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Second-order initial value problems with singularities

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Dedicated to Professor Ivan Kiguradze.

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Abstract

Using barrier strip arguments, we investigate the existence of $C[0,T]\cap C^2(0,T]$-solutions to the initial value problem $x''=f(t,x,x')$, $x(0)=A$, $\lim_{t\to 0^+}x'(t)=B$, which may be singular at $x=A$ and $x'=B$.

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1 Introduction

In this paper we study the solvability of initial value problems (IVPs) of the form

$$x''=f(t,x,x'), \tag{1.1}$$

$$x(0)=A, \qquad \lim_{t\to 0^+}x'(t)=B, \quad B>0. \tag{1.2}$$

Here the scalar function $f(t,x,p)$ is defined on a set of the form $(D_t\times D_x\times D_p)\setminus(S_A\cup S_B)$, where $D_t,D_x,D_p\subseteq\mathbb{R}$, $S_A=\mathcal{T}_1\times\{A\}\times\mathcal{P}$, $S_B=\mathcal{T}_2\times\mathcal{X}\times\{B\}$, $\mathcal{T}_i\subseteq D_t$, $i=1,2$, $\mathcal{X}\subseteq D_x$, $\mathcal{P}\subseteq D_p$, and so it may be singular at $x=A$ and $p=B$.

IVPs of the form

$$\bigl(\varphi(t)x'(t)\bigr)'=\varphi(t)f\bigl(x(t)\bigr),$$

$$x(0)=A, \qquad x'(0)=0,$$

have been investigated by Rachůnková and Tomeček [1–3]. For example in [1], the authors have discussed the set of all solutions to this problem with a singularity at $t=0$. Here $A<0$, $\varphi\in C[0,\infty)\cap C^1(0,\infty)$ with $\varphi(0)=0$, $\varphi'(t)>0$ for $t\in(0,\infty)$ and $\lim_{t\to\infty}\frac{\varphi'(t)}{\varphi(t)}=0$, f is locally Lipschitz on $(-\infty,L]$ with the properties $f(L)=0$ and $xf(x)<0$ for $x\in(-\infty,0)\cup(0,L)$, where $L>0$ is a suitable constant.

Agarwal and O'Regan [4] have studied the problem

$$x''=\varphi(t)f\bigl(t,x,x'\bigr), \quad t\in(0,T],$$

$$x(0)=x'(0)=0,$$

where $f(t,x,p)$ may be singular at $x = 0$ and/or $p = 0$. The obtained results give a positive $C^1[0,T] \cap C^2(0,T]$-solution under the assumptions that $\varphi \in C[0,T]$, $\varphi(t) > 0$ for $t \in (0,T]$, $f : [0,T] \times (0,\infty)^2 \to (0,\infty)$ is continuous and

$$f(t,x,p) \leq \left[g(x) + h(x)\right]\left[r(p) + w(p)\right] \quad \text{for } (t,x,p) \in [0,T] \times (0,\infty)^2,$$

where g, h, r, and w are suitable functions.

IVPs of the form

$$x''(t) = f\big(t,x(t),x'(t)\big), \quad 0 < t < 1,$$
$$x(0) = x'(0) = 0,$$

where $f(t,x,p) \in C((0,1) \times (0,\infty)^2)$, maybe singular at $t = 0$, $t = 1$, $x = 0$ or $p = 0$, have been studied by Yang [5, 6]. The solvability in $C^1[0,1]$ and $C[0,1] \cap C^2(0,1)$ is established in these works, respectively, under the assumption that

$$0 < f(t,x,p) \leq k(t)F(x)G(y) \quad \text{for } (t,x,p) \in (0,1) \times (0,\infty)^2,$$

where k, F, and G are suitable functions.

The solvability of various IVPs has been studied also by Bobisud and O'Regan [7], Bobisud and Lee [8], Cabada and Heikkilä [9], Cabada *et al.* [10, 11], Cid [12], Maagli and Masmoudi [13], and Zhao [14]. Existence results for problem (1.1), (1.2) with a singularity at the initial value of x' have been reported in Kelevedjiev-Popivanov [15].

Here, as usual, we use regularization and sequential techniques. Namely, we proceed as follows. First, by means of the topological transversality theorem [16], we prove an existence result guaranteeing $C^2[a,T]$-solutions to the nonsingular IVP for equations of the form (1.1) with boundary conditions

$$x(a) = A, \qquad x'(a) = B.$$

Moreover, we establish the needed *a priori* bounds by the barrier strips technique. Further, the obtained existence theorem assures $C^2[0,T]$-solutions for each nonsingular IVP included in the family

$$\begin{aligned} &x'' = f\big(t,x,x'\big), \\ &x(0) = A + n^{-1}, \qquad x'(0) = B - n^{-1}, \end{aligned} \tag{1.3}$$

where $n \in \mathbb{N}$ is suitable. Finally, we apply the Arzela-Ascoli theorem on the sequence $\{x_n\}$ of $C^2[0,T]$-solutions thus constructed to (1.3) to extract a uniformly convergent subsequence and show that its limit is a $C[0,T] \cap C^2(0,T]$-solution to singular problem (1.1), (1.2). In the case $A \geq 0$, $B \geq 0$ we establish $C[0,T] \cap C^2(0,T]$-solutions with important properties - monotony and positivity.

We have used variants of the approach described above for various boundary value problems (BVPs); see Grammatikopoulos *et al.* [17], Kelevedjiev and Popivanov [18] and Palamides *et al.* [19]. For example in [17], we have established the existence of positive solutions to the BVP

$$g(t,x,x',x'') = 0, \quad t \in (0,1),$$
$$x(0) = 0, \qquad x'(1) = B, \quad B > 0,$$

which may be singular at $x = 0$. Note that despite the more general equation of this problem, the conditions imposed here as well as the results obtained are not consequences of those in [17].

2 Topological transversality theorem

In this short section we state our main tools - the topological transversality theorem and a theorem giving an important property of the constant maps.

So, let X be a metric space and Y be a convex subset of a Banach space E. Let $U \subset Y$ be open in Y. The compact map $F : \overline{U} \to Y$ is called *admissible* if it is fixed point free on ∂U. We denote the set of all such maps by $\mathbf{L}_{\partial u}(\overline{U}, Y)$.

A map F in $\mathbf{L}_{\partial u}(\overline{U}, Y)$ is *essential* if every map G in $\mathbf{L}_{\partial u}(\overline{U}, Y)$ such that $G|\partial U = F|\partial U$ has a fixed point in U. It is clear, in particular, every essential map has a fixed point in U.

Theorem 2.1 ([16, Chapter I, Theorem 2.2]) *Let $p \in U$ be fixed and $F \in \mathbf{L}_{\partial u}(\overline{U}, Y)$ be the constant map $F(x) = p$ for $x \in \overline{U}$. Then F is essential.*

We say that the homotopy $\{\mathrm{H}_\lambda : X \to Y\}$, $0 \le \lambda \le 1$, is compact if the map $\mathrm{H}(x,\lambda) : X \times [0,1] \to Y$ given by $\mathrm{H}(x,\lambda) \equiv \mathrm{H}_\lambda(x)$ for $(x,\lambda) \in X \times [0,1]$ is compact.

Theorem 2.2 ([16, Chapter I, Theorem 2.6]) *Let Y be a convex subset of a Banach space E and $U \subset Y$ be open. Suppose*:

(i) *$F, G : \overline{U} \to Y$ are compact maps.*

(ii) *$G \in \mathbf{L}_{\partial U}(\overline{U}, Y)$ is essential.*

(iii) *$\mathrm{H}(x,\lambda)$, $\lambda \in [0,1]$, is a compact homotopy joining F and G, i.e.*

$$\mathrm{H}(x,1) = F(x) \quad \textit{and} \quad \mathrm{H}(x,0) = G(x).$$

(iv) *$\mathrm{H}(x,\lambda)$, $\lambda \in [0,1]$, is fixed point free on ∂U.*

Then $\mathrm{H}(x,\lambda)$, $\lambda \in [0,1]$, has at least one fixed point in U and in particular there is a $x_0 \in U$ such that $x_0 = F(x_0)$.

3 Nonsingular problem

Consider the IVP

$$\begin{cases} x'' = f(t,x,x'), \\ x(a) = A, \qquad x'(a) = B, \quad B \ge 0, \end{cases} \tag{3.1}$$

where $f : D_t \times D_x \times D_p \to \mathbb{R}$, $D_t, D_x, D_p \subseteq \mathbb{R}$.

We include this problem into the following family of regular IVPs constructed for $\lambda \in [0,1]$

$$\begin{cases} x'' = \lambda f(t,x,x'), \\ x(a) = A, \qquad x'(a) = B, \end{cases} \tag{3.2}$$

and suppose the following.

(R) There exist constants $T > a$, m_1, $\overline{m}_1$, M_1, $\overline{M}_1$, and a sufficiently small $\tau > 0$ such that

$$m_1 \geq 0, \qquad \overline{M}_1 - \tau \geq M_1 \geq B \geq m_1 \geq \overline{m}_1 + \tau,$$
$$[a,T] \subseteq D_t, \qquad [A-\tau, M_0+\tau] \subseteq D_x, \qquad [\overline{m}_1, \overline{M}_1] \subseteq D_p,$$

where $M_0 = A + M_1(T-a)$,

$$\begin{aligned} &f(t,x,p) \in C\big([a,T] \times [A-\tau, M_0+\tau] \times [m_1-\tau, M_1+\tau]\big), \\ &f(t,x,p) \leq 0 \quad \text{for } (t,x,p) \in [a,T] \times D_x \times [M_1, \overline{M}_1], \\ &f(t,x,p) \geq 0 \quad \text{for } (t,x,p) \in [a,T] \times D_{M_0} \times [\overline{m}_1, m_1], \end{aligned} \tag{3.3}$$

where $D_{M_0} = D_x \cap (-\infty, M_0]$.

Our first result ensures bounds for the eventual C^2-solutions to (3.2). We need them to prepare the application of the topological transversality theorem.

Lemma 3.1 *Let* (R) *hold. Then each solution* $x \in C^2[a,T]$ *to the family* $(3.2)_\lambda$, $\lambda \in [0,1]$, *satisfies the bounds*

$$A \leq x(t) \leq M_0, \qquad m_1 \leq x'(t) \leq M_1, \qquad m_2 \leq x''(t) \leq M_2 \quad \textit{for } t \in [a,T],$$

where

$$m_2 = \min\big\{f(t,x,p) : (t,x,p) \in [a,T] \times [A, M_0] \times [m_1, M_1]\big\},$$
$$M_2 = \max\big\{f(t,x,p) : (t,x,p) \in [a,T] \times [A, M_0] \times [m_1, M_1]\big\}.$$

Proof Suppose that the set

$$S_- = \big\{t \in [a,T] : M_1 < x'(t) \leq \overline{M}_1\big\}$$

is not empty. Then

$$x'(a) = B \leq M_1 \quad \text{and} \quad x' \in C[a,T]$$

imply that there exists an interval $[\alpha, \beta] \subset S_-$ such that

$$x'(\alpha) < x'(\beta).$$

This inequality and the continuity of $x'(t)$ guarantee the existence of some $\gamma \in [\alpha, \beta]$ for which

$$x''(\gamma) > 0.$$

Since $x(t)$, $t \in [a,T]$, is a solution of the differential equation, we have $(t, x(t), x'(t)) \in [a,T] \times D_x \times D_p$. In particular for γ we have

$$\big(\gamma, x(\gamma), x'(\gamma)\big) \in S_- \times D_x \times (M_1, \overline{M}_1].$$

Thus, we apply (R) to conclude that

$$x''(\gamma) = \lambda f\big(\gamma, x(\gamma), x'(\gamma)\big) \le 0,$$

which contradicts the inequality $x''(\gamma) > 0$. This has been established above. Thus, S_- is empty and as a result

$$x'(t) \le M_1 \quad \text{for } t \in [a, T].$$

Now, by the mean value theorem for each $t \in (a, T]$ there exists a $\xi \in (a, t)$ such that

$$x(t) - x(a) = x'(\xi)(t - a),$$

which yields

$$x(t) \le M_0 \quad \text{for } t \in [a, T].$$

This allows us to use (3.3) to show similarly to above that the set

$$S_+ = \big\{t \in [a, T] : \overline{m}_1 \le x'(t) < m_1\big\}$$

is empty. Hence,

$$0 \le m_1 \le x'(t) \quad \text{for } t \in [a, T]$$

and so

$$A \le x(t) \quad \text{for } t \in [a, T].$$

To estimate $x''(t)$, we observe firstly that (R) implies in particular

$$f(t, x, M_1) \le 0 \quad \text{for } (t, x) \in [a, T] \times [A, M_0]$$

and

$$f(t, x, m_1) \ge 0 \quad \text{for } (t, x) \in [a, T] \times [A, M_0],$$

which yield $m_2 \le 0$ and $M_2 \ge 0$. Multiplying both sides of the inequality $\lambda \le 1$ by m_2 and M_2, we get, respectively, $m_2 \le \lambda m_2$ and $\lambda M_2 \le M_2$. On the other hand, we have established

$$x(t) \in [A, M_0] \quad \text{and} \quad x'(t) \in [m_1, M_1] \quad \text{for } t \in [a, T].$$

Thus,

$$m_2 \le \lambda m_2 \le \lambda f\big(t, x(t), x'(t)\big) \le \lambda M_2 \le M_2 \quad \text{for } t \in [a, T]$$

and each $\lambda \in [0,1]$ and so

$$x''(t) \in [m_2, M_2] \quad \text{for } t \in [a,T]. \qquad \square$$

Let us mention that some analogous results have been obtained in Kelevedjiev [20]. For completeness of our explanations, we present the full proofs here.

Now we prove an existence result guaranteeing the solvability of IVP (3.1).

Theorem 3.2 *Let* (R) *hold. Then nonsingular problem* (3.1) *has at least one non-decreasing solution in* $C^2[a,T]$.

Proof Preparing the application of Theorem 2.2, we define first the set

$$U = \{x \in C_I^2[a,T] : A - \tau < x < M_0 + \tau, m_1 - \tau < x' < M_1 + \tau, m_2 - \tau < x'' < M_2 + \tau\},$$

where $C_I^2[a,T] = \{x \in C^2[a,T] : x(a) = A, x'(a) = B\}$. It is important to notice that according to Lemma 3.1 all $C^2[a,T]$-solutions to family (3.2) are interior points of U. Further, we introduce the continuous maps

$$j : C_I^2[a,T] \to C^1[a,T] \quad \text{by } jx = x,$$
$$V : C_I^2[a,T] \to C[a,T] \quad \text{by } Vx = x'',$$

and for $t \in [a,T]$ and $x(t) \in j(\overline{U})$ the map

$$\Phi : C^1[a,T] \to C[a,T] \quad \text{by } (\Phi x)(t) = f(t, x(t), x'(t)).$$

Clearly, the map Φ is also continuous since, by assumption, the function $f(t, x(t), x'(t))$ is continuous on $[a,T]$ if

$$x(t) \in [m_0 - \tau, M_0 + \tau] \quad \text{and} \quad x'(t) \in [m_1 - \tau, M_1 + \tau] \quad \text{for } t \in [a,T].$$

In addition we verify that V^{-1} exists and is also continuous. To this aim we introduce the linear map

$$W : C_{I_0}^2[a,T] \to C[a,T],$$

defined by $Wx = x''$, where $C_{I_0}^2[a,T] = \{x \in C^2[a,T] : x(a) = 0, x'(a) = 0\}$. It is one-to-one because each function $x \in C_{I_0}^2[a,T]$ has a unique image, and each function $y \in C[a,T]$ has a unique inverse image which is the unique solution to the IVP

$$x'' = y, \qquad x(a) = 0, \qquad x'(a) = 0.$$

It is not hard to see that W is bounded and so, by the bounded inverse theorem, the map W^{-1} exists and is linear and bounded. Thus, it is continuous. Now, using W^{-1}, we define

$$V^{-1} : C[a,T] \to C_I^2[a,T] \quad \text{by } (V^{-1}y)(t) = \ell(t) + (W^{-1}y)(t),$$

where $\ell(t) = B(t-a) + A$ is the unique solution of the problem

$$x'' = 0, \qquad x(a) = A, \qquad x'(a) = B.$$

Clearly, V^{-1} is continuous since W^{-1} is continuous.

We already can introduce a homotopy

$$\mathrm{H} : \overline{U} \times [0,1] \to C_I^2[a,T],$$

defined by

$$\mathrm{H}(x,\lambda) \equiv \mathrm{H}_\lambda(x) \equiv \lambda V^{-1}\Phi j(x) + (1-\lambda)\ell.$$

It is well known that j is completely continuous, that is, j maps each bounded subset of $C_I^2[a,T]$ into a compact subset of $C^1[a,T]$. Thus, the image $j(\overline{U})$ of the bounded set U is compact. Now, from the continuity of Φ and V^{-1} it follows that the sets $\Phi(j(\overline{U}))$ and $V^{-1}(\Phi(j(\overline{U})))$ are also compact. In summary, we have established that the homotopy is compact. On the other hand, for its fixed points we have

$$\lambda V^{-1}\Phi j(x) + (1-\lambda)\ell = x$$

and

$$Vx = \lambda \Phi j(x),$$

which is the operator form of family (3.2). So, each fixed point of H_λ is a solution to (3.2), which, according to Lemma 3.1, lies in U. Consequently, the homotopy is fixed point free on ∂U.

Finally, $\mathrm{H}_0(x)$ is a constant map mapping each function $x \in \overline{U}$ to $\ell(t)$. Thus, according to Theorem 2.1, $\mathrm{H}_0(x) = \ell$ is essential.

So, all assumptions of Theorem 2.2 are fulfilled. Hence $\mathrm{H}_1(x)$ has a fixed point in U which means that the IVP of (3.2) obtained for $\lambda = 1$ (*i.e.* (3.1)) has at least one solution $x(t)$ in $C^2[a,T]$. From Lemma 3.1 we know that

$$x'(t) \geq m_1 \geq 0 \quad \text{for } t \in [a,T],$$

from which its monotony follows. □

The validity of the following results follows similarly.

Theorem 3.3 *Let $B > 0$ and let* (R) *hold for $m_1 > 0$. Then problem* (3.1) *has at least one strictly increasing solution in $C^2[a,T]$.*

Theorem 3.4 *Let $A > 0$ $(A = 0)$ and let* (R) *hold for $m_1 = 0$. Then problem* (3.1) *has at least one positive* (*nonnegative*) *non-decreasing solution in $C^2[a,T]$.*

Theorem 3.5 *Let $A \geq 0$, $B > 0$ and let* (R) *hold for $m_1 > 0$. Then problem* (3.1) *has at least one strictly increasing solution in $C^2[a,T]$ with positive values for $t \in (a,T)$.*

4 A problem singular at x and x'

In this section we study the solvability of singular IVP (1.1), (1.2) under the following assumptions.

(S_1) There are constants $T > 0$, m_1, $\overline{m}_1$ and a sufficiently small $\nu > 0$ such that

$$m_1 > 0, \qquad B > m_1 \geq \overline{m}_1 + \nu,$$
$$[0,T] \subseteq D_t, \qquad (A, \tilde{M}_0 + \nu] \subseteq D_x, \qquad [\overline{m}_1, B) \subseteq D_p,$$

where $\tilde{M}_0 = A + BT + 1$,

$$f(t,x,p) \in C\big([0,T] \times (A, \tilde{M}_0 + \nu] \times [m_1 - \nu, B)\big), \tag{4.1}$$
$$f(t,x,p) \leq 0 \quad \text{for } (t,x,p) \in \big([0,T] \times D_x \times [m_1, B)\big) \backslash S_A \tag{4.2}$$

and

$$f(t,x,p) \geq 0 \quad \text{for } (t,x,p) \in \big([0,T] \times D_{\tilde{M}_0} \times [\overline{m}_1, m_1]\big) \backslash S_A,$$

where $D_{\tilde{M}_0} = (-\infty, \tilde{M}_0] \cap D_x$.

(S_2) For some $\alpha \in (0,T]$ and $\mu \in (m_1, B)$ there exists a constant $k < 0$ such that $k\alpha + B > \mu$ and

$$f(t,x,p) \leq k < 0 \quad \text{for } (t,x,p) \in [0,\alpha] \times (A, \tilde{M}_0] \times [\mu, B),$$

where T, m_1 and $\tilde{M}_0$ are as in (S_1).

Now, for $n \geq n_{\alpha,\mu}$, where $n_{\alpha,\mu} > \max\{\alpha^{-1}, (B + k\alpha - \mu)^{-1}\}$, and α, μ, and k are as in (S_2), we construct the following family of regular IVPs:

$$\begin{cases} x'' = f(t,x,x'), \\ x(0) = A + n^{-1}, \qquad x'(0) = B - n^{-1}. \end{cases} \tag{4.3}$$

Notice, for $n \geq n_{\alpha,\mu}$, that we have $B - n^{-1} > \mu - k\alpha > \mu > m_1 > 0$.

Lemma 4.1 *Let* (S_1) *and* (S_2) *hold and let* $x_n \in C^2[0,T]$, $n \geq n_{\alpha,\mu}$, *be a solution to* (4.3) *such that*

$$A < x_n(t) \leq \tilde{M}_0 \quad \text{and} \quad m_1 \leq x_n'(t) < B \quad \text{for } t \in [0,T].$$

Then the following bound is satisfied for each $n \geq n_{\alpha,\mu}$:

$$x_n'(t) < \phi_\alpha(t) < B \quad \text{for } t \in (0,T],$$

where $\phi_\alpha(t) = \begin{cases} kt + B, & t \in [0,\alpha], \\ k\alpha + B, & t \in (\alpha, T]. \end{cases}$

Proof Since for each $n \geq n_{\alpha,\mu}$ we have

$$x_n'(0) = B - n^{-1} > \mu - k\alpha > \mu,$$

we will consider the proof for an arbitrary fixed $n \geq n_{\alpha,\mu}$, considering two cases. Namely, $x_n'(t) > \mu$ for $t \in [0,\alpha]$ is the first case and the second one is $x_n'(t) > \mu$ for $t \in [0,\beta)$ with $x_n'(\beta) = \mu$ for some $\beta \in (0,\alpha]$.

Case 1. From $\mu < x_n'(t) \leq B$, $t \in [0,\alpha]$, and (S_2) we have

$$x_n''(t) = f\bigl(t, x_n(t), x_n'(t)\bigr) \leq k \quad \text{for } t \in [0,\alpha],$$

i.e. $x_n''(t) \leq k$ for $t \in [0,\alpha]$. Integrating the last inequality from 0 to t we get

$$x_n'(t) - x_n'(0) \leq kt, \quad t \in [0,\alpha],$$

which yields

$$x_n'(t) \leq kt + B - n^{-1} \quad \text{for } t \in [0,\alpha].$$

Now $m_1 \leq x_n'(t) < B$, $t \in [0,T]$, and (4.2) imply

$$x_n''(t) = f\bigl(t, x_n(t), x_n'(t)\bigr) \leq 0 \quad \text{for } t \in [0,T].$$

In particular $x_n''(t) \leq 0$ for $t \in [\alpha, T]$, thus

$$x_n'(t) \leq x_n'(\alpha) \leq k\alpha + B - n^{-1} \quad \text{for } t \in (\alpha, T].$$

Case 2. As in the first case, we derive

$$x_n'(t) \leq kt + B - n^{-1} \quad \text{for } t \in [0,\beta].$$

On the other hand, since $m_1 \leq x_n'(t) < B$ for $t \in [\beta, T]$, again from (4.2) it follows that

$$x_n''(t) = f\bigl(t, x_n(t), x_n'(t)\bigr) \leq 0 \quad \text{for } t \in [\beta, T],$$

which yields

$$x_n'(t) \leq x_n'(\beta) = \mu < k\alpha + B - n^{-1} \leq kt + B - n^{-1} \quad \text{for } t \in [\beta,\alpha]$$

and

$$x_n'(t) < k\alpha + B - n^{-1} \quad \text{for } t \in (\alpha, T].$$

So, as a result of the considered cases we get

$$x_n'(t) \leq \begin{cases} kt + B - n^{-1}, & t \in [0,\alpha], \\ k\alpha + B - n^{-1}, & t \in (\alpha, T] \end{cases} < \phi_\alpha(t) \quad \text{for } t \in [0,T] \text{ and } n \geq n_{\alpha,\mu},$$

from which the assertion follows immediately. □

Having this lemma, we prove the basic result of this section.

Theorem 4.2 *Let* (S_1) *and* (S_2) *hold. Then singular IVP* (1.1), (1.2) *has at least one strictly increasing solution in* $C[0,T] \cap C^2(0,T]$ *such that*

$$m_1 t + A \le x(t) \le Bt + A \quad \text{for } t \in [0,T], \qquad m_1 \le x'(t) < B \quad \text{for } t \in (0,T].$$

Proof For each fixed $n \ge n_{\alpha,\mu}$ introduce $\tau = \min\{(2n)^{-1}, \nu\}$,

$$M_1 = B - n^{-1}, \qquad \overline{M}_1 = B - (2n)^{-1} \quad \text{and} \quad M_0 = \left(B - n^{-1}\right)T + A + 1 < \tilde{M}_0$$

having the properties

$$\overline{M}_1 - \tau > M_1 = B - n^{-1} > \mu - k\alpha > \mu > m_1 \ge \overline{m}_1 + \tau,$$
$$[0,T] \subseteq D_t, \qquad \left[A + n^{-1} - \tau, M_0 + \tau\right] \subseteq (A, \tilde{M}_0 + \tau] \subseteq D_x$$

and $[\overline{m}_1, \overline{M}_1] \subseteq D_p$ since $\overline{M}_1 = B - (2n)^{-1} < B$. Besides,

$$f(t,x,p) \le 0 \quad \text{for } (t,x,p) \in \left([0,T] \times D_x \times [M_1, \overline{M}_1]\right) \backslash S_A,$$
$$f(t,x,p) \ge 0 \quad \text{for } (t,x,p) \in \left([0,T] \times \left(D_x \times (-\infty, M_0]\right) \times [\overline{m}_1, m_1]\right) \backslash S_A$$

and, in view of (4.1),

$$f(t,x,p) \in C\left([0,T] \times \left[A + n^{-1} - \tau, M_0 + \nu\right] \times [m_1 - \tau, M_1 + \tau]\right).$$

All this implies that for each $n \ge n_{\alpha,\mu}$ the corresponding IVP of family (4.3) satisfies (R). Thus, we apply Theorem 3.2 to conclude that (4.3) has a solution $x_n \in C^2[0,T]$ for each $n \ge n_{\alpha,\mu}$. We can use also Lemma 3.1 to conclude that for each $n \ge n_{\alpha,\mu}$ and $t \in [0,T]$ we have

$$A < A + n^{-1} \le x_n(t) \le M_0 < \tilde{M}_0 \tag{4.4}$$

and

$$m_1 \le x_n'(t) \le B - n^{-1} < B.$$

Now, these bounds allow the application of Lemma 4.1 from which one infers that for each $n \ge n_{\alpha,\mu}$ and $t \in [0,T]$ the bounds

$$m_1 \le x_n'(t) < \phi_\alpha(t) \le B \tag{4.5}$$

hold. For later use, integrating the least inequality from 0 to t, $t \in (0,T]$, we get

$$m_1 t + A + n^{-1} \le x_n(t) < Bt + A + n^{-1} \quad \text{for } t \in [0,T] \tag{4.6}$$

and $n \ge n_{\alpha,\mu}$.

We consider firstly the sequence $\{x_n\}$ of $C^2[0,T]$-solutions of (4.3) only for each $n \ge n_{\alpha,\mu}$. Clearly, for each $n \ge n_{\alpha,\mu}$ we have in particular

$$x_n'(t) \ge m_1 > 0 \quad \text{for } t \in [\alpha, T],$$

which together with (4.6) gives

$$x_n(t) \geq x_n(\alpha) \geq m_1\alpha + A + n^{-1} > A_1 > A \quad \text{for } t \in [\alpha, T],$$

where $A_1 = m_1\alpha + A$. On combining the last inequality and (4.4) we obtain

$$A_1 < x_n(t) < \tilde{M}_0 \quad \text{for } t \in [\alpha, T], n \geq n_{\alpha,\mu}. \tag{4.7}$$

From (4.5) we have in addition

$$m_1 \leq x_n'(t) < \phi_\alpha(\alpha) = B + k\alpha \quad \text{for } t \in [\alpha, T], n \geq n_{\alpha,\mu}. \tag{4.8}$$

Now, using the fact that (4.1) implies continuity of $f(t,x,p)$ on the compact set $[\alpha, T] \times [A_1, \tilde{M}_0] \times [m_1, \phi_\alpha(\alpha)]$ and keeping in mind that for each $n \geq n_{\alpha,\mu}$

$$x_n''(t) = f\bigl(t, x_n(t), x_n'(t)\bigr) \quad \text{for } t \in [\alpha, T],$$

we conclude that there is a constant M_2, independent of n, such that

$$\bigl|x_n''(t)\bigr| \leq M_2 \quad \text{for } t \in [\alpha, T] \text{ and } n \geq n_{\alpha,\mu}.$$

Using the obtained *a priori* bounds for $x_n(t)$, $x_n'(t)$ and $x_n''(t)$ on the interval $[\alpha, T]$, we apply the Arzela-Ascoli theorem to conclude that there exists a subsequence $\{x_{n_k}\}$, $k \in \mathbb{N}$, $n_k \geq n_{\alpha,\mu}$, of $\{x_n\}$ and a function $x_\alpha \in C^1[\alpha, T]$ such that

$$\|x_{n_k} - x_\alpha\|_1 \to 0 \quad \text{on the interval } [\alpha, T],$$

i.e., the sequences $\{x_{n_k}\}$ and $\{x_{n_k}'\}$ converge uniformly on the interval $[\alpha, T]$ to x_α and x_α', respectively. Obviously, (4.7) and (4.8) are valid in particular for the elements of $\{x_{n_k}\}$ and $\{x_{n_k}'\}$, respectively, from which, letting $k \to \infty$, one finds

$$A_1 \leq x_\alpha(t) \leq \tilde{M}_0 \quad \text{for } t \in [\alpha, T],$$
$$m_1 \leq x_\alpha'(t) \leq \phi_\alpha(\alpha) < B \quad \text{for } t \in [\alpha, T].$$

Clearly, the functions $x_{n_k}(t)$, $k \in \mathbb{N}$, $n_k \geq n_{\alpha,\mu}$, satisfy integral equations of the form

$$x_{n_k}'(t) = x_{n_k}'(\alpha) + \int_\alpha^t f\bigl(s, x_{n_k}(s), x_{n_k}'(s)\bigr)\, ds, \quad t \in (\alpha, T].$$

Now, since $f(t,x,p)$ is uniformly continuous on the compact set $[\alpha, T] \times [A_1, \tilde{M}_0] \times [m_1, \phi_\alpha(\alpha)]$, from the uniform convergence of $\{x_{n_k}\}$ it follows that the sequence $\{f(s, x_{n_k}(s), x_{n_k}'(s))\}$, $n_k \geq n_{\alpha,\mu}$ is uniformly convergent on $[\alpha, T]$ to the function $f(s, x_\alpha(s), x_\alpha'(s))$, which means

$$\lim_{k\to\infty} \int_\alpha^t f\bigl(s, x_{n_k}(s), x_{n_k}'(s)\bigr)\, ds = \int_\alpha^t f\bigl(s, x_\alpha(s), x_\alpha'(s)\bigr)\, ds$$

for each $t \in (\alpha, T]$. Returning to the integral equation and letting $k \to \infty$ yield

$$x'_\alpha(t) = x'_\alpha(\alpha) + \int_\alpha^t f\bigl(s, x_\alpha(s), x'_\alpha(s)\bigr)\, ds, \quad t \in (\alpha, T],$$

which implies that $x_\alpha(t)$ is a $C^2(\alpha, T]$-solution to the differential equation $x'' = f(t, x, x')$ on $(\alpha, T]$. Besides, (4.6) implies

$$m_1 t + A \le x_\alpha(t) \le Bt + A \quad \text{for } t \in [\alpha, T].$$

Further, we observe that if the condition (S_2) holds for some $\alpha > 0$, then it is true also for an arbitrary $\alpha_0 \in (0, \alpha)$. We will use this fact considering a sequence $\{\alpha_i\} \subset (0, \alpha)$, $i \in \mathbb{N}$, with the properties

$$\alpha_{i+1} < \alpha_i \quad \text{for } i \in \mathbb{N} \text{ and } \lim_{i\to\infty} \alpha_i = 0.$$

For each $i \in \mathbb{N}$ we consider sequences

$$\{x_{i,n_k}\}, \quad n_k \ge n_{i+1,\mu}, k \in \mathbb{N}, n_{i+1,\mu} > \max\bigl\{\alpha_{i+1}^{-1}, (B + k\alpha_{i+1} - \mu)^{-1}\bigr\},$$

on the interval $[\alpha_{i+1}, T]$. Thus, we establish that each sequence $\{x_{i,n_k}\}$ has a subsequence $\{x_{i+1,n_k}\}$, $k \in \mathbb{N}$, $n_k \ge n_{i+1,\mu}$, converging uniformly on the interval $[\alpha_{i+1}, T]$ to any function $x_{\alpha_{i+1}}(t)$, $t \in [\alpha_{i+1}, T]$, that is,

$$\|x_{i+1,n_k} - x_{\alpha_{i+1}}\|_1 \to 0 \quad \text{on } [\alpha_{i+1}, T], \tag{4.9}$$

which is a $C^2(\alpha_{i+1}, T]$-solution to the differential equation $x''(t) = f(t, x(t), x'(t))$ on $(\alpha_{i+1}, T]$ and

$$m_1 t + A \le x_{\alpha_{i+1}}(t) \le Bt + A \quad \text{for } t \in [\alpha_{i+1}, T],$$
$$m_1 \le x'_{\alpha_{i+1}}(t) \le \phi_\alpha(\alpha_{i+1}) < B \quad \text{for } t \in [\alpha_{i+1}, T],$$
$$x_{\alpha_{i+1}}(t) = x_{\alpha_i}(t) \quad \text{and} \quad x'_{\alpha_{i+1}}(t) = x'_{\alpha_i}(t) \quad \text{for } t \in [\alpha_i, T].$$

The properties of the functions from $\{x_{\alpha_i}\}$, $i \in \mathbb{N}$, imply that there exists a function $x_0(t)$ which is a $C^2(0, T]$-solution to the equation $x'' = f(t, x, x')$ on the interval $(0, T]$ and is such that

$$m_1 t + A \le x_0(t) \le Bt + A \quad \text{for } t \in (0, T],$$

hence $\lim_{t\to 0^+} x_0(t) = A$,

$$\begin{aligned} &m_1 \le x'_0(t) \le \phi_\alpha(t) < B \quad \text{for } t \in (0, T], \\ &x_0(t) = x_{\alpha_i}(t) \quad \text{for } t \in [\alpha_i, T] \text{ and } i \in \mathbb{N}, \\ &x'_0(t) = x'_{\alpha_i}(t) \quad \text{for } t \in [\alpha_i, T] \text{ and } i \in \mathbb{N}. \end{aligned} \tag{4.10}$$

We have to show also that

$$\lim_{t\to 0^+} x_0'(t) = B. \tag{4.11}$$

Reasoning by contradiction, assume that there exists a sufficiently small $\varepsilon > 0$ such that for every $\delta > 0$ there is a $t \in (0,\delta)$ such that

$$x_0'(t) < B - \varepsilon.$$

In other words, assume that for every sequence $\{\delta_j\} \subset (0,T]$, $j \in \mathbb{N}$, with $\lim_{j\to\infty} \delta_j = 0$, there exists a sequence $\{t_j\}$ having the properties $t_j \in (0,\delta_j)$, $\lim_{j\to\infty} t_j = 0$ and

$$x_0'(t_j) < B - \varepsilon. \tag{4.12}$$

It is clear that every interval $(0,\delta_j)$, $j \in \mathbb{N}$, contains a subsequence of $\{t_j\}$ converging to 0. Besides, from (4.9) and (4.10) it follows that for every $j \in \mathbb{N}$ there are $i_j, n_j \in \mathbb{N}$ such that $\alpha_{i_j} < \delta_j$ and

$$\| x_{i,n_k}' - x_0' \| \to 0 \quad \text{on } [\alpha_i, \delta_j) \tag{4.13}$$

for all $i > i_j$ and all $n_k \geq \max\{n_{i,\mu}, n_j\}$. Moreover, since the accumulation point of $\{t_j\}$ is 0, for each sufficiently large $j \in \mathbb{N}$ there is a $t_j \in [\alpha_i, \delta_j)$ where $i > i_j$. In summary, for every sufficiently large $j \in \mathbb{N}$, that is, for every sufficiently small $\delta_j > 0$, there are $i_j, n_j \in \mathbb{N}$ such that for all $i > i_j$ and $n_k \geq \max\{n_{i,\mu}, n_j\}$ from (4.12) and (4.13) we have

$$x_{i,n_k}'(t_j) < B - \varepsilon,$$

which contradicts to the fact that $x_{i,n_k}'(0) = B - n_k^{-1}$ and $x_{i,n_k}' \in C[0,T]$. This contradiction proves that (4.11) is true.

Now, it is easy to verify that the function

$$x(t) = \begin{cases} A, & t = 0, \\ x_0(t), & t \in (0,T], \end{cases}$$

is a $C[0,T] \cap C^2(0,T]$-solution to (1.1), (1.2). This function is strictly increasing because $x'(t) = x_0'(t) \geq m_1 > 0$ for $t \in (0,T]$, and the bounds for $x(t)$ and $x'(t)$ follows immediately from the corresponding bounds for $x_0(t)$ and $x_0'(t)$. □

The following results provide information about the presence of other useful properties of the assured solutions. Their correctness follows directly from Theorem 4.2.

Theorem 4.3 *Let* $A \geq 0$ *and let* (S_1) *and* (S_2) *hold. Then the singular IVP* (1.1), (1.2) *has at least one strictly increasing solution in* $C[0,T] \cap C^2(0,T]$ *with positive values for* $t \in (0,T]$.

5 Examples

Example 5.1 Consider the IVP

$$x'' = \frac{\sqrt{b^2 - x^2}}{\sqrt{c^2 - t^2}} P_k(x'),$$

$$x(0) = 0, \qquad x'(0) = B, \quad B > 0,$$

where $b, c \in (0, \infty)$, and the polynomial $P_k(p)$, $k \geq 2$, has simple zeroes p_1 and p_2 such that

$$0 < p_1 < B < p_2.$$

Let us note that here $D_t = (-c, c)$, $D_x = [-b, b]$ and $D_p = \mathbb{R}$.

Clearly, there is a sufficiently small $\theta > 0$ such that

$$0 < p_1 - \theta, \qquad p_1 + \theta \leq B \leq p_2 - \theta$$

and $P_k(p) \neq 0$ for $p \in [p_1 - \theta, p_1) \cup (p_1, p_1 + \theta] \cup [p_2 - \theta, p_2) \cup (p_2, p_2 + \theta]$.

We will show that all assumptions of Theorem 3.2 are fulfilled in the case

$$P_k(p) > 0 \quad \text{for } p \in [p_1 - \theta, p_1) \quad \text{and} \quad P_k(p) < 0 \quad \text{for } p \in (p_2, p_2 + \theta];$$

the other cases as regards the sign of $P_k(p)$ around p_1 and p_2 may be treated similarly. For this case choose $\tau = \theta/2$, $m_1 = p_1 > 0$ and $M_1 = p_2$. Next, using the requirement $[A - \tau, M_0 + \tau] \subseteq [-b, b]$, *i.e.* $[-\theta/2, p_2 T_0 + \theta/2] \subseteq [-b, b]$, we get the following conditions for θ and T:

$$-\theta/2 \geq -b \quad \text{and} \quad p_2 T_0 + \theta/2 \leq b,$$

which yield $\theta \in (0, 2b]$ and $T \leq \frac{2b-\theta}{2p_2}$. Besides, $[0, T] \subseteq (-c, c)$ yields $T < C$. Thus, $0 < T < \min\{c, \frac{2b-\theta}{2p_2}\}$. Now, choosing

$$\overline{m}_1 = p_1 - \theta \quad \text{and} \quad \overline{M}_1 = p_2 + \theta,$$

we really can apply Theorem 3.2 to conclude that the considered problem has a strictly increasing solution $x \in C^2[0, T]$ with $x(t) > 0$ on $t \in (0, T]$ for each $T < \min\{c, \frac{2b-\theta}{2p_2}\}$.

Example 5.2 Consider the IVP

$$x'' = \frac{(x' - 5)(15 - x')}{(x - 2)^2(x' - 10)},$$
$$x(0) = 2, \qquad \lim_{t \to 0^+} x'(t) = 10.$$

Notice that here

$$S_A = \mathbb{R} \times \{2\} \times \big((-\infty, 10) \cup (10, \infty)\big), \qquad S_B = \mathbb{R} \times \big((-\infty, 2) \cup (2, \infty)\big) \times \{10\}.$$

It is easy to check that (S_1) holds, for example, for $\overline{m}_1 = 4$, $m_1 = 5$, $\nu = 0.1$, and an arbitrary fixed $T > 0$, moreover, $\tilde{M}_0 = 10T + 3$. Besides, for $k = -24/(10T + 1)^2$, $\alpha = T/100$ and $\mu = 9$, for example, we have

$$k\alpha + B = -24T/100(10T + 1)^2 + 10 > 9 = \mu$$

and $f(t, x, p) \leq -24/(10T + 1)^2$ on $[0, T/100] \times (2, 10T + 3] \times [9, 10)$, which means that ($S_2$) also holds. By Theorem 4.3, the considered IVP has at least one positive strictly increasing solution in $C[0, T] \cap C^2(0, T]$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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References

1. Rachůnková, I, Tomeček, J: Bubble-type solutions of non-linear singular problem. Math. Comput. Model. **51**, 658-669 (2010)
2. Rachůnková, I, Tomeček, J: Homoclinic solutions of singular nonautonomous second-order differential equations. Bound. Value Probl. **2009**, Article ID 959636 (2009)
3. Rachůnková, I, Tomeček, J: Strictly increasing solutions of a nonlinear singular differential equation arising in hydrodynamics. Nonlinear Anal. **72**, 2114-2118 (2010)
4. Agarwal, RP, O'Regan, D: Second-order initial value problems of singular type. J. Math. Anal. Appl. **229**, 441-451 (1999)
5. Yang, G: Minimal positive solutions to some singular second-order differential equations. J. Math. Anal. Appl. **266**, 479-491 (2002)
6. Yang, G: Positive solutions of some second-order nonlinear singular differential equations. Comput. Math. Appl. **45**, 605-614 (2003)
7. Bobisud, LE, O'Regan, D: Existence of solutions to some singular initial value problems. J. Math. Anal. Appl. **133**, 215-230 (1988)
8. Bobisud, LE, Lee, YS: Existence of monotone or positive solutions of singular second-order sublinear differential equations. J. Math. Anal. Appl. **159**, 449-468 (1991)
9. Cabada, A, Heikkilä, S: Extremality results for discontinuous explicit and implicit diffusion problems. J. Comput. Appl. Math. **143**, 69-80 (2002)
10. Cabada, A, Cid, JA, Pouso, RL: Positive solutions for a class of singular differential equations arising in diffusion processes. Dyn. Contin. Discrete Impuls. Syst., Ser. A Math. Anal. **12**, 329-342 (2005)
11. Cabada, A, Nieto, JJ, Pouso, RL: Approximate solutions to a new class of nonlinear diffusion problems. J. Comput. Appl. Math. **108**, 219-231 (1999)
12. Cid, JA: Extremal positive solutions for a class of singular and discontinuous second order problems. Nonlinear Anal. **51**, 1055-1072 (2002)
13. Maagli, H, Masmoudi, S: Existence theorem of nonlinear singular boundary value problem. Nonlinear Anal. **46**, 465-473 (2001)
14. Zhao, Z: Positive solutions of nonlinear second order ordinary differential equations. Proc. Am. Math. Soc. **121**, 465-469 (1994)
15. Kelevedjiev, P, Popivanov, N: On the solvability of a second-order initial value problem. Paper presented at the 40th international conference on the applications of mathematics in engineering and economics, Technical University of Sofia, Sozopol, 8-13 June 2014
16. Granas, A, Guenther, RB, Lee, JW: Nonlinear boundary value problems for ordinary differential equations. Diss. Math. **244**, 1-128 (1985)
17. Grammatikopoulos, MK, Kelevedjiev, PS, Popivanov, N: On the solvability of a singular boundary-value problem for the equation $f(t,x,x',x'') = 0$. J. Math. Sci. **149**, 1504-1516 (2008)
18. Kelevedjiev, P, Popivanov, N: Second order boundary value problems with nonlinear two-point boundary conditions. Georgian Math. J. **7**, 677-688 (2000)
19. Palamides, P, Kelevedjiev, P, Popivanov, N: On the solvability of a Neumann boundary value problem for the differential equation $f(t,x,x',x'') = 0$. Bound. Value Probl. **2012**, 77 (2012). doi:10.1186/1687-2770-2012-77
20. Kelevedjiev, P: Positive solutions of nonsingular and singular second order initial value problems. Int. Electron. J. Pure Appl. Math. **2**, 117-127 (2010)

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