned} &a(\lambda) = f(1,\lambda)\theta'(1,\lambda) - f'(1,\lambda)\theta(1,\lambda), \\ &b(\lambda) = f'(1,\lambda)\varphi(1,\lambda) - f(1,\lambda)\varphi'(1,\lambda). \end{aligned} \tag{1.17}$$

Further, (1.1), for $0 \leq x \leq 1$, takes the form $-y'' + q(x)y = -\lambda^2 y$, and the fundamental system of solution of this follows from [4, p.18] by the representation

$$\begin{aligned} &\varphi(x,\lambda) = \frac{\sinh\lambda x}{\lambda} + \int_0^x B(x,t)\frac{\sinh\lambda t}{\lambda}\,dt, \\ &A(x,x) = \frac{1}{2}\int_0^x q(t)\,dt, \qquad A(x,0) = 0, \qquad A(0,0) = h, \end{aligned} \tag{1.18}$$

$$\begin{aligned} &\theta(x,\lambda) = \cosh\lambda x + \int_0^x A(x,t)\cosh\lambda t\,dt, \\ &B(x,x) = \frac{1}{2}\int_0^x q(t)\,dt, \qquad \frac{\partial B}{\partial t}\Big|_{t=0} = 0, \qquad B(0,0) = 1. \end{aligned} \tag{1.19}$$

Now we find the characteristic equation of the eigenvalues of (1.1)-(1.2). Since the solution (1.15) belongs to $L_2(0,\infty)$, $\operatorname{Im}\lambda > 0$ it follows that, for $\mu = \lambda^2$ to be an eigenvalue, it must satisfy the initial condition (1.2), namely

$$f'(0,\lambda) - hf(0,\lambda) = 0. \tag{1.20}$$

From (1.15) and (1.16) we have

$$f'(0,\lambda) - hf(0,\lambda) = b(\lambda) = f'(1,\lambda)\varphi(1,\lambda) - f(1,\lambda)\varphi'(1,\lambda). \tag{1.21}$$

In the following lemmas we study some properties of the eigenvalues of problem (1.1)-(1.2).

Lemma 1.1 *Under the conditions* $q(x) > 0$ $(0 < x < \infty)$, *the roots of* (1.20), *for* $\operatorname{Im}\lambda > 0$, *are simple and lie only on the imaginary axis.*

Proof Let λ_o, where $\operatorname{Im}\lambda_o > 0$, be a zero of the function $f'(0,\lambda) - hf(0,\lambda)$, so that

$$f'(0,\lambda_o) - hf(0,\lambda_o) = 0. \tag{1.22}$$

We prove that $\lambda_o = i\tau_o$, $\tau_o > 0$. Since $f(x,\lambda_o)$ is a solution of (1.1) we have

$$-f''(x,\lambda_o) + q(x)f(x,\lambda_o) = \lambda_o^2\rho(x)f(x,\lambda_o), \tag{1.23}$$

multiplying both sides of this by $\overline{f(x,\lambda_o)}$ and integrating both sides from 0 to ∞, we have

$$-\int_0^\infty f''(x,\lambda_o)\overline{f(x,\lambda_o)}\,dx + \int_0^\infty q(x)\left|f(x,\lambda_o)\right|^2 dx = \lambda_o^2\int_0^\infty \rho(x)\left|f(x,\lambda_o)\right|^2 dx.$$

Integrating the first integral by parts and using (1.22), (1.15) we obtain

$$\lambda_o^2 = \frac{\int_0^\infty\{|f'(x,\lambda_o)|^2 + q(x)|f(x,\lambda_o)|^2\}\,dx + h|f(0,\lambda_o)|^2}{\int_0^\infty \rho(x)|f(x,\lambda_o)|^2\,dx}, \tag{1.24}$$

where $\int_0^\infty \rho(x)|f(x,\lambda_o)|^2\,dx \neq 0$, from which we deduce that λ_o^2 is real and hence λ_o is pure imaginary. We turn now to the proof that the roots are simple from (1.22), this is carried out by proving that $f'(0,\lambda) - hf(0,\lambda) = 0$ implies $[\dot{f}'(0,\lambda) - h\dot{f}(0,\lambda)] \neq 0$, where 'dot' denotes differentiation with respect to λ.

Integrating the difference $[\dot{f}(x,\lambda) \times (1.23)] - [f(x,\lambda) \times \frac{d}{d\lambda}(1.23)]$ with respect to x from 0 to ∞ and using (1.20) we get after some calculation that

$$f(0,\lambda)\big[\dot{f}'(0,\lambda) - h\dot{f}(0,\lambda)\big] = 2\lambda\int_0^\infty \rho(x)f^2(x,\lambda)\,dx. \tag{1.25}$$

We prove the reality of $f(x,\lambda)$.

For $x > 1$, $\lambda = i\tau$ the function $f(x,\lambda) = e^{-\tau x} + \int_x^\infty K(x,t)e^{-\tau t}\,dt$ is real because reality of $K(x,t)$ comes from the reality of $q(x)$.

To prove that, for $0 \leq x < 1$, we observe that φ and θ are real. Let $\lambda = i\tau$; since $\varphi(x,\lambda)$ is a solution of (1.1)-(1.2), we have

$$-\varphi''(x,\lambda) + q(x)\varphi(x,\lambda) = -\tau^2\rho(x)\varphi(x,\lambda), \qquad \varphi(0,\lambda) = 1, \qquad \varphi'(0,\lambda) = h. \tag{1.26}$$

Taking the conjugate of (1.26) we have

$$-\overline{\varphi''(x,\lambda)} + q(x)\overline{\varphi(x,\lambda)} = -\tau^2\rho(x)\overline{\varphi(x,\lambda)}, \qquad \overline{\varphi(0,\lambda)} = 1, \qquad \overline{\varphi'(0,\lambda)} = h. \tag{1.27}$$

It is clear, from (1.26) and (1.27), that $\varphi(x,\lambda) = \overline{\varphi(x,\lambda)}$. In a similar way we can prove that $\theta(x,\lambda)$ is also real so that the solution $f(x,\lambda)$ for $0 \leq x < 1$ is real from which we have $f^2(x,\lambda) = |f(x,\lambda)|^2$ and (1.25) takes the form

$$f(0,\lambda)\big[\dot{f}'(0,\lambda) - h\dot{f}(0,\lambda)\big] = 2\lambda\int_0^\infty \rho(x)\left|f(x,\lambda)\right|^2 dx. \tag{1.28}$$

From (1.28) we see that $\frac{d}{d\lambda}[f'(0,\lambda) - hf(0,\lambda)] \neq 0$, which completes the proof. □

Remark 1 For $\operatorname{Im}\lambda_n > 0$ and $f'(0,\lambda_n) - hf(0,\lambda_n) = 0$, the function $f(0,\lambda_n)$ is the eigenfunction of problem (1.1)-(1.2) that corresponds to the negative eigenvalues $\mu_n = \lambda_n^2 = -\chi_n^2$.

Lemma 1.2 *For all* $\operatorname{Re}\lambda \neq 0$ *the function* $f'(0,\lambda) - hf(0,\lambda)$ *does not tend to zero*, *i.e*.

$$f'(0,\lambda) - hf(0,\lambda) \neq 0, \qquad \operatorname{Re}\lambda \neq 0, \quad -\infty < \lambda < \infty. \tag{1.29}$$

Proof Since the function $f(x,\lambda)$ is the solution of (1.1), $f(x,-\lambda)$ is also a solution, and it can be shown that these two solutions are linearly independent and their Wronskian is

$$W\big[f(x,\lambda), f(x,-\lambda)\big] = -2i\lambda, \tag{1.30}$$

so that $W[f(x,\lambda), f(x,-\lambda)] \neq 0$, for $\operatorname{Re}\lambda \neq 0$, so that $f(x,\lambda)$ and $f(x,-\lambda)$ is a fundamental system of solutions of (1.1). In particular, putting $x = 0$ into (1.30) we have

$$f(0,\lambda)f'(0,-\lambda) - f'(0,\lambda)f(0,-\lambda) = -2i\lambda. \tag{1.31}$$

To prove that $f'(0,\lambda) - hf(0,\lambda) \neq 0$, $\operatorname{Re}\lambda \neq 0$, $-\infty < \lambda < \infty$, assume to the contrary *i.e.* $f'(0,\lambda) - hf(0,\lambda) = 0$, $\operatorname{Re}\lambda \neq 0$, $-\infty < \lambda < \infty$. From (1.31) and (1.20) we reach to contradiction to the assumption, and, consequently, we deduce that $f'(0,\lambda) - hf(0,\lambda) \neq 0$, $\operatorname{Re}\lambda \neq 0$, $-\infty < \lambda < \infty$. Notice that $\overline{f(x,\lambda)} = f(x,-\lambda)$. □

Lemma 1.3 *For all* $\operatorname{Re}\lambda \neq 0$ *the following equality holds*:

$$\frac{2i\lambda\varphi(x,\lambda)}{f'(0,\lambda) - hf(0,\lambda)} = f(x,-\lambda) - S(\lambda)f(x,\lambda), \tag{1.32}$$

where $\varphi(x,\lambda)$ *is the solution of problem* (1.1)-(1.2) *and the function*

$$S(\lambda) = \frac{f'(0,-\lambda) - hf(0,-\lambda)}{f'(0,\lambda) - hf(0,\lambda)} \tag{1.33}$$

satisfies the properties

$$\overline{S(\lambda)} = S(-\lambda), \qquad \big|S(\lambda)\big| = 1, \quad -\infty < \lambda < \infty. \tag{1.34}$$

It should be noted here that the function $S(\lambda)$ defined by (1.33) is called the scattering function of problem (1.1)-(1.2) and the function $f'(0,\lambda) - hf(0,\lambda)$ is called the denominator of $S(\lambda)$.

Proof As mentioned before (1.30) for all $\operatorname{Re}\lambda \neq 0$, $f(x,\lambda)$ and $f(x,-\lambda)$ is a fundamental system of solutions of (1.1)-(1.2), so that any linear combination of them is again a solution of (1.1)-(1.2):

$$\varphi(x,\lambda) = A(\lambda)f(x,\lambda) + B(\lambda)f(x,-\lambda), \tag{1.35}$$

where $A(\lambda)$, $B(\lambda)$ are calculated from the initial conditions $\varphi(0,\lambda) = 1$, $\varphi'(0,\lambda) = h$ in the form

$$A(\lambda) = \frac{f'(0,-\lambda) - hf(0,-\lambda)}{-2i\lambda}, \qquad B(\lambda) = \frac{f'(0,\lambda) - hf(0,\lambda)}{-2i\lambda}. \tag{1.36}$$

Substituting (1.36) into (1.35) we arrive at the required formula (1.32). Further, since $\overline{f(x,\lambda)} = f(x,-\lambda)$, it follows from (1.33) that

$$S(\lambda) = \frac{f'(0,-\lambda) - hf(0,-\lambda)}{f'(0,\lambda) - hf(0,\lambda)} = \frac{\overline{f'(0,\lambda) - hf(0,\lambda)}}{f'(0,\lambda) - hf(0,\lambda)},$$

from which we have

$$\left|S(\lambda)\right| = \frac{\overline{|f'(0,\lambda) - hf(0,\lambda)|}}{|f'(0,\lambda) - hf(0,\lambda)|} = 1,$$

and

$$\overline{S(\lambda)} = \overline{\left(\frac{f'(0,-\lambda) - hf(0,-\lambda)}{f'(0,\lambda) - hf(0,\lambda)}\right)} = \frac{f'(0,\lambda) - hf(0,\lambda)}{f'(0,-\lambda) - hf(0,-\lambda)} = S(-\lambda). \qquad \square$$

2 The asymptotic formulas of eigenvalues and normalizing numbers

The eigenvalues $\mu = \lambda^2$ of problem (1.1)-(1.2) are the roots of the equation

$$f'(0,\lambda) - hf(0,\lambda) = 0, \quad \operatorname{Im}\lambda > 0. \tag{2.1}$$

In the following we prove that (2.1) has an infinite number of roots and find their asymptotic formula. From (1.15), (1.17), (1.18), and (1.19) we have

$$f'(0,\lambda) - hf(0,\lambda) = b(\lambda) = f(1,\lambda)\varphi'(1,\lambda) - f'(1,\lambda)\varphi(1,\lambda). \tag{2.2}$$

Now, we calculate the asymptotic formula of $f(1,\lambda)$, $f'(1,\lambda)$, $\varphi(1,\lambda)$ and $\varphi'(1,\lambda)$. Integrating (1.15) by parts we have, for $x \geq 1$, $\operatorname{Im}\lambda > 0$,

$$f(x,\lambda) = e^{i\lambda x} - \frac{K(x,x)}{i\lambda}e^{i\lambda x} - \int_x^\infty \frac{e^{i\lambda x}}{i\lambda}\frac{\partial K(x,x)}{\partial t}\,dt, \tag{2.3}$$

$$f'(x,\lambda) = i\lambda e^{i\lambda x} - K(x,x)e^{i\lambda x} + \int_x^\infty \frac{\partial K(x,x)}{\partial x}e^{i\lambda x}\,dt. \tag{2.4}$$

Similarly from (1.18) we have

$$\varphi(x,\lambda) = \cosh\lambda x + A(x,x)\frac{\sinh\lambda x}{\lambda} - \frac{1}{\lambda}\int_0^x \frac{\partial A(x,t)}{\partial t}\sinh\lambda t\,dt, \tag{2.5}$$

$$\varphi'(x,\lambda) = \lambda\sinh\lambda x + A(x,x)\cosh\lambda x + \int_0^x \frac{\partial A(x,t)}{\partial x}\cosh\lambda t\,dt. \tag{2.6}$$

The following group of inequalities follows from (2.3)-(2.6):

$$f(1,\lambda) = e^{i\lambda} + \mathrm{O}\left(\frac{e^{-\operatorname{Im}\lambda}}{\lambda}\right), \tag{2.7}$$

$$f'(1,\lambda) = i\lambda e^{i\lambda} + \mathrm{O}\left(e^{-\operatorname{Im}\lambda}\right), \tag{2.8}$$

$$\varphi(1,\lambda) = \cosh\lambda + \mathrm{O}\left(\frac{e^{|\operatorname{Re}\lambda|}}{\lambda}\right), \tag{2.9}$$

$$\varphi'(1,\lambda) = \lambda \sinh\lambda + O\bigl(e^{|\operatorname{Re}\lambda|}\bigr). \tag{2.10}$$

Substituting (2.7)-(2.10) into (2.2), we obtain

$$\begin{aligned} &f'(0,\lambda) - hf(0,\lambda) = -i\lambda e^{i\lambda}[\cosh\lambda + i\sinh\lambda] + O\bigl(e^{-\operatorname{Im}\lambda + |\operatorname{Re}\lambda|}\bigr), \\ &\quad \lambda \neq 0, \operatorname{Im}\lambda > 0; \end{aligned} \tag{2.11}$$

comparing (1.10) and (2.11) we see that $f'(0,\lambda) - hf(0,\lambda)$ and $f_o(0,\lambda) = e^{i\lambda}[\cosh\lambda + i\sinh\lambda]$ have the same number of zeros inside the quadratic contour Γ_n where $\{\Gamma_n : |\operatorname{Re}\lambda| \le \pi(n - \frac{1}{4}), 0 < \operatorname{Im}\lambda \le \pi(n - \frac{1}{4})\}$, but since $f_o(0,\lambda)$ has exactly n zeros, namely $\lambda_k^o = i\pi(k - \frac{1}{4})$, $k = 1,2,\ldots,n$, $f'(0,\lambda) - hf(0,\lambda)$ has an infinite number of zeros, as $n \to \infty$, with limiting point at infinity. Denote by λ_n the zeros of $f'(0,\lambda) - hf(0,\lambda) = 0$, so that, by the Rouche theorem, we have

$$\lambda_n = i\left(n + \frac{1}{4}\right) + \varepsilon_n. \tag{2.12}$$

To make (2.12) more accurate, we must refine (2.11). With the aid of Lemma 1.1, λ_n lies on the imaginary axis, so that it is sufficient to know the asymptotic of $f'(0,\lambda) - hf(0,\lambda)$ for small λ. Let $\lambda = i\tau$, $\tau > 0$, we find the asymptotic formula of $f'(0,i\tau) - hf(0,i\tau)$ for $\tau \to \infty$. From (2.3), (2.4), (2.5), and (2.6), we have

$$\begin{aligned} &f(1,i\tau) = e^{-\tau} + K(1,1)\frac{e^{-\tau}}{\tau} + o\left(\frac{e^{-\tau}}{\tau}\right), \\ &7f'(1,i\tau) = -\tau e^{-\tau} - K(1,1)e^{-\tau} + o\bigl(e^{-\tau}\bigr), \\ &\varphi(1,i\tau) = \cos\tau + A(1,1)\frac{\sin\tau}{\tau} + o\left(\frac{1}{\tau}\right), \\ &\varphi'(1,i\tau) = -\tau\sin\tau + A(1,1)\cos\tau + o(1), \end{aligned} \tag{2.13}$$

substituting (2.13) into $f'(0,i\tau) - hf(0,i\tau) = 0$, and putting $\lambda_n = i\tau_n$ we have

$$\cos\tau_n - \sin\tau_n + \frac{A(1,1) + K(1,1)}{\tau_n}(\cos\tau_n + \sin\tau_n) + o\left(\frac{e^{-\tau_n}}{\tau_n}\right) = 0, \tag{2.14}$$

and from this and by virtue of the inequality $|\cos\tau_n| \ge \delta > 0$ $\forall n$, we have

$$\begin{aligned} &1 - \tan\tau_n + \frac{\alpha}{\tau_n} + \frac{\beta}{\tau_n}\tan + o\left(\frac{1}{\tau_n}\right) = 0, \\ &\quad \text{where } \alpha = A(1,1) + K(1,1), \beta = A(1,1) - K(1,1). \end{aligned} \tag{2.15}$$

From (2.12), it is easy to see that

$$\begin{aligned} &\tau_n = \left(n + \frac{1}{4}\right) + \varepsilon_n, \\ &\tan\tau_n = 1 + 2\varepsilon_n + O\left(\frac{1}{n^3}\right), \\ &\frac{1}{\tau_n} = \frac{1}{n\pi} + O\left(\frac{1}{n}\right). \end{aligned} \tag{2.16}$$

The estimation of ε_n follows from (2.15) and (2.16) in the form

$$\varepsilon_n = -1 + \frac{\beta - \alpha}{n\pi} + \mathrm{o}\left(\frac{1}{n}\right). \tag{2.17}$$

Therefore

$$\tau_n = \pi\left(n + \frac{1}{4}\right) - 1 + \frac{c_o}{n\pi} + \mathrm{o}\left(\frac{1}{n}\right), \quad c_o = \frac{1}{\pi}\int_0^1 q(t)\,dt. \tag{2.18}$$

Finally

$$\lambda_n = i\left[\pi\left(n + \frac{1}{4}\right) - 1 + \frac{c_o}{n\pi} + \mathrm{o}\left(\frac{1}{n}\right)\right], \quad c_o = \frac{1}{\pi}\int_0^1 q(t)\,dt. \tag{2.19}$$

Definition (The normalizing numbers) The numbers

$$a_n \stackrel{\text{def}}{=} \int_0^\infty \rho(x)\left|f(x,\lambda_n)\right|^2 dx \tag{2.20}$$

are called the normalizing numbers of problem (1.1)-(1.2) (notice that $f(x,\lambda_n)$ are the eigenfunctions of problem (1.1)-(1.2) corresponding to the eigenvalues λ_n). From (1.28) and the reality of $f(x,\lambda_n)$, we have

$$a_n \stackrel{\text{def}}{=} \int_0^\infty \rho(x)\left|f(x,\lambda_n)\right|^2 dx = -\frac{[\dot{f}'(0,\lambda_n) - h\dot{f}(0,\lambda_n)]f(0,\lambda_n)}{2\lambda_n}. \tag{2.21}$$

To evaluate the asymptotic formula of a_n we evaluate the asymptotic formula of the right hand side of (2.21). From (1.15), (1.17) we have

$$\left[\dot{f}'(0,\lambda_n) - h\dot{f}(0,\lambda_n)\right]f(0,\lambda_n) = \left[\dot{b}(\lambda) + (1+h)\dot{a}(\lambda)\right]a(\lambda), \tag{2.22}$$

where dots and dashes denote the differentiation with respect to λ and x, respectively, $a(\lambda)$ and $b(\lambda)$ are given by (1.17)

$$a(\lambda) = f(1,\lambda)\theta'(1,\lambda) - f'(1,\lambda)\theta(1,\lambda),$$
$$b(\lambda) = f'(1,\lambda)\varphi(1,\lambda) - f(1,\lambda)\varphi'(1,\lambda),$$

from which it follows that

$$\begin{aligned} \dot{a}(\lambda) &= \dot{f}(1,\lambda)\theta'(1,\lambda) + f(1,\lambda)\dot{\theta}'(1,\lambda) - \dot{f}'(1,\lambda)\theta(1,\lambda) - f'(1,\lambda)\dot{\theta}(1,\lambda), \\ \dot{b}(\lambda) &= \dot{f}'(1,\lambda)\varphi(1,\lambda) + f'(1,\lambda)\dot{\varphi}(1,\lambda) - \dot{f}(1,\lambda)\varphi'(1,\lambda) - f(1,\lambda)\dot{\varphi}'(1,\lambda). \end{aligned} \tag{2.23}$$

From (1.18), using integration by parts and then putting $x = 1$, $\lambda = i\tau$, we obtain

$$\begin{aligned}
\varphi(1,i\tau) &= \cos\tau + A(1,1)\frac{\sin\tau}{\tau} + A_t(1,1)\frac{\cos\tau}{\tau^2} + \mathrm{o}\left(\frac{1}{\tau^2}\right),\\
\varphi'(1,i\tau) &= -\tau\sin\tau + A(1,1)\cos\tau + A_t(1,1)\frac{\sin\tau}{\tau} + \mathrm{o}\left(\frac{1}{\tau}\right),\\
\dot{\varphi}(1,i\tau) &= -\sin\tau + A(1,1)\frac{\cos\tau}{\tau} - \left[A(1,1) + A_t(1,1)\right]\frac{\sin\tau}{\tau^2} + \mathrm{o}\left(\frac{1}{\tau^2}\right),\\
\dot{\varphi}'(1,i\tau) &= -\tau\cos\tau - \left[1 + A(1,1)\right]\sin\tau - A_t(1,1)\frac{\cos\tau}{\tau} + A_t(1,1)\frac{\sin\tau}{\tau^2} + \mathrm{o}\left(\frac{1}{\tau^2}\right).
\end{aligned} \tag{2.24}$$

From (1.19), carrying out a similar calculation with respect to θ, we obtain

$$\begin{aligned}
l\theta(1,i\tau) &= \frac{\sin\tau}{\tau} - B(1,1)\frac{\cos\tau}{\tau^2} + \mathrm{o}\left(\frac{1}{\tau^2}\right),\\
\theta'(1,i\tau) &= \cos\tau + B(1,1)\frac{\sin\tau}{\tau} + \mathrm{o}\left(\frac{1}{\tau}\right),\\
\dot{\theta}(1,i\tau) &= \frac{\cos\tau}{\tau} - \left[1 + B(1,1)\right]\frac{\sin\tau}{\tau^2} + 2B(1,1)\frac{\cos\tau}{\tau^3} + \mathrm{o}\left(\frac{1}{\tau^3}\right),\\
\dot{\theta}'(1,i\tau) &= -\sin\tau + B(1,1)\frac{\cos\tau}{\tau} - B(1,1)\frac{\sin\tau}{\tau^2} + \mathrm{o}\left(\frac{1}{\tau^2}\right).
\end{aligned} \tag{2.25}$$

With the aid of (1.15), similar expressions can be calculated with respect to $f(1,i\tau)$:

$$\begin{aligned}
f(1,i\tau) &= e^{-\tau} + K(1,1)\frac{e^{-\tau}}{\tau} + \mathrm{o}\left(\frac{e^{-\tau}}{\tau}\right),\\
f'(1,i\tau) &= -\tau e^{-\tau} - K(1,1)e^{-\tau} + \mathrm{o}\left(e^{-\tau}\right),\\
\dot{f}(1,i\tau) &= -e^{-\tau} - K(1,1)\frac{e^{-\tau}}{\tau} - K(1,1)\frac{e^{-\tau}}{\tau^2} + \mathrm{o}\left(\frac{e^{-\tau}}{\tau^2}\right),\\
\dot{f}'(1,i\tau) &= \tau e^{-\tau} - e^{-\tau} + K(1,1)e^{-\tau} + \mathrm{o}\left(e^{-\tau}\right).
\end{aligned} \tag{2.26}$$

From (2.21) and (2.22), the normalizing numbers a_n can be written in the form

$$a_n = -\frac{[\dot{b}(\lambda) + (1+h)\dot{a}(\lambda)]a(\lambda)}{2\lambda_n}. \tag{2.27}$$

We substitute (2.23), (2.24), (2.25), and (2.26) into (2.27), $\lambda_n = i\tau_n$, and we find

$$\begin{aligned}
a_n = -e^{-2\tau_n}\biggl\{&\cos\tau_n\sin\tau_n + \cos^2\tau_n + \frac{\alpha+\beta}{2\tau_n}\cos\tau_n\sin\tau_n + \frac{\beta}{2\tau_n}\sin^2\tau_n\\
&+ \frac{\alpha}{2\tau_n}\cos^2\tau_n + \frac{\alpha_1}{\tau_n}\cos\tau_n\sin\tau_n + \frac{\beta_1}{\tau_n}\cos^2\tau_n + \mathrm{o}\left(\frac{1}{\tau_n}\right)\biggr\},
\end{aligned} \tag{2.28}$$

where $\alpha = 1 + 2K(1,1) + 2A(1,1)$, $\beta = 1 + 2A(1,1)$, $\alpha_1 = B(1,1) + K(1,1)$, and $\beta_1 = K(1,1) - B(1,1)$. Further, from (2.16) and (2.17) we have

$$\begin{aligned}
\frac{1}{\tau_n} &= \frac{1}{n\pi}\left[1 + \mathrm{o}\left(\frac{1}{\tau_n}\right)\right], \\
\cos^2\tau_n &= \frac{3}{2} - \frac{\beta-\alpha}{n\pi} + \mathrm{o}\left(\frac{1}{\tau_n}\right), \\
\sin^2\tau_n &= \frac{1}{2} - \frac{\beta-\alpha}{n\pi} + \mathrm{o}\left(\frac{1}{\tau_n}\right), \\
\cos\tau_n\sin\tau_n &= \frac{1}{2}\left[1 + \mathrm{o}\left(\frac{1}{\tau_n}\right)\right].
\end{aligned} \tag{2.29}$$

By substituting from (2.29) into (2.28) we obtain the required asymptotic formula for a_n:

$$a_n = -e^{-2\tau_n}\left\{2 + \frac{c_1}{n} + \mathrm{o}\left(\frac{1}{n}\right)\right\}, \tag{2.30}$$

where

$$c_1 = \frac{8h+4}{4\pi} + \frac{3}{8\pi}\int_0^1 q(t)\,dt + \frac{9}{8\pi}\int_1^\infty q(t)\,dt.$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The two authors typed read and approved the final manuscript also they contributed to each part of this work equally.

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