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Three periodic solutions for a class of ordinary p-Hamiltonian systems

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Abstract
We study the p-Hamiltonian systems $-(|u'|^{p-2}u')' + A(t)|u|^{p-2}u = \nabla F(t,u) + \lambda\nabla G(t,u)$, $u(0) - u(T) = u'(0) - u'(T) = 0$. Three periodic solutions are obtained by using a three critical points theorem.
MSC: 34K13; 34B15; 58E30

Keywords: p-Hamiltonian systems; three periodic solutions; three critical points theorem

1 Introduction

Consider the p-Hamiltonian systems

$$\begin{cases} -(|u'|^{p-2}u')' + A(t)|u|^{p-2}u = \nabla F(t,u) + \lambda\nabla G(t,u), \\ u(0) - u(T) = u'(0) - u'(T) = 0, \end{cases} \tag{1.1}$$

where $p > 1$, $T > 0$, $\lambda \in (-\infty, +\infty)$, $F : [0, T] \times \mathbf{R}^N \to \mathbf{R}$ is a function such that $F(\cdot, x)$ is continuous in $[0, T]$ for all $x \in \mathbf{R}^N$ and $F(\cdot, x)$ is a C^1-function in $\mathbf{R}^N$ for almost every $t \in [0, T]$, and $G : [0, T] \times \mathbf{R}^N \to \mathbf{R}$ is measurable in $[0, T]$ and $C^1 \in \mathbf{R}^N$. $A = (a_{ij}(t))_{N\times N}$ is symmetric, $A \in C([0, T], \mathbf{R}^{N\times N})$, and there exists a positive constant λ_1 such that $(A(t)|x|^{p-2}x, x) \geq \lambda_1^p|x|^p$ for all $x \in \mathbf{R}^N$ and $t \in [0, T]$, that is, $A(t)$ is positive definite for all $t \in [0, T]$.

In recent years, the three critical points theorem of Ricceri [1] has widely been used to solve differential equations; see [2–4] and references therein.

In [5], Li *et al.* have studied the three periodic solutions for p-Hamiltonian systems

$$\begin{cases} -(|u'|^{p-2}u')' + A(t)|u|^{p-2}u = \lambda\nabla F(t,u) + \mu\nabla G(t,u), \\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{cases} \tag{1.2}$$

Their technical approach is based on two general three critical points theorems obtained by Averna and Bonanno [6] and Ricceri [4].

In [7], Shang and Zhang obtained three solutions for a perturbed Dirichlet boundary value problem involving the p-Laplacian by using the following Theorem A. In this paper, we generalize the results in [7] on problem (1.1).

Theorem A [1, 7] *Let X be a separable and reflexive real Banach space, and let $\phi, \psi : X \to \mathbf{R}$ be two continuously Gâteaux differentiable functionals. Assume that ψ is sequentially* *weakly lower semicontinuous and even that ϕ is sequentially weakly continuous and odd, and that, for some $b > 0$ and for each $\lambda \in [-b, b]$, the functional $\psi + \lambda\phi$ satisfies the Palais-Smale condition and*

$$\lim_{\|x\|\to\infty} \bigl[\psi(x) + \lambda\phi(x)\bigr] = +\infty.$$

Finally, assume that there exists $k > 0$ such that

$$\inf_{x\in X} \psi(x) < \inf_{|\phi(x)|<k} \psi(u).$$

Then, for every $b > 0$, there exist an open interval $\Lambda \subset [-b, b]$ and a positive real number σ, such that for every $\lambda \in \Lambda$, the equation

$$\psi'(x) + \lambda\phi'(x) = 0$$

admits at least three solutions whose norms are smaller than σ.

2 Proofs of theorems

First, we give some notations and definitions. Let

$$W_T^{1,p} = \bigl\{u : [0, T] \to \mathbf{R}^N \mid u \text{ is absolutely continuous}, u(0) = u(T), u' \in L^p\bigl(0, T; \mathbf{R}^N\bigr)\bigr\}$$

and is endowed with the norm

$$\|u\| = \left(\int_0^T \bigl|u'(t)\bigr|^p\,dt + \int_0^T \bigl(A(t)\bigl|u(t)\bigr|^{p-2}u(t), u(t)\bigr)\,dt\right)^{\frac{1}{p}}.$$

Let $\varphi_\lambda : W_T^{1,p} \to \mathbf{R}$ be defined by the energy functional

$$\varphi_\lambda(u) = \psi(u) + \lambda\phi(u), \tag{2.1}$$

where $\psi(u) = \frac{1}{p}\|u\|^p - \int_0^T F(t, u(t))\,dt$, $\phi(u) = \int_0^T G(t, u(t))\,dt$.

Then $\varphi_\lambda \in C'(W_T^{1,p}, \mathbf{R})$ and one can check that

$$\begin{aligned}\bigl\langle\varphi_\lambda'(u), v\bigr\rangle = \int_0^T &\bigl[\bigl(\bigl|u'(t)\bigr|^{p-2}u'(t), v'(t)\bigr) - \bigl(\nabla F\bigl(t, u(t)\bigr), v(t)\bigr) \\ &- \lambda\bigl(\nabla G\bigl(t, u(t)\bigr), v(t)\bigr)\bigr]\,dt,\end{aligned} \tag{2.2}$$

for all $u, v \in W_T^{1,p}$. It is well known that the T-periodic solutions of problem (1.1) correspond to the critical points of φ_λ.

As $A(t)$ is positive definite for all $t \in [0, T]$, we have Lemma 2.1.

Lemma 2.1 *For each $u \in W_T^{1,p}$,*

$$\lambda_1\|u\|_{L^p} \le \|u\|, \tag{2.3}$$

where $\|u\|_{L^p} = \int_0^T |u(t)|^p\,dt$.

Theorem 2.1 *Suppose that F and G satisfy the following conditions*:

(H1) $\lim_{|x|\to\infty}\frac{|\nabla F(t,x)|}{|x|^{p-1}} = 0$, *for a.e.* $t \in [0,T]$;

(H2) $\lim_{|x|\to 0}\frac{|\nabla F(t,x)|}{|x|^{p-1}} = 0$, *for a.e.* $t \in [0,T]$;

(H3) $\lim_{|x|\to 0}\frac{F(t,x)}{|x|^{p}} = \infty$, *for a.e.* $t \in [0,T]$;

(H4) $|\nabla G(t,x)| \le c(1+|x|^{q-1})$, $\forall x \in \mathbf{R}^N$, *a.e.* $t\in[0,T]$, *for some* $c>0$ *and* $1 \le q < p$;

(H5) $F(t,\cdot)$ *is even and* $G(t,\cdot)$ *is odd for a.e.* $t \in [0,T]$.

Then, for every $b > 0$, there exist an open interval $\Lambda \subset [-b,b]$ and a positive real number σ, such that for every $\lambda \in \Lambda$, problem (1.1) *admits at least three solutions whose norms are smaller than σ.*

Proof By (H1) and (H2), given $\varepsilon > 0$, we may find a constant $C_\varepsilon > 0$ such that

$$\left|\nabla F(t,x)\right| \le C_\varepsilon + \varepsilon|x|^{p-1}, \quad \text{for every } x \in \mathbf{R}^N, \text{ a.e. } t \in [0,T], \tag{2.4}$$

$$\left|F(t,x)\right| \le C_\varepsilon + \frac{\varepsilon}{p}|x|^{p}, \quad \text{for every } x \in \mathbf{R}^N, \text{ a.e. } t \in [0,T], \tag{2.5}$$

and so the functional $\psi(u)$ is continuously Gâteaux differentiable functional and sequentially weakly continuous in the space $W_T^{1,p}$. Also, by (H4), we know $\phi(u)$ is sequentially weakly continuous. According to (H4), we get

$$\left|G(t,x)\right| \le c|x| + \frac{c}{p}|x|^{q}, \quad \text{for every } x \in \mathbf{R}^N, \text{ a.e. } t \in [0,T]. \tag{2.6}$$

For $\forall\lambda \in \mathbf{R}$, from the inequality (2.5) and (2.6), we deduce that

$$\begin{aligned}\psi(u) + \lambda\phi(u) &\ge \frac{1}{p}\|u\|^p - \int_0^T \left(C_\varepsilon + \frac{\varepsilon}{p}\left|u(t)\right|^p\right)dt - \lambda\int_0^T\left(c\left|u(t)\right| + \frac{c}{q}\left|u(t)\right|^q\right)dt \\ &\ge \frac{1}{p}\left(1-\frac{\varepsilon}{\lambda_1}\right)\|u\|^p - \frac{c\lambda}{q\lambda_1}T^{\frac{p-q}{q}}\|u\|^q - \frac{c\lambda}{\lambda_1}T^{\frac{p-1}{p}}\|u\| - \varepsilon T.\end{aligned}$$

Since $p > q$, ε small enough, we have

$$\lim_{\|u\|\to\infty}\left[\psi(u) + \lambda\phi(u)\right] = +\infty. \tag{2.7}$$

Now, we prove that φ_λ satisfies the (PS) condition.

Suppose $\{u_n\}$ is a (PS) sequence of φ_λ, that is, there exists $C > 0$ such that

$$\varphi_\lambda(u_n) \to C, \qquad \varphi_\lambda'(u_n) \to 0 \quad \text{as } n \to \infty.$$

Assume that $\|u_n\| \to \infty$. By (2.7), which contradicts $\varphi_\lambda(u_n) \to C$. Thus $\{u_n\}$ is bounded. We may assume that there exists $u_0 \in W_T^{1,p}$ satisfying

$$u_n \rightharpoonup u_0, \quad \text{weakly in } W_T^{1,p}, \qquad u_n \to u_0, \quad \text{strongly in } L^p[0,T],$$

$$u_n(x) \to u_0(x), \quad \text{a.e. } t \in [0,T].$$

Observe that

$$\begin{aligned}
&\langle \varphi_\lambda'(u_n), u_n - u_0 \rangle \\
&\quad = \int_0^T \big[\big(|u_n'(t)|^{p-2} u_n'(t), u_n'(t) - u_0'(t) \big) + \big(A(t)|u_n(t)|^{p-2} u_n(t), u_n(t) - u_0(t) \big) \big]\, dt \\
&\qquad - \int_0^T \big(\big(\nabla F(t, u_n(t)), u_n(t) - u_0(t) \big) \big)\, dt \\
&\qquad - \lambda \int_0^T \big(\nabla G(t, u_n(t)), u_n(t) - u_0(t) \big)\, dt. \qquad (2.8)
\end{aligned}$$

We already know that

$$\langle \varphi_\lambda'(u_n), u_n - u_0 \rangle \to 0, \quad \text{as } n \to \infty. \qquad (2.9)$$

By (2.4) and (H4) we have

$$\begin{aligned}
&\int_0^T \big(\nabla F(t, u_n(t)), u_n(t) - u_0(t) \big)\, dt \to 0, \quad \text{as } n \to \infty, \\
&\int_0^T \big(\nabla G(t, u_n(t)), u_n(t) - u_0(t) \big)\, dt \to 0, \quad \text{as } n \to \infty.
\end{aligned}$$

Using this, (2.8), and (2.9) we obtain

$$\begin{aligned}
&\int_0^T \big[\big(|u_n'(t)|^{p-2} u_n'(t), u_n'(t) - u_0'(t) \big) + \big(A(t)|u_n(t)|^{p-2} u_n(t), u_n(t) - u_0(t) \big) \big]\, dt \to 0, \\
&\quad \text{as } n \to \infty.
\end{aligned}$$

This together with the weak convergence of $u_n \rightharpoonup u_0$ in $W_T^{1,p}$ implies that

$$u_n \to u_0, \quad \text{strongly in } W_T^{1,p}.$$

Hence, φ_λ satisfies the (PS) condition. Next, we want to prove that

$$\inf_{u \in W_T^{1,p}} \psi(u) < 0. \qquad (2.10)$$

Owing to the assumption (H3), we can find $\delta > 0$, for $L > 0$, such that

$$|F(t,x)| > L|x|, \quad \text{for } 0 < |x| \le \delta, \text{ and a.e. } t \in [0,T].$$

We choose a function $0 \ne v \in C_0^\infty([0,T])$, put $L > \|v\|^p / (p \int_0^T |v|^p\, dt)$, and we take $\varepsilon > 0$ small. Then we obtain

$$\begin{aligned}
\psi(\varepsilon v) &= \frac{1}{p}\|\varepsilon v\|^p - \int_0^T F(t, \varepsilon v(t))\, dt \\
&\le \frac{\varepsilon^p}{p}\|v\|^p - L\varepsilon^p \int_0^T |v(t)|^p\, dt < 0.
\end{aligned}$$

Thus (2.10) holds.

From (H2), $\forall\varepsilon > 0$, $\exists\rho_0(\varepsilon) > 0$ such that

$$\left|\nabla F(t,x)\right| \le \varepsilon |x|^{p-1}, \quad \text{if } 0 < \rho = |x| < \rho_0(\varepsilon).$$

Thus

$$\int_0^T F\bigl(t,u(t)\bigr)\,dt \le \frac{\varepsilon}{p}\int_0^T \left|u(t)\right|^p\,dt \le \frac{\varepsilon}{p\lambda_1}\|u\|^p.$$

Choose $\varepsilon = \lambda_1/2$, one has

$$\begin{aligned}\psi(u) &= \frac{1}{p}\|u\|^p - \frac{\varepsilon}{p\lambda_1}\|u\|^p \\ &= \frac{1}{2p}\|u\|^p > 0.\end{aligned}$$

Hence, there exists $k > 0$ such that

$$\inf_{|\phi(u)|<k} \psi(u) = 0.$$

So we have

$$\inf_{u\in W_T^{1,p}} \psi(u) < \inf_{|\phi(u)|<k} \psi(u).$$

The condition (H5) implies ψ is even and ϕ is odd. All the assumptions of Theorem A are verified. Thus, for every $b > 0$ there exist an open interval $\Lambda \subset [-b,b]$ and a positive real number σ, such that for every $\lambda \in \Lambda$, problem (1.1) admits at least three weak solutions in $W_T^{1,p}$ whose norms are smaller than σ. □

Theorem 2.2 *If F and G satisfy assumptions* (H1)-(H2), (H4)-(H5), *and the following condition* (H3′):

(H3′) *there is a constant* $B_1 = \sup\{1/\int_0^T |u(t)|^p\,dt : \|u\| = 1\}$, $B_2 \ge 0$, *such that*

$$F(t,x) \ge 2B_1\frac{|x|^p}{p} - B_2, \quad \textit{for } x \in \mathbf{R}^N, \textit{ a.e. } t \in [0,T].$$

Then, for every $b > 0$, *there exist an open interval* $\Lambda \subset [-b,b]$ *and a positive real number* σ, *such that for every* $\lambda \in \Lambda$, *problem* (1.1) *admits at least three solutions whose norms are smaller than* σ.

Proof The proof is similar to the one of Theorem 2.1. So we give only a sketch of it. By the proof of Theorem 2.1, the functional ψ and ϕ are sequentially weakly lower semicontinuous and continuously Gâteaux differentiable in $W_T^{1,p}$, ψ is even and ϕ is odd. For every $\lambda \in \mathbf{R}$, the functional $\psi + \lambda\phi$ satisfies the (PS) condition and

$$\lim_{\|u\|\to\infty} (\psi + \lambda\phi) = +\infty.$$

To this end, we choose a function $v \in W_T^{1,p}$ with $\|v\| = 1$. By condition (H3), a simple calculation shows that, as $s \to \infty$,

$$\begin{aligned}\psi(sv) &= \frac{1}{p}\|sv\|^p - \int_0^T F\bigl(t, sv(t)\bigr)\,dt \\ &\le \frac{s^p}{p}\|v\|^p - 2\frac{s^p B_1}{p}\int_0^T \bigl|v(t)\bigr|^p\,dt + B_2 T \\ &\le -\frac{s^p}{p} + B_2 T \to -\infty. \end{aligned} \tag{2.11}$$

Then (2.11) implies that $\psi(sv) < 0$ for $s > 0$ large enough. So, we choose large enough, $s_0 > 0$, let $u_1 = s_0 v$, such that $\psi(u_1) < 0$. Thus, we get

$$\inf_{u \in W_T^{1,p}} \psi(u) < 0.$$

By the proof of Theorem 2.1 we know that there exists $k > 0$, such that

$$\inf_{u \in W_T^{1,p}} \psi(u) < \inf_{|\phi(u)| < k} \psi(u).$$

According to Theorem A, for every $b > 0$ there exist an open interval $\Lambda \subset [-b, b]$ and a positive real number σ, such that for every $\lambda \in \Lambda$, problem (1.1) admits at least three weak solutions in $W_T^{1,p}$ whose norms are smaller than σ. □

Competing interests
The author declares that they have no competing interests.

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References

1. Bonanno, G: Some remarks on a three critical points theorem. Nonlinear Anal. **54**, 651-665 (2003)
2. Afrouzi, GA, Heidarkhani, S: Three solutions for a Dirichlet boundary value problem involving the *p*-Laplacian. Nonlinear Anal. **66**, 2281-2288 (2007)
3. Ricceri, B: On a three critical points theorem. Arch. Math. **75**, 220-226 (2000)
4. Ricceri, B: A three critical points theorem revisited. Nonlinear Anal. **70**, 3084-3089 (2009)
5. Li, C, Ou, Z-Q, Tang, C: Three periodic solutions for *p*-Hamiltonian systems. Nonlinear Anal. **74**, 1596-1606 (2011)
6. Averna, D, Bonanno, G: A three critical point theorems and its applications to the ordinary Dirichlet problem. Topol. Methods Nonlinear Anal. **22**, 93-103 (2003)
7. Shang, X, Zhang, J: Three solutions for a perturbed Dirichlet boundary value problem involving the *p*-Laplacian. Nonlinear Anal. **72**, 1417-1422 (2010)

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