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On exact solution of unsteady MHD flow of a viscous fluid in an orthogonal rheometer

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Abstract

This paper studies the unsteady MHD flow of a viscous fluid in which each point of the parallel planes are subject to the non-torsional oscillations in their own planes. The streamlines at any given time are concentric circles. Exact solutions are obtained and the loci Γ of the centres of these concentric circles are discussed. It is shown that the motion so obtained gives three infinite sets of exact solutions in the geometry of an orthogonal rheometer in which the above non-torsional oscillations are superposed on the disks. These solutions reduce to a single unique solution when symmetric solutions are looked for. Some interesting special cases are also obtained from these solutions.

Keywords: viscous fluid; MHD flow; orthogonal rheometer; eccentric rotation; exact solutions

1 Introduction

Berker [1] has defined the 'pseudo plane motions' of the first kind that: if the streamlines in a plane flow are contained in parallel planes but the velocity components are dependent on the coordinate normal to the planes. Berker [2] has obtained a class of exact solutions to the Navier-Stokes equations belonging to the above type of flows. The ingenious analysis of Berker [3, 4] of the flow of an incompressible viscous fluid confined between two coaxially or non-coaxially rotating infinite parallel disks has shown the existence of an infinite number of exact solutions characterizing the asymmetric and symmetric motions. The symmetric solutions presented by Berker [3] belong to a new class and cannot be obtained as limiting cases of any of the known solutions. It was observed that these results are relevant in understanding the motions of the fluid in a device called an orthogonal rheometer.

Owing to the importance of exact solutions, many authors [5–9] have analyzed the flows in this geometry for different types of fluids. Rao and Kasiviswanathan [10] have investigated the unsteady viscous flow in the orthogonal rheometer and established the existence of three infinite sets of exact solutions. The flow of an elastico-viscous fluid in the above geometry has been investigated by Abbott and Walters [11]. They obtained the unique exact solution of the problem and it was considered as a particular case of the infinite set of solutions which satisfies the condition for symmetric solutions. Thornley [12] has considered an exact solution for the flow of an incompressible fluid confined between two infinite disks when both the fluid and the disk were in rigid-body rotation with constant

angular velocity and one of the disks is used to perform non-torsional oscillations in its own plane in a rotating frame of reference.

The MHD flow of Newtonian and non-Newtonian fluids between two non-coaxial rotating disks has been one of the subjects which has attracted attention of many workers. Kasiviswanathan and Rao [13] investigated the unsteady MHD flow of a viscous, homogeneous, incompressible, electrically conducting fluid between two infinite, parallel, insulated disks rotating with the same angular velocity about two non-coincident axes in which the disks were subjected to non-torsional oscillations. Using the Laplace transform two sets of infinite solutions are presented and these solutions reduce to unique solution when symmetric solutions are looked for. Mohanty [14] obtained an exact solution for the viscous, homogeneous, incompressible, conducting fluid between eccentrically rotating disks having the same angular velocity in the presence of uniform transverse magnetic field. Further, Erkman [15] studied the same problem but for an induced magnetic field. Murthy and Ram [16] examined an exact solution for the case of flow due to non-coaxial rotations of a porous disk and a fluid at infinity under the application of a uniform transverse magnetic field. Rao and Rao [17] also considered the same physical situation as discussed by [14] but they considered the second grade fluid. Ersoy [18] obtained an exact solution of an Oldroyd-B fluid between two infinite, parallel, insulated disks rotating about non-coincident axes in the presence of a uniform transverse magnetic field. Moreover, Rajagopal [19] reviewed problems related to both Newtonian and non-Newtonian fluids between two parallel disks rotating about the same or distinct axes with the same and different angular velocities.

In this paper we have generalised the work of Rao and Kasiviswanathan [10] to the case of unsteady MHD flow of a conducting, incompressible viscous fluid between two infinite, parallel, insulated disks rotating with the same angular velocity about non-coincident axes normal to the disks. Each point of the parallel planes is subjected to non-torsional oscillations in their own planes. The streamlines at a given instant are concentric circles in each plane parallel to a fixed plane π_0 and each point of the plane is performing non-torsional oscillations. Exact solutions are obtained and the loci of the centres of concentric circles are discussed. In the geometry of an orthogonal rheometer where non-torsional oscillations are imposed on the disks relative to the basic rotations, the existence of three infinite sets of exact solutions is established. Three cases arise *i.e.* when: (i) $\omega > \delta$, (ii) $\omega < \delta$ and (iii) $\omega = \delta$. These solutions reduce to a single unique solution when symmetric solutions are looked for.

2 Governing equations

Consider the flow of an incompressible viscous fluid filling the entire three-dimensional space in which the streamlines at a given instant are concentric circles in each plane parallel to a fixed plane π_0 and each point of the plane is performing non-torsional oscillations. A uniform magnetic field B_0 is applied along the z-axis. The unsteady flow of an incompressible fluid in the presence of a magnetic field is governed by the following equations:

$$\rho \frac{D\mathbf{V}}{Dt} = \text{div}\,\mathbf{T} + \mathbf{J} \times \mathbf{B}, \tag{1}$$

$$\nabla \cdot \mathbf{V} = 0, \tag{2}$$

where $\mathbf{V}$ is the velocity field, $\mathbf{T}$ the Cauchy stress tensor, $\mathbf{J}$ the electric current density, ρ the density and D/Dt the material time derivative. In addition, $\mathbf{B}$ ($= \mathbf{B}_0 + \mathbf{b}$) is the total magnetic field in which $\mathbf{b}$ and $\mathbf{B}_0$ denote the induced magnetic field and imposed magnetic field, respectively. In the absence of displacement currents, the Maxwell equations [20, 21] and the modified Ohm law [22] can be written as

$$\left.\begin{array}{c} \nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{B} = \mu_m \mathbf{J}, \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \\ \mathbf{J} = \sigma [\mathbf{E} + \mathbf{V} \times \mathbf{B}], \end{array}\right\} \tag{3}$$

in which $\mathbf{E}$ is the electric field, μ_m the magnetic permeability and σ the electric field conductivity. It is assumed that all the thermo-physical fluid properties are constant throughout the flow field region. Also, the induced magnetic field $\mathbf{b}$ is negligible compared with the applied magnetic field $\mathbf{B}_0$. It is further supposed that no energy is added or extracted from the fluid by the electric field which implies that there is no electric field present in the fluid flow region. Under these assumptions the electromagnetic force $\mathbf{J} \times \mathbf{B}$ takes the form

$$\mathbf{J} \times \mathbf{B} = -\sigma B_0^2 \mathbf{V}. \tag{4}$$

The velocity field is chosen as follows:

$$u = u(x, y, z, t), \qquad v = v(x, y, z, t), \qquad w = 0, \tag{5}$$

where u, v, w are the components of velocity in the Cartesian coordinate system (x, y, z). For this motion, the streamlines for all time t are contained in planes parallel to the fixed plane π_0 and the plane Oxy is parallel to this fixed plane. The constitutive equation of the viscous fluid is

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 + \rho\mathbf{f}, \tag{6}$$

with p being the pressure, $\mathbf{I}$ the unit tensor, $\mathbf{A}_1$ the Rivilin-Ericksen tensor, μ the dynamic viscosity and $\mathbf{f}$ the body force. The Rivilin-Ericksen tensor $\mathbf{A}_1$ is defined by

$$\mathbf{A}_1 = \mathbf{L} + \mathbf{L}^T, \qquad \mathbf{L} = \nabla\mathbf{V}, \tag{7}$$

in which T denotes the transpose. The Navier-Stokes equations governing the flow in view of Eqs. (1) and (4)-(7) are given by

$$\psi_{yt} + \psi_y\psi_{yx} - \psi_x\psi_{yy} = -\phi_x + \nu(\psi_{xx} + \psi_{yy} + \psi_{zz})_y - \frac{\sigma B_0^2 \psi_y}{\rho}, \tag{8}$$

$$\psi_{xt} + \psi_y\psi_{xx} - \psi_x\psi_{yx} = \phi_y + \nu(\psi_{xx} + \psi_{yy} + \psi_{zz})_x - \frac{\sigma B_0^2 \psi_x}{\rho}, \tag{9}$$

$$u = \psi_y, \qquad v = -\psi_x, \tag{10}$$

$$\phi = \frac{p}{\rho} + \Omega, \tag{11}$$

where $\psi(x, y, z, t)$ is the stream function, Ω the potential due to external force and ν the kinematic coefficient of viscosity. The subscript denotes the partial derivatives with respect to corresponding variables throughout the paper. From the equation of motion in the z-direction one finds ϕ to be independent of z. In the present analysis the authors are interested in getting solutions of an unsteady MHD fluid motion in which the streamlines at any instant are concentric circles in each plane with normal line parallel to the Oz-axis. Therefore, consider

$$x = f(z, t), \qquad y = g(z, t), \tag{12}$$

and let these be the loci Γ of the centres of these circles which are fixed points at a given instant t and belong to the instantaneous axis of the vortex. The condition that Γ must not be a straight line perpendicular to the Oxy plane gives the inequality:

$$f_z^2 + g_z^2 \not\cong 0. \tag{13}$$

Following an analysis similar to Berker [4], the general form of the stream function for this type of unsteady MHD flow is given by

$$\psi = K\left[\left(x - f(z, t)\right)^2 + \left(y - g(z, t)\right)^2\right]. \tag{14}$$

This stream function ψ has to satisfy Eqs. (8) and (9). According to Eq. (10), the velocity components are given by the equations:

$$u = 2K\left(y - g(z, t)\right), \qquad v = -2K\left(x - f(z, t)\right). \tag{15}$$

Let c be any constant and let π be the plane $z = c$. Equation (15) shows that at a given time t_0, the fluid particles belonging to the plane π move like the points of a solid plate which rotates around a fixed point with coordinates $(f(c, t_0), g(c, t_0), c)$ and with constant angular velocity ω, given by

$$\omega = -2K. \tag{16}$$

This angular velocity is same for all planes π which are parallel to the plane Oxy. Thus (15) should be interpreted as a motion due to rigid rotation superposed by some unsteady motion.

3 Exact solutions

Eliminating ϕ from (8) and (9) by differentiating with respect to z and making use of (14) and (16) we obtain

$$\nu W_{zzz} - \left(\frac{\sigma B_0^2}{\rho} + i\omega\right) W_z = W_{zt}, \tag{17}$$

where

$$W(z, t) = f(z, t) + ig(z, t). \tag{18}$$

We investigate the unsteady MHD motion generated by subjecting each of the parallel planes to a non-torsional oscillation of the type $e^{i\delta t}$ or $e^{-i\delta t}$, where δ is the frequency of oscillation. Letting

$$W = F(z)e^{-i\delta t}, \tag{19}$$

Eq. (17) takes the form

$$F_{zzz} - \left(N + i\frac{\omega - \delta}{\nu}\right)F_z = 0, \tag{20}$$

where

$$N = \frac{\sigma B_0^2}{\rho \nu}. \tag{21}$$

The general solutions of the differential equation (20) are

$$\left.\begin{array}{l} F(z) = A + B\exp(\sqrt{N + 2i\beta_1^2}\,z) + C\exp(-\sqrt{N + 2i\beta_1^2}\,z), \quad \omega > \delta, \\ F(z) = A + B\exp(\sqrt{N - 2i\beta_2^2}\,z) + C\exp(-\sqrt{N - 2i\beta_2^2}\,z), \quad \omega < \delta, \\ F(z) = A + B\exp(\sqrt{N}\,z) + C\exp(-\sqrt{N}\,z), \quad \omega = \delta, \end{array}\right\} \tag{22}$$

where

$$\left.\begin{array}{l} \beta_1^2 = (\omega - \delta)/2\nu, \\ \beta_2^2 = (\delta - \omega)/2\nu, \end{array}\right\} \tag{23}$$

and the arbitrary complex constants A, B and C are assumed to be

$$A = A_1 + iA_2, \qquad B = B_1 + iB_2, \qquad C = C_1 + iC_2. \tag{24}$$

The solutions for non-torsional oscillation of the type $e^{i\delta t}$ are obtained by replacing δ by $-\delta$ at appropriate places in the above equations. The general solutions of the differential equation (17) in view of Eqs. (18), (19), (22) and (24) are given by

$$\begin{aligned} f = {} & f_0 + (A_1 \cos \delta t + A_2 \sin \delta t) + e^{\beta_{r1}z}\left[B_1 \cos(\beta_{i1}z - \delta t) - B_2 \sin(\beta_{i1}z - \delta t)\right] \\ & + e^{-\beta_{r1}z}\left[C_1 \cos(\beta_{i1}z + \delta t) + C_2 \sin(\beta_{i1}z + \delta t)\right], \end{aligned} \tag{25}$$

$$\begin{aligned} g = {} & g_0 + (A_2 \cos \delta t - A_1 \sin \delta t) + e^{\beta_{r1}z}\left[B_1 \sin(\beta_{i1}z - \delta t) + B_2 \cos(\beta_{i1}z - \delta t)\right] \\ & + e^{-\beta_{r1}z}\left[C_2 \cos(\beta_{i1}z + \delta t) - C_1 \sin(\beta_{i1}z + \delta t)\right], \end{aligned} \tag{26}$$

where f_0 and g_0 denote solutions due to steady rotation and

$$\beta_{r1} + i\beta_{i1} = \sqrt{N + 2i\beta_1^2}. \tag{27}$$

Similarly, expressions for $\omega < \delta$ and $\omega = \delta$ can be obtained. It is worthy to mention that the solutions (25) and (26) reduce to the solutions presented by Rao and Kasiviswanathan [10] when $B_0 = 0$.

Now we will establish that by making a suitable change of axes, that is, the change that satisfies the condition that the plane Oxy be parallel to the plane π_0, it is possible to reduce

the number of arbitrary constants contained in the curve Γ given in Eqs. (25) and (26). The change to be made is the result of transformations described below for $\omega > \delta$.

(i) A translation parallel to the plane Oxy and defined by the equations:

$$x_1 = x - (A_1 \cos \delta t + A_2 \sin \delta t), \qquad y_1 = y - (A_2 \cos \delta t - A_1 \sin \delta t). \tag{28}$$

This transformation gets rid of the constants A_1 and A_2 contained in Eqs. (25) and (26).

(ii) If $B_1^2 + B_2^2 > 0$ and $C_1^2 + C_2^2 > 0$, a translation parallel to the axis Oz and a rotation about this axis. This translation and rotation is so chosen that the equation of the curve Γ in the new axes reduces to the following form:

$$\Gamma_1: \quad \left.\begin{array}{l} x = ax_0 + a(x_B \cos \delta t + y_B \sin \delta t), \\ y = ay_0 + a(y_B \cos \delta t - x_B \sin \delta t), \end{array}\right\} \tag{29}$$

where

$$\begin{aligned} x_0 &= \cosh(\beta_0 z) \cos(\beta_0 z + n), & y_0 &= \sinh(\beta_0 z) \sin(\beta_0 z + n), \\ x_B &= \cosh(\beta_{i1} z) \cos(\beta_{i1} z + n), & y_B &= \sinh(\beta_{i1} z) \sin(\beta_{i1} z + n) \end{aligned} \tag{30}$$

with $\beta_1, a > 0$, $\beta_0 = \sqrt{\omega/2\upsilon} > 0$. Equation (29) contains only two arbitrary real constants a and n.

(iii) If $B_1^2 + B_2^2 > 0$ and $C_1 = C_2 = 0$, a translation parallel to the axis Oz. This translation is so chosen that the equations of the curve Γ with respect to the new axes reduce to the following form:

$$\Gamma_2: \quad \left.\begin{array}{l} x = ax_0 + a\exp(\beta_{r1} z) \cos(\beta_{i1} z - \delta t), \\ y = ay_0 + a\exp(\beta_{r1} z) \sin(\beta_{i1} z + \delta t), \end{array}\right\} \tag{31}$$

where

$$x_0 = \exp(\beta_0 z) \cos(\beta_0 z), \qquad y_0 = \exp(\beta_0 z) \sin(\beta_0 z), \tag{32}$$

with $a > 0$, $\beta_0 \neq 0$, $\beta_1 \neq 0$. Equation (31) involves only one arbitrary real constant a.

(iv) If $C_1^2 + C_2^2 > 0$ and $B_1 = 0 = B_2$, again a translation parallel to the axis Oz. This translation is so chosen that the equations of the curve Γ in the new axes reduce to the following form:

$$\Gamma_3: \quad \left.\begin{array}{l} x = ax_0 - a\exp(-\beta_{r1} z) \cos(\beta_{i1} z + \delta t), \\ y = ay_0 - a\exp(-\beta_{r1} z) \sin(\beta_{i1} z - \delta t), \end{array}\right\} \tag{33}$$

where

$$x_0 = \exp(-\beta_0 z) \cos(\beta_0 z), \qquad y_0 = -\exp(-\beta_0 z) \sin(\beta_0 z), \tag{34}$$

with $a > 0$, $\beta_0 \neq 0$, $\beta_1 \neq 0$. Equation (33) again contains only one constant a. It is worth mentioning that the curves Γ_1-Γ_3 reduce to the curves given by Rao and Kasiviswanathan [10] when $B_0 = 0$.

(v) Lastly, when $\omega = \delta$ the curve Γ by a proper choice of axes can be written as

$$\Gamma_4: \quad \left. \begin{array}{l} x = (az + bz^2)\cos(\omega t + n), \\ y = -(az + bz^2)\sin(\omega t + n), \end{array} \right\} \tag{35}$$

with $a, b > 0$; n is any arbitrary constant. This transformation has no counterpart in the corresponding steady problem. Likewise, curves Γ_5 and Γ_6, similar to the curves Γ_1 and Γ_2, can be obtained for $\omega < \delta$.

Equations (29) of the curve Γ_1 in cylindrical coordinates can be written in the form

$$r = \frac{a}{2} e^{(\beta_0 z)} \big[1 + f_1(\beta_0, \beta_1, z, t) \big]^{1/2}, \tag{36}$$

where f_1 is negligible for large z or for very small δ. So for large z (36) reduces to

$$r = \frac{a}{2} e^{\beta_0 z}. \tag{37}$$

Let C_1 be the projection of the curve Γ_1 on the *Oxy* plane. The equation of C_1 in polar coordinates is

$$r = \frac{a}{2} e^\theta, \qquad \theta = \beta_0 z. \tag{38}$$

Therefore, the curve C_1 tends towards the logarithmic spiral (38) for large value of z, having the properties given by Berker [4]. A similar discussion holds for the curves Γ_2-Γ_6. It can be further noted that $e^{\beta_0 z}$ dominates over the other terms, the curve Γ_1 is wrapped around the surface of revolution S_1 the axis of which is *Oz* and the meridian (in cylindrical coordinates) of which is given in Eq. (36).

4 Unsteady MHD flow in an orthogonal rheometer

Consider the unsteady MHD flow of a conducting, incompressible viscous fluid between two infinite, parallel, insulated disks D_1 and D_2 rotating with the same angular velocity ω about non-coincident axes z' and z'' normal to the planes of the disks and performing non-torsional oscillations in their own planes. This model represents the motion of the fluid in the instrument called an orthogonal rheometer.

We choose the Cartesian coordinate system with the z-axis perpendicular to the disks and the lower and the upper disks are located at $z = -h$ and $z = h$, respectively. Let the lower disk rotate about the point $P_1(-x_1, -y_1, -h)$ and the upper disk about $P_2(x_1, y_1, h)$. Taking the middle point O of $P_1 P_2$ as the origin and perpendicular axes *Ox* and *Oy* lying in the plane $z = 0$ so that the z-axis is perpendicular to these axes. The superposed unsteady motion is created by subjecting each point of the lower and upper disk to non-torsional oscillations of the type $\overline{a}e^{i\delta t} + \overline{b}e^{-i\delta t}$ and $\overline{c}e^{i\delta t} + \overline{d}e^{-i\delta t}$, respectively, where $\overline{a}$, $\overline{b}$, $\overline{c}$ and $\overline{d}$, are given complex constants. The governing equation for this motion is given by Eq. (17) whereas the boundary conditions are

$$W = -(x_1 + iy_1) + \overline{a}e^{i\delta t} + \overline{b}e^{-i\delta t}, \quad \text{on } z = -h, \tag{39}$$

$$W = (x_1 + iy_1) + \overline{c}e^{i\delta t} + \overline{d}e^{-i\delta t}, \quad \text{on } z = h. \tag{40}$$

Assume the solution for Eq. (17) to be of the form

$$W = \overline{a}F_1(z)e^{i\delta t} + \overline{b}F_2(z)e^{-i\delta t} + F_3(z), \tag{41}$$

where the first two terms correspond to the oscillatory motion and the third term gives the solution due to the basic rigid rotation. The corresponding boundary conditions are

$$F_1(-h) = 1, \qquad F_2(-h) = 1, \qquad F_3(-h) = -(x_1 + iy_1), \tag{42}$$

and

$$F_1(h) = \frac{\overline{c}}{\overline{a}}, \qquad F_2(h) = \frac{\overline{d}}{\overline{b}}, \qquad F_3(h) = x_1 + iy_1. \tag{43}$$

The governing differential equation (17) is of the third order, giving three sets of infinite solutions; one corresponding to the steady part and the other two corresponding to the oscillatory parts, proportional to $e^{i\delta t}$ and $e^{-i\delta t}$. As there are only two boundary conditions (42) and (43) to determine each complex function, therefore its solution is not unique. We introduce one more arbitrary boundary condition in addition to the boundary conditions (42) and (43) in order to determine the solution. Let x_p and y_p be the coordinates of the point of intersection of the curve Γ and the plane Oxy and also the non-torsional oscillations are prescribed on the plane Oxy. Then the boundary conditions will be of the form

$$W(0) = x_P + iy_P + a_0 e^{i\delta t} + b_0 e^{-i\delta t}$$

giving

$$F_1(0) = \frac{a_0}{\overline{a}}, \qquad F_2(0) = \frac{b_0}{\overline{b}}, \qquad F_3(0) = x_p + iy_p. \tag{44}$$

The solutions of the differential equation (17), satisfying the boundary conditions (42)-(44), are given by

$$
\begin{aligned}
W = {} & \left\{ \frac{\overline{c} + \overline{a} - 2a_0}{2} \frac{\cosh\sqrt{N + \lambda_3^2}z - 1}{\cosh\sqrt{N + \lambda_3^2}h - 1} + \frac{\overline{c} - \overline{a}}{2} \frac{\sinh\sqrt{N + \lambda_3^2}z}{\sinh\sqrt{N + \lambda_3^2}h} + a_0 \right\} e^{i\delta t} \\
& + \left\{ \frac{\overline{d} + \overline{b} - 2b_0}{2} \frac{\cosh\sqrt{N + \lambda_1^2}z - 1}{\cosh\sqrt{N + \lambda_1^2}h - 1} + \frac{\overline{d} - \overline{b}}{2} \frac{\sinh\sqrt{N + \lambda_1^2}z}{\sinh\sqrt{N + \lambda_1^2}h} + b_0 \right\} e^{-i\delta t} \\
& + (x_p + iy_p)\left\{ 1 - \frac{\cosh\sqrt{N + \lambda_0^2}z - 1}{\cosh\sqrt{N + \lambda_0^2}h - 1} \right\} + (x_1 + iy_1)\frac{\sinh\sqrt{N + \lambda_0^2}z}{\sinh\sqrt{N + \lambda_0}h},
\end{aligned}
\tag{45}
$$

where

$$\lambda_0^2 = 2i\beta_0^2, \qquad \lambda_1^2 = 2i\beta_1^2 = -2i\beta_2^2, \qquad \lambda_3^2 = 2i\beta_3^2, \qquad \beta_0^2 = \frac{\omega}{2\nu},$$

$$\beta_1^2 = (\omega - \delta)/2\nu, \qquad \beta_2^2 = (\delta - \omega)/2\nu, \qquad \beta_3^2 = (\omega + \delta)/2\nu. \tag{46}$$

Setting

$$\frac{\overline{c} - \overline{a}}{2} = a_1 + ia_2, \qquad \frac{\overline{c} + \overline{a} - 2a_0}{2} = b_1 + ib_2, \qquad a_0 = a_1' + ia_2',$$

$$\frac{\overline{d} - \overline{b}}{2} = c_1 + ic_2, \qquad \frac{\overline{d} + \overline{b} - 2b_0}{2} = d_1 + id_2, \qquad b_0 = b_1' + ib_2', \tag{47}$$

in Eq. (45) and equating the real and imaginary parts, we get for $\omega > \delta$

$$f = \big[(c_2 R_1 + c_1 Q_1 + d_1 T_1 + d_2 S_1) - (a_2 R_3 + a_1 Q_3 + b_2 S_3 + b_1 T_3) - a_2' + b_2'\big] \sin \delta t$$

$$+ \big[(a_1 R_3 - a_2 Q_3 + b_1 S_3 - b_2 T_3) + (c_1 R_1 - c_2 Q_1 + d_1 S_1 - d_2 T_1) + a_1' + b_1'\big] \cos \delta t$$

$$+ x_1 R_0 - y_1 Q_0 + x_p (1 - S_0) + y_p T_0 \tag{48}$$

and

$$g = \big[(a_1 R_3 - a_2 Q_3 + b_1 S_3 - b_2 T_3) - (c_1 R_1 - c_2 Q_1 + d_1 S_1 - d_2 T_1) + a_1' - b_1'\big] \sin \delta t$$

$$+ \big[(a_2 R_3 + a_1 Q_3 + b_2 S_3 + b_1 T_3) + (c_2 R_1 + c_1 Q_1 + d_1 T_1 + d_2 S_1) + a_2' + b_2'\big] \cos \delta t$$

$$+ x_1 Q_0 + y_1 R_0 - x_p T_0 + y_p (1 - S_0), \tag{49}$$

where

$$\frac{\sinh \sqrt{N + \lambda_j^2} z}{\sinh \sqrt{N + \lambda_j^2} h} = R_j \pm iQ_j, \qquad \frac{\cosh \sqrt{N + \lambda_j^2} z - 1}{\cosh \sqrt{N + \lambda_j^2} h - 1} = S_j \pm iT_j, \quad j = 0, 1, 3. \tag{50}$$

The minus sign is taken when $j = 2$. Thus we have

$$R_j = \big[\xi_j(z)\xi_j(h) + \eta_j(z)\eta_j(h)\big]/\Delta_{1j}, \qquad Q_j = \big[\eta_j(z)\xi_j(h) - \xi_j(z)\eta_j(h)\big]/\Delta_{1j},$$

$$S_j = \big[\phi_j(z)\phi_j(h) + \chi_j(z)\chi_j(h)\big]/\Delta_{2j}, \qquad T_j = \big[\chi_j(z)\phi_j(h) - \phi_j(z)\chi_j(h)\big]/\Delta_{2j}, \tag{51}$$

$$\xi_j(z) = \sinh(\beta_{rj} z) \cos(\beta_{ij} z), \qquad \eta_j(z) = \cosh(\beta_{rj} z) \sin(\beta_{ij} z),$$

$$\phi_j(z) = \cosh(\beta_{rj} z) \cos \beta_{ij} z - 1, \qquad \chi_j(z) = \sinh(\beta_{rj} z) \sin(\beta_{ij} z), \tag{52}$$

$$\Delta_{1j} = \xi_j^2(h) + \eta_j^2(h), \qquad \Delta_{2j} = \phi_j^2(h) + \chi_j^2(h),$$

where $j = 0, 1, 2, 3$. Similarly, we can get the solutions for $\omega < \delta$. The solution for the case $\omega = \delta$ is obtained from Eqs. (48) and (49) by taking the limit $\delta \longrightarrow \omega$. In this case we obtain

$$R_1 = R_2 = \frac{\sinh \sqrt{N} z}{\sinh \sqrt{N} h}, \qquad S_1 = S_2 = \frac{\cosh \sqrt{N} z - 1}{\cosh \sqrt{N} h - 1}, \qquad Q_1 = Q_2 = T_1 = T_2 = 0. \tag{53}$$

We have thus obtained the unsteady MHD motion of viscous fluid in the orthogonal rheometer for the three different cases of ω, *i.e.*, (i) $\omega > \delta$, (ii) $\omega < \delta$, (iii) $\omega = \delta$. The solutions for f and g contain six arbitrary constants, namely a_1, a_2, b_1, b_2, x_p and y_p, giving three sets of infinite solutions. Hence the existence of three sets of infinite solutions for the unsteady MHD motion of the fluid in an orthogonal rheometer in which the non-torsional oscillations are superposed on the disks are established. These solutions for f and g reduce to the solutions of Rao and Kasiviswanathan [10] for $B_0 \to 0$.

5 Results and discussion

In view of Eqs. (15) and (16) the velocity components are given by

$$u = -\omega\big(y - g(z,t)\big), \qquad v = \omega\big(x - f(z,t)\big), \qquad w = 0. \tag{54}$$

Then the components (ξ, η, ζ) of the vorticity are given by

$$\xi = \omega f_z, \qquad \eta = \omega g_z, \qquad \zeta = 2\omega, \tag{55}$$

where f and g are given by Eqs. (25) and (26). Substituting Eqs. (14) and (16) in Eqs. (8) and (9), we get

$$
\begin{aligned}
\phi_x &= \omega^2 x + \omega\left(v g_{zz} - \omega f - g_t - \frac{\sigma B_0^2}{\rho}g\right) + \frac{\sigma B_0^2}{\rho}\omega y, \\
\phi_y &= \omega^2 y - \omega\left(v f_{zz} + \omega g - f_t - \frac{\sigma B_0^2}{\rho}f\right) - \frac{\sigma B_0^2}{\rho}\omega x.
\end{aligned}
\tag{56}
$$

Using Eqs. (11), (25) and (26) in (56) we obtain

$$
\begin{aligned}
p + \rho\Omega &= \frac{\rho\omega^2}{2}\left(1 - \frac{\sigma^2 B_0^4}{\rho^2\omega^2}\right)\left[\frac{(x - X_0)^2}{(1 + \frac{\sigma B_0^2}{\rho\omega})} + \frac{(y - Y_0)^2}{(1 - \frac{\sigma B_0^2}{\rho\omega})}\right] + p_0, \quad B_0^2 < \frac{\rho\omega}{\sigma}, \\
p + \rho\Omega &= \frac{\rho\omega^2}{2}\left(\frac{\sigma^2 B_0^4}{\rho^2\omega^2} - 1\right)\left[\frac{(y - Y_0)^2}{(\frac{\sigma B_0^2}{\rho\omega} - 1)} - \frac{(x + X_0)^2}{(\frac{\sigma B_0^2}{\rho\omega} + 1)}\right] + p_0, \quad B_0^2 > \frac{\rho\omega}{\sigma}, \\
p + \rho\Omega &= \rho\omega^2\big[(x - X_0)^2 - 4Y_0 y\big] + p_0, \quad \frac{\sigma^2 B_0^4}{\rho^2\omega^2} = 1,
\end{aligned}
\tag{57}
$$

where p_0 is an arbitrary constant, $X_0(h,t)$, $Y_0(h,t)$ are given by

$$
\begin{aligned}
X_0 = {} & x_p\big(1 + \phi_0(h)/\Delta_{20}\big) + y_p\chi_0(h)/\Delta_{20} \\
& - \left[\left(1 + \frac{\delta}{\omega}\right)\{(b_2\chi_3(h) + b_1\phi_3(h))/\Delta_{23} - a_1'\}\right. \\
& \left. + \left(1 - \frac{\delta}{\omega}\right)\{(d_1\phi_1(h) + d_2\chi_1(h))/\Delta_{21} - b_1'\}\right]\cos\delta t \\
& - \left[\left(1 + \frac{\delta}{\omega}\right)\{(b_1\chi_3(h) - b_2\phi_3(h))/\Delta_{23} + a_2'\}\right. \\
& \left. + \left(1 - \frac{\delta}{\omega}\right)\{(d_2\phi_1(h) - d_1\chi_1(h))/\Delta_{21} - b_2'\}\right]\sin\delta t \\
& - \frac{N v}{\omega}\Big[-y_p\big(1 - \phi_0(h)/\Delta_{20}\big) - x_p\chi_0(h) \\
& + \{(b_2\phi_3(h) - b_1\chi_3(h))/\Delta_{23} + (d_1\chi_1(h) - d_2\phi_1(h))/\Delta_{21} - a_2' - b_2'\}\cos\delta t \\
& + \{(b_1\phi_3(h) + b_2\chi_3(h))/\Delta_{23} - (d_1\phi_1(h) + d_2\chi_1(h))/\Delta_{21} - a_1' + b_1'\}\sin\delta t\Big], \\
Y_0 = {} & y_p\big(1 + \phi_0(h)/\Delta_{20}\big) - x_p\chi_0(h)/\Delta_{20} \\
& + \left[\left(1 + \frac{\delta}{\omega}\right)\{(b_1\chi_3(h) - b_2\phi_3(h))/\Delta_{23} + a_2'\}\right.
\end{aligned}
\tag{58}
$$

$$+ \left(1 - \frac{\delta}{\omega}\right)\left\{\left(d_1\chi_1(h) - d_2\phi_1(h)\right)/\Delta_{21} + b_2'\right\}\right]\cos\delta t$$

$$+ \left[\left(1 + \frac{\delta}{\omega}\right)\left\{a_1' - \left(b_2\chi_3(h) + b_1\phi_3(h)\right)/\Delta_{23}\right\}\right.$$

$$+ \left(1 - \frac{\delta}{\omega}\right)\left\{\left(d_1\phi_1(h) + d_2\chi_1(h)\right)/\Delta_{21} - b_1'\right\}\right]\sin\delta t$$

$$+ \frac{N\nu}{\omega}\left[-x_p\left(1 + \phi_0(h)/\Delta_{20}\right) - y_p\chi_0(h)/\Delta_{20}\right.$$

$$+ \left\{\left(b_1\phi_3(h) + b_2\chi_3(h)\right)/\Delta_{23} + \left(d_1\phi_1(h) + d_2\chi_1(h)\right)/\Delta_{21} - a_1' - b_1'\right\}\cos\delta t$$

$$+ \left\{\left(d_2\phi_1(h) - d_1\chi_1(h)\right)/\Delta_{21} - \left(b_2\phi_3(h) - b_1\chi_3(h)\right)/\Delta_{23} + a_2' - b_2'\right\}\sin\delta t\right]. \tag{59}$$

The results (57)-(59) reduce to the results of Rao and Kasiviswanathan [10] in the absence of MHD and Berker [4] in the absence of MHD and oscillations.

The components of stress t_1 applied by the fluid at any point (x, y, h) on the upper disk in (x, y, z) directions are given by

$$t_{1x} = -\mu(u_z + w_x), \qquad t_{1y} = -\mu(v_z + w_y), \qquad t_{1z} = p - 2\mu w_z, \tag{60}$$

where p is the pressure and μ the coefficient of viscosity. Using the values of (u, v, w) given in Eq. (54), we obtain from Eq. (60)

$$t_{1x} = -\mu\omega g_z(t, h), \qquad t_{1y} = \mu\omega f_z(t, h), \qquad t_{1z} = p, \tag{61}$$

where f and g are defined in Eqs. (48) and (49). Similarly, the stress applied on the lower plate at any point $(x, y, -h)$ can be calculated.

A number of interesting special cases can be derived from the general solutions given by Eq. (45). A solution is said to be symmetric solution with respect to a given point O, if the velocity field is a symmetric vector field with respect to the point O. Therefore, the symmetric solution has to satisfy the following condition:

$$\mathbf{V}(x, y, z) = -\mathbf{V}(x, y, z). \tag{62}$$

Among the solutions obtained for the motion in the orthogonal rheometer, we need to find whether there exist solutions which are symmetric with respect to: (i) the point O, (ii) the mid point of the line segment P_1P_2. The velocity field given by Eq. (54) satisfies the condition (62), provided

$$f(-z, t) = -f(z, t) \quad \text{and} \quad g(-z, t) = -g(z, t), \tag{63}$$

which implies that f and g are odd functions of z and

$$f(0) = 0, \qquad g(0) = 0. \tag{64}$$

The latter equation shows that the curve Γ must pass through the origin O of the axes. Therefore, for a symmetric solution the constants in Eq. (45) should satisfy

$$\bar{c} + \bar{a} - 2a_0 = \bar{d} + \bar{b} - 2b_0 = 0, \qquad a_0 = b_0 = x_p = y_p = 0. \tag{65}$$

Thus, for a symmetric solution all the arbitrary constants given in the boundary condition (44) vanish and the solution thus obtained is unique. This symmetric solution is given by

$$W = \left\{ \frac{\overline{c} - \overline{a}}{2} \frac{\sinh \sqrt{N + \lambda_3^2} z}{\sinh \sqrt{N + \lambda_3^2} h} \right\} e^{i\delta t} + \left\{ \frac{\overline{d} - \overline{b}}{2} \frac{\sinh \sqrt{N + \lambda_1^2} z}{\sinh \sqrt{N + \lambda_1^2} h} \right\} e^{-i\delta t}$$

$$+ (x_1 + iy_1) \frac{\sinh \sqrt{N + \lambda_0^2} z}{\sinh \sqrt{N + \lambda_0^2} h}. \tag{66}$$

This solution includes the symmetric solution of Kasiviswanathan and Rao [13] as a special case. Taking $y_1 = 0$ and $y_P = 0$ in Eq. (45), we obtain the solutions of the MHD flow due to non-torsionally oscillating eccentric rotating disks given by Kasiviswanathan and Rao [13]. In the limit $B_0 \to 0$ in Eq. (45), we obtain solutions given by Rao and Kasiviswanathan [10]. By taking $B_0 \to 0$, $\delta \to 0$ (together with the amplitudes of the oscillations) in Eq. (45), we get

$$W(z) = (x_1 + iy_1) \frac{\sinh \lambda_0 z}{\sinh \lambda_0 h} + (x_P + iy_P) \left[1 - \frac{\cosh \lambda_0 z - 1}{\cosh \lambda_0 z - 1} \right], \tag{67}$$

which corresponds to the solutions given by Berker [4]. As $\delta \to 0$, $B_0 \to 0$, $\omega \to 0$ in Eq. (45), we get

$$W(z) = \left[\frac{\overline{c} + \overline{a} - 2a_0}{2} + \frac{\overline{d} + \overline{b} - 2b_0}{2} - (x_P + iy_P) \right] \frac{z^2}{h^2}$$

$$+ \left[\frac{\overline{c} - \overline{a}}{2} + \frac{\overline{d} - \overline{b}}{2} + x_1 + iy_1 \right] \frac{z}{h} + (x_P + iy_P + a_0 + b_0), \tag{68}$$

and the corresponding boundary conditions are

$$W(h) = x_1 + iy_1 + \overline{c} + \overline{d}, \qquad W(-h) = -(x_1 + iy_1) + \overline{a} + \overline{b},$$

$$W(0) = x_P + iy_P + a_0 + b_0. \tag{69}$$

This case corresponds to the flow of a fluid confined between two plates moving with velocities of different magnitudes and direction. It reduces to the solutions given by Kasiviswanathan and Rao [13] for $y_P = 0$ and $y_1 = 0$. Since in the limit $\omega \to 0$, all the velocity components given in Eq. (15) become zero; therefore we cannot take this velocity field for the above problem. However, in order to understand the nature and physical significance of this case, we choose the velocity field given by Kasiviswanathan and Rao [13]. We have

$$u = f(z), \qquad v = g(z), \qquad w = 0. \tag{70}$$

In the absence of a magnetic field, Eq. (1) in view of Eq. (70) yields

$$\mu f_{zz} - p_x = 0, \qquad \mu g_{zz} - p_y = 0, \qquad p_z = 0. \tag{71}$$

Eliminating the pressure gradient we obtain

$$W_{zzz} = 0. \tag{72}$$

The boundary conditions are given by

$$W(h) = u_1 + iu_1' = U_1, \qquad W(-h) = u_2 + iu_2' = U_2,$$
$$W(0) = u_3 + iu_3' = U_3, \tag{73}$$

where U_3 is chosen arbitrarily. The solution of Eq. (72) subject to the boundary conditions (73) yields

$$W(z) = \frac{1}{2}[U_1 + U_2 - 2U_3]\frac{z^2}{h^2} + \frac{1}{2}[U_1 - U_2]\frac{z}{h} + U_3, \tag{74}$$

which is similar to the solution given in Eq. (68). The solution given by the above equation yields a set of infinite number of solutions as U_3 is chosen arbitrarily. For $U_1 = -U_2$ and $U_3 = 0$, Eq. (74) reduces to a simple shearing motion and the solution so obtained is a symmetric solution. The stream function for this flow is given by

$$\psi(z) = yf(z) - xg(z) + C, \tag{75}$$

and the streamlines are given by a family of parallel straight lines in each of the plane where $z = $ constant. In Eq. (75), C denotes an arbitrary constant.

6 Conclusion

In this study we have considered the unsteady MHD flow of a viscous fluid in which each point of the parallel planes is subjected to the non-torsional oscillations in their own planes. Exact solutions for the described model are presented. The major findings of the present work are summarised as follows:

- The existence of three sets of infinite numbers of solutions for the unsteady MHD motion of a viscous fluid in an orthogonal rheometer in which the non-torsional oscillations are superposed on the disks is established. These solutions reduce to a single unique solution when symmetric solutions are looked for.
- The results of Rao and Kasiviswanathan [10] are recovered for $B_0 = 0$ and Berker [4] in the absence of MHD and oscillations.
- The present work also includes the solutions of Kasiviswanathan and Rao [13] as a special case.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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