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Biharmonic equations with improved subcritical polynomial growth and subcritical exponential growth

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Abstract
The main purpose of this paper is to establish the existence of two nontrivial solutions and the existence of infinitely many solutions for a class of fourth-order elliptic equations with subcritical polynomial growth and subcritical exponential growth by using a suitable version of the mountain pass theorem and the symmetric mountain pass theorem.

Keywords: mountain pass theorem; Adams-type inequality; subcritical polynomial growth; subcritical exponential growth

1 Introduction

Consider the following Navier boundary value problem:

$$\begin{cases} \triangle^2 u(x) + c\triangle u = f(x,u), & \text{in } \Omega; \\ u = \triangle u = 0, & \text{in } \partial\Omega, \end{cases} \tag{1}$$

where $\triangle^2$ is the biharmonic operator and Ω is a bounded smooth domain in $\mathbb{R}^N$ $(N \geq 4)$.

In problem (1), let $f(x,u) = b[(u+1)^+ - 1]$, then we get the following Dirichlet problem:

$$\begin{cases} \triangle^2 u(x) + c\triangle u = b[(u+1)^+ - 1], & \text{in } \Omega; \\ u = \triangle u = 0, & \text{in } \partial\Omega, \end{cases} \tag{2}$$

where $u^+ = \max\{u,0\}$ and $b \in \mathbb{R}$. We let λ_k $(k = 1,2,\ldots)$ denote the eigenvalues of $-\triangle$ in $H_0^1(\Omega)$.

Thus, fourth-order problems with $N > 4$ have been studied by many authors. In [1], Lazer and McKenna pointed out that this type of nonlinearity furnishes a model to study traveling waves in suspension bridges. Since then, more general nonlinear fourth-order elliptic boundary value problems have been studied. For problem (2), Lazer and McKenna [2] proved the existence of $2k-1$ solutions when $N = 1$, and $b > \lambda_k(\lambda_k - c)$ by the global bifurcation method. In [3], Tarantello found a negative solution when $b \geq \lambda_1(\lambda_1 - c)$ by a degree argument. For problem (1) when $f(x,u) = bg(x,u)$, Micheletti and Pistoia [4] proved that there exist two or three solutions for a more general nonlinearity g by the variational method. Xu and Zhang [5] discussed the problem when f satisfies the local superlinearity and sublinearity. Zhang [6] proved the existence of solutions for a more general nonlinearity $f(x,u)$ under some weaker assumptions. Zhang and Li [7] proved the existence of multiple nontrivial solutions by means of Morse theory and local linking. An and Liu [8] and Liu and Wang [9] also obtained the existence result for nontrivial solutions when f is asymptotically linear at positive infinity.

We noticed that almost all of works (see [4–9]) mentioned above involve the nonlinear term $f(x,u)$ of a subcritical (polynomial) growth, say,

(SCP): there exist positive constants c_1 and c_2 and $q_0 \in (1, p^* - 1)$ such that

$$\left|f(x,t)\right| \le c_1 + c_2|t|^{q_0} \quad \text{for all } t \in \mathbb{R} \text{ and } x \in \Omega,$$

where $p^* = 2N/(N-4)$ denotes the critical Sobolev exponent. One of the main reasons to assume this condition (SCP) is that they can use the Sobolev compact embedding $H^2(\Omega) \cap H_0^1(\Omega) \hookrightarrow L^q(\Omega)$ $(1 \le q < p^*)$. At that time, it is easy to see that seeking a weak solution of problem (1) is equivalent to finding a nonzero critical points of the following functional on $H^2(\Omega) \cap H_0^1(\Omega)$:

$$I(u) = \frac{1}{2}\int_\Omega \left(|\triangle u|^2 - c|\nabla u|^2\right) dx - \int_\Omega F(x,u)\, dx, \quad \text{where } F(x,u) = \int_0^u f(x,t)\, dt. \tag{3}$$

In this paper, stimulated by Lam and Lu [10], our first main results will be to study problem (1) in the improved subcritical polynomial growth

$$\text{(SCPI):} \quad \lim_{t\to\infty} \frac{f(x,t)}{|t|^{p^*-1}} = 0$$

which is much weaker than (SCP). Note that in this case, we do not have the Sobolev compact embedding anymore. Our work is to study problem (1) when nonlinearity f does not satisfy the (AR) condition, *i.e.*, for some $\theta > 2$ and $\gamma > 0$,

$$0 < \theta F(x,t) \le f(x,t)t \quad \text{for all } |t| \ge \gamma \text{ and } x \in \Omega. \tag{AR}$$

In fact, this condition was studied by Liu and Wang in [11] in the case of Laplacian by the Nehari manifold approach. However, we will use a suitable version of the mountain pass theorem to get the nontrivial solution to problem (1) in the general case $N > 4$. We will also use the symmetric mountain pass theorem to get infinitely many solutions for problem (1) in the general case $N > 4$ when nonlinearity f is odd.

Let us now state our results. In this paper, we always assume that $f(x,t) \in C(\bar{\Omega} \times \mathbb{R})$. The conditions imposed on $f(x,t)$ are as follows:

(H_1) $f(x,t)t \ge 0$ for all $x \in \Omega$, $t \in \mathbb{R}$;
(H_2) $\lim_{|t|\to 0} \frac{f(x,t)}{t} = f_0$ uniformly for $x \in \Omega$, where f_0 is a constant;
(H_3) $\lim_{|t|\to\infty} \frac{f(x,t)}{t} = +\infty$ uniformly for $x \in \Omega$;
(H_4) $\frac{f(x,t)}{|t|}$ is nondecreasing in $t \in \mathbb{R}$ for any $x \in \Omega$.

Let $0 < \mu_1 < \mu_2 < \cdots < \mu_k < \cdots$ be the eigenvalues of $(\triangle^2 - c\triangle, H^2(\Omega) \cap H_0^1(\Omega))$ and $\varphi_1(x) > 0$ be the eigenfunction corresponding to μ_1. Let E_{μ_k} denote the eigenspace associated to μ_k. In fact, $\mu_k = \lambda_k(\lambda_k - c)$. Throughout this paper, we denote by $|\cdot|_p$ the $L^p(\Omega)$

norm, $c < \lambda_1$ in $\triangle^2 - c\triangle$ and the norm of u in $H^2(\Omega) \cap H_0^1(\Omega)$ will be defined by

$$\|u\| := \left(\int_\Omega \left(|\triangle u|^2 - c|\nabla u|^2 \right) dx \right)^{\frac{1}{2}}.$$

We also define $E = H^2(\Omega) \cap H_0^1(\Omega)$.

Theorem 1.1 *Let $N > 4$ and assume that f has the improved subcritical polynomial growth on Ω (condition* (SCPI)*) and satisfies* (H_1)-(H_4). *If $f_0 < \mu_1$, then problem* (1) *has at least two nontrivial solutions.*

Theorem 1.2 *Let $N > 4$ and assume that f has the improved subcritical polynomial growth on Ω (condition* (SCPI)*), is odd in t and satisfies* (H_3) *and* (H_4). *If $f(x,0) = 0$, then problem* (1) *has infinitely many nontrivial solutions.*

In the case of $N = 4$, we have $p^* = +\infty$. So it is necessary to introduce the definition of the subcritical (exponential) growth in this case. By the improved Adams inequality (see [12]) for the fourth-order derivative, namely,

$$\sup_{u \in E, \|u\| \leq 1} \int_\Omega e^{32\pi^2 u^2}\, dx \leq C|\Omega|.$$

So, we now define the subcritical (exponential) growth in this case as follows:

(SCE): f has subcritical (exponential) growth on Ω, *i.e.*, $\lim_{t\to\infty} \frac{|f(x,t)|}{\exp(\alpha t^2)} = 0$ uniformly on $x \in \Omega$ for all $\alpha > 0$.

When $N = 4$ and f has the subcritical (exponential) growth (SCE), our work is still to study problem (1) without the (AR) condition. Our results are as follows.

Theorem 1.3 *Let $N = 4$ and assume that f has the subcritical exponential growth on Ω (condition* (SCE)*) and satisfies* (H_1)-(H_4). *If $f_0 < \mu_1$, then problem* (1) *has at least two nontrivial solutions.*

Theorem 1.4 *Let $N = 4$ and assume that f has the subcritical exponential growth on Ω (condition* (SCE)*), is odd in t and satisfies* (H_3) *and* (H_4). *If $f(x,0) = 0$, then problem* (1) *has infinitely many nontrivial solutions.*

2 Preliminaries and auxiliary lemmas

Definition 2.1 Let $(E, \|\cdot\|_E)$ be a real Banach space with its dual space $(E^*, \|\cdot\|_{E^*})$ and $I \in C^1(E, \mathbb{R})$. For $c^* \in \mathbb{R}$, we say that I satisfies the $(PS)_{c^*}$ condition if for any sequence $\{x_n\} \subset E$ with

$$I(x_n) \to c^*, \qquad DI(x_n) \to 0 \quad \text{in } E^*,$$

there is a subsequence $\{x_{n_k}\}$ such that $\{x_{n_k}\}$ converges strongly in E. Also, we say that I satisfies the $(C)_{c^*}$ condition if for any sequence $\{x_n\} \subset E$ with

$$I(x_n) \to c^*, \qquad \left\|DI(x_n)\right\|_{E^*}\left(1 + \|x_n\|_E\right) \to 0,$$

there is a subsequence $\{x_{n_k}\}$ such that $\{x_{n_k}\}$ converges strongly in E.

We have the following version of the mountain pass theorem (see [13]).

Proposition 2.1 *Let E be a real Banach space and suppose that $I \in C^1(E,R)$ satisfies the condition*

$$\max\{I(0), I(u_1)\} \leq \alpha < \beta \leq \inf_{\|u\|=\rho} I(u)$$

for some $\alpha < \beta$, $\rho > 0$ and $u_1 \in E$ with $\|u_1\| > \rho$. Let $c^* \geq \beta$ be characterized by

$$c^* = \inf_{\gamma\in\Gamma} \max_{0\leq t\leq 1} I(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1],E), \gamma(0) = 0, \gamma(1) = u_1\}$ is the set of continuous paths joining 0 and u_1. Then there exists a sequence $\{u_n\} \subset E$ such that

$$I(u_n) \to c^* \geq \beta \quad \text{and} \quad (1 + \|u_n\|)\|I'(u_n)\|_{E^*} \to 0 \quad \text{as } n \to \infty.$$

Consider the following problem:

$$\begin{cases} \triangle^2 u + c\triangle u = f_+(x,u), & x \in \Omega, \\ u|_{\partial\Omega} = \triangle u|_{\partial\Omega} = 0, \end{cases}$$

where

$$f_+(x,t) = \begin{cases} f(x,t), & t > 0, \\ 0, & t \leq 0. \end{cases}$$

Define a functional $I_+ : E \to \mathbb{R}$ by

$$I_+(u) = \frac{1}{2}\int_\Omega \left(|\triangle u|^2 - c|\nabla u|^2\right) dx - \int_\Omega F_+(x,u)\,dx,$$

where $F_+(x,t) = \int_0^t f_+(x,s)\,ds$, then $I_+ \in C^1(E,\mathbb{R})$.

Lemma 2.1 *Let $N > 4$ and $\varphi_1 > 0$ be a μ_1-eigenfunction with $\|\varphi_1\| = 1$ and assume that* (H_2), (H_3) *and* (SCPI) *hold. If $f_0 < \mu_1$, then*:

(i) *There exist $\rho, \alpha > 0$ such that $I_+(u) \geq \alpha$ for all $u \in E$ with $\|u\| = \rho$.*
(ii) *$I_+(t\varphi_1) \to -\infty$ as $t \to +\infty$.*

Proof By (SCPI), (H_2) and (H_3), for any $\varepsilon > 0$, there exist $A_1 = A_1(\varepsilon)$, $B_1 = B_1(\varepsilon)$ and $l > 2\mu_1$ such that for all $(x,s) \in \Omega \times \mathbb{R}$,

$$F_+(x,s) \leq \frac{1}{2}(f_0 + \varepsilon)s^2 + A_1 s^{p^*}, \tag{4}$$

$$F_+(x,s) \geq \frac{1}{2}ls^2 - B_1. \tag{5}$$

Choose $\varepsilon > 0$ such that $(f_0 + \varepsilon) < \mu_1$. By (4), the Poincaré inequality and the Sobolev inequality $|u|_{p^*}^{p^*} \leq K\|u\|^{p^*}$, we get

$$I_+(u) \geq \frac{1}{2}\|u\|^2 - \frac{f_0 + \varepsilon}{2}|u|_2^2 - A_1|u|_{p^*}^{p^*} \geq \frac{1}{2}\left(1 - \frac{f_0 + \varepsilon}{\mu_1}\right)\|u\|^2 - A_1 K\|u\|^{p^*}.$$

So, part (i) is proved if we choose $\|u\| = \rho > 0$ small enough.

On the other hand, from (5) we have

$$I_+(t\varphi_1) \le \frac{1}{2}\left(1 - \frac{l}{\mu_1}\right)t^2 + B_1|\Omega| \to -\infty \quad \text{as } t \to -\infty.$$

Thus part (ii) is proved. □

Lemma 2.2 (see [12]) *Let $\Omega \subset \mathbb{R}^4$ be a bounded domain. Then there exists a constant $C > 0$ such that*

$$\sup_{u\in E, \|u\|\le 1} \int_\Omega e^{32\pi^2 u^2}\,dx \le C|\Omega|,$$

and this inequality is sharp.

Lemma 2.3 *Let $N = 4$ and $\varphi_1 > 0$ be a μ_1-eigenfunction with $\|\varphi_1\| = 1$ and assume that* (H_2), (H_3) *and* (SCE) *hold. If $f_0 < \mu_1$, then*:

(i) *There exist $\rho, \alpha > 0$ such that $I_+(u) \ge \alpha$ for all $u \in E$ with $\|u\| = \rho$.*
(ii) *$I_+(t\varphi_1) \to -\infty$ as $t \to +\infty$.*

Proof By (SCE), (H_2) and (H_3), for any $\varepsilon > 0$, there exist $A_1 = A_1(\varepsilon)$, $B_1 = B_1(\varepsilon)$, $\kappa > 0$, $q > 2$ and $l > 2\mu_1$ such that for all $(x, s) \in \Omega \times \mathbb{R}$,

$$F_+(x,s) \le \frac{1}{2}(f_0 + \varepsilon)s^2 + A_1 \exp\bigl(\kappa|s|^2\bigr)s^q, \tag{6}$$

$$F_+(x,s) \ge \frac{1}{2}ls^2 - B_1. \tag{7}$$

Choose $\varepsilon > 0$ such that $(f_0 + \varepsilon) < \mu_1$. By (6), the Holder inequality and Lemma 2.2, we get

$$\begin{aligned} I_+(u) &\ge \frac{1}{2}\|u\|^2 - \frac{f_0+\varepsilon}{2}|u|_2^2 - A_1\int_\Omega \exp\bigl(\kappa|u|^2\bigr)|u|^q\,dx \\ &\ge \frac{1}{2}\left(1 - \frac{f_0+\varepsilon}{\mu_1}\right)\|u\|^2 - A_1\left(\int_\Omega \exp\left(\kappa r\|u\|^2\left(\frac{|u|}{\|u\|}\right)^2\right)dx\right)^{\frac{1}{r}}\left(\int_\Omega |u|^{r'q}\,dx\right)^{\frac{1}{r'}} \\ &\ge \frac{1}{2}\left(1 - \frac{f_0+\varepsilon}{\mu_1}\right)\|u\|^2 - C\|u\|^q, \end{aligned}$$

where $r > 1$ is sufficiently close to 1, $\|u\| \le \sigma$ and $\kappa r\sigma^2 < 32\pi^2$. So, part (i) is proved if we choose $\|u\| = \rho > 0$ small enough.

On the other hand, from (7) we have

$$I_+(t\varphi_1) \le \frac{1}{2}\left(1 - \frac{l}{\mu_1}\right)|t|^2 + B_1|\Omega| \to -\infty \quad \text{as } t \to -\infty.$$

Thus part (ii) is proved. □

Lemma 2.4 *For the functional I defined by* (3), *if condition* (H_4) *holds, and for any $\{u_n\} \in E$ with*

$$\bigl\langle I'(u_n), u_n\bigr\rangle \to 0 \quad \textit{as } n \to \infty,$$

then there is a subsequence, still denoted by $\{u_n\}$, *such that*

$$I(tu_n) \le \frac{1+t^2}{2n} + I(u_n) \quad \textit{for all } t \in \mathbb{R} \textit{ and } n \in N.$$

Proof This lemma is essentially due to [14]. We omit it here. □

3 Proofs of the main results

Proof of Theorem 1.1 By Lemma 2.1 and Proposition 2.1, there exists a sequence $\{u_n\} \subset E$ such that

$$I_+(u_n) = \frac{1}{2}\|u_n\|^2 - \int_\Omega F_+(x, u_n)\,dx = c^* + o(1), \tag{8}$$

$$\bigl(1 + \|u_n\|\bigr)\bigl\|I'_+(u_n)\bigr\|_E \to 0 \quad \text{as } n \to \infty. \tag{9}$$

Clearly, (9) implies that

$$\bigl\langle I'_+(u_n), u_n\bigr\rangle = \|u_n\|^2 - \int_\Omega f_+\bigl(x, u_n(x)\bigr)u_n\,dx = o(1). \tag{10}$$

To complete our proof, we first need to verify that $\{u_n\}$ is bounded in E. Assume $\|u_n\| \to +\infty$ as $n \to \infty$. Let

$$s_n = \frac{2\sqrt{c^*}}{\|u_n\|}, \qquad w_n = s_n u_n = \frac{2\sqrt{c^*}u_n}{\|u_n\|}. \tag{11}$$

Since $\{w_n\}$ is bounded in E, it is possible to extract a subsequence (denoted also by $\{w_n\}$) such that

$$\begin{aligned} &w_n \rightharpoonup w_0 \quad \text{in } E,\\ &w_n^+ \to w_0^+ \quad \text{in } L^2(\Omega),\\ &w_n^+(x) \to w_0^+(x) \quad \text{a.e. } x \in \Omega,\\ &\bigl|w_n^+(x)\bigr| \le h(x) \quad \text{a.e. } x \in \Omega, \end{aligned}$$

where $w_n^+ = \max\{w_n, 0\}$, $w_0 \in E$ and $h \in L^2(\Omega)$.

We claim that if $\|u_n\| \to +\infty$ as $n \to +\infty$, then $w^+(x) \equiv 0$. In fact, we set $\Omega_1 = \{x \in \Omega : w^+ = 0\}$, $\Omega_2 = \{x \in \Omega : w^+ > 0\}$. Obviously, by (11), $u_n^+ \to +\infty$ a.e. in Ω_2, noticing condition (H_3), then for any given $K > 0$, we have

$$\lim_{n\to+\infty} \frac{f(x, u_n^+)}{u_n^+}\bigl(w_n^+(x)\bigr)^2 \ge Kw^+(x)^2 \quad \text{for a.e. } x \in \Omega_2. \tag{12}$$

From (10), (11) and (12), we obtain

$$\begin{aligned} 4c^* &= \lim_{n\to+\infty}\|w_n\|^2 = \lim_{n\to+\infty}\int_\Omega \frac{f(x, u_n^+)}{u_n^+}\bigl(w_n^+\bigr)^2\,dx\\ &\ge \int_{\Omega_2}\lim_{n\to+\infty}\frac{f(x, u_n^+)}{u_n^+}\bigl(w_n^+\bigr)^2\,dx \ge K\int_{\Omega_2}\bigl(w^+\bigr)^2\,dx. \end{aligned}$$

Noticing that $w^+ > 0$ in Ω_2 and $K > 0$ can be chosen large enough, so $|\Omega_2| = 0$ and $w^+ \equiv 0$ in Ω. However, if $w^+ \equiv 0$, then $\lim_{n\to+\infty} \int_\Omega F(x, w_n^+)\,dx = 0$ and consequently

$$I_+(w_n) = \frac{1}{2}\|w_n\|^2 + o(1) = 2c^* + o(1). \tag{13}$$

By $\|u_n\| \to +\infty$ as $n \to +\infty$ and in view of (11), we observe that $s_n \to 0$, then it follows from Lemma 2.4 and (8) that

$$I_+(w_n) = I_+(s_n u_n) \le \frac{1+s_n^2}{2n} + I_+(u_n) \to c^* > 0 \quad \text{as } n \to +\infty. \tag{14}$$

Clearly, (13) and (14) are contradictory. So $\{u_n\}$ is bounded in E.

Next, we prove that $\{u_n\}$ has a convergence subsequence. In fact, we can suppose that

$$\begin{aligned}
&u_n \rightharpoonup u \quad \text{in } E,\\
&u_n \to u \quad \text{in } L^q(\Omega), \forall 1 \le q < p^*,\\
&u_n(x) \to u(x) \quad \text{a.e. } x \in \Omega.
\end{aligned}$$

Now, since f has the improved subcritical growth on Ω, for every $\varepsilon > 0$, we can find a constant $C(\varepsilon) > 0$ such that

$$f_+(x, s) \le C(\varepsilon) + \varepsilon|s|^{p^*-1}, \quad \forall (x, s) \in \Omega \times \mathbb{R},$$

then

$$\begin{aligned}
&\left|\int_\Omega f_+(x, u_n)(u_n - u)\,dx\right|\\
&\quad\le C(\varepsilon)\int_\Omega |u_n - u|\,dx + \varepsilon\int_\Omega |u_n - u||u_n|^{p^*-1}\,dx\\
&\quad\le C(\varepsilon)\int_\Omega |u_n - u|\,dx + \varepsilon\left(\int_\Omega \left(|u_n|^{p^*-1}\right)^{\frac{p^*}{p^*-1}}\,dx\right)^{\frac{p^*-1}{p^*}}\left(\int_\Omega |u_n - u|^{p^*}\right)^{\frac{1}{p^*}}\\
&\quad\le C(\varepsilon)\int_\Omega |u_n - u|\,dx + \varepsilon C(\Omega).
\end{aligned}$$

Similarly, since $u_n \rightharpoonup u$ in E, $\int_\Omega |u_n - u|\,dx \to 0$. Since $\varepsilon > 0$ is arbitrary, we can conclude that

$$\int_\Omega \left(f_+(x, u_n) - f_+(x, u)\right)(u_n - u)\,dx \to 0 \quad \text{as } n \to \infty. \tag{15}$$

By (10), we have

$$\left\langle I_+'(u_n) - I_+'(u), (u_n - u)\right\rangle \to 0 \quad \text{as } n \to \infty. \tag{16}$$

From (15) and (16), we obtain

$$\int_\Omega \left[\left|\triangle(u_n - u)\right|^2 - c\left|\nabla(u_n - u)\right|^2\right]dx \to 0 \quad \text{as } n \to \infty.$$

So we have $u_n \to u$ in E which means that I_+ satisfies $(C)_{c^*}$. Thus, from the strong maximum principle, we obtain that the functional I_+ has a positive critical point u_1, *i.e.*, u_1

is a positive solution of problem (1). Similarly, we also obtain a negative solution u_2 for problem (1). □

Proof of Theorem 1.2 It follows from the assumptions that I is even. Obviously, $I \in C^1(E,\mathbb{R})$ and $I(0) = 0$. By the proof of Theorem 1.1, we easily prove that $I(u)$ satisfies condition $(C)_{c^*}$ ($c^* > 0$). Now, we can prove the theorem by using the symmetric mountain pass theorem in [15–17].

Step 1. We claim that condition (i) holds in Theorem 9.12 (see [16]). Let $V_1 = E_{\mu_1} \oplus E_{\mu_2} \oplus \cdots \oplus E_{\mu_k}$, $V_2 = E \setminus V_1$. For all $u \in V_2$, by (SCPI), we have

$$\begin{aligned} I(u) &= \frac{1}{2}\int_\Omega \left(|\triangle u|^2 - c|\nabla u|^2\right) dx - \int_\Omega F(x,u)\, dx \\ &\geq \frac{1}{2}\int_\Omega \left(|\triangle u|^2 - c|\nabla u|^2\right) dx - c_3 \int_\Omega |u|^{p^*}\, dx - c_4 \\ &\geq \|u\|^2 \left(\frac{1}{2} - c_5 \lambda_{k+1}^{-(1-a)p^*/2} \|u\|^{p^*-2}\right) - c_6, \end{aligned}$$

where $a \in (0,1)$ is defined by

$$\frac{1}{p^*} = a\left(\frac{1}{2} - \frac{1}{N}\right) + (1-a)\frac{1}{2}.$$

Choose $\rho = \rho(k) = \|u\|$ so that the coefficient of ρ^2 in the above formula is $\frac{1}{4}$. Therefore

$$I(u) \geq \frac{1}{4}\rho^2 - c_6 \tag{17}$$

for $u \in \partial B_\rho \cap V_2$. Since $\lambda_k \to \infty$ as $k \to \infty$, $\rho(k) \to \infty$ as $k \to \infty$. Choose k so that $\frac{1}{4}\rho^2 > 2c_6$. Consequently

$$I(u) \geq \frac{1}{8}\rho^2 \equiv \alpha. \tag{18}$$

Hence, our claim holds.

Step 2. We claim that condition (ii) holds in Theorem 9.12 (see [16]). By (H_3), there exists large enough M such that

$$F(x,t) \geq Mt^2 - c_7, \quad x \in \Omega, t \in \mathbb{R}.$$

So, for any $u \in E \setminus \{0\}$, we have

$$\begin{aligned} I(tu) &= \frac{1}{2}t^2 \int_\Omega \left(|\triangle u|^2 - c|\nabla u|^2\right) dx - \int_\Omega F(x,tu)\, dx \\ &\leq \frac{1}{2}t^2\|u\|^2 - Mt^2 \int_\Omega u^2\, dx + c_7|\Omega| \to -\infty \quad \text{as } t \to +\infty. \end{aligned}$$

Hence, for every finite dimension subspace $\tilde{E} \subset E$, there exists $R = R(\tilde{E})$ such that

$$I(u) \leq 0, \quad u \in \tilde{E} \setminus B_R(\tilde{E})$$

and our claim holds. □

Proof of Theorem 1.3 By Lemma 2.3, the geometry conditions of the mountain pass theorem (see Proposition 2.1) for the functional I_+ hold. So, we only need to verify condition $(C)_{c^*}$. Similar to the previous part of the proof of Theorem 1.1, we easily know that $(C)_{c^*}$ sequence $\{u_n\}$ is bounded in E. Next, we prove that $\{u_n\}$ has a convergence subsequence. Without loss of generality, suppose that

$$\begin{aligned}
&\|u_n\| \le \beta, \\
&u_n \rightharpoonup u \quad \text{in } E, \\
&u_n \to u \quad \text{in } L^q(\Omega), \forall q \ge 1, \\
&u_n(x) \to u(x) \quad \text{a.e. } x \in \Omega.
\end{aligned}$$

Now, since f_+ has the subcritical exponential growth (SCE) on Ω, we can find a constant $C_\beta > 0$ such that

$$\left|f_+(x,t)\right| \le C_\beta \exp\left(\frac{32\pi^2}{2\beta^2}|t|^2\right), \quad \forall (x,t) \in \Omega \times \mathbb{R}.$$

Thus, by the Adams-type inequality (see Lemma 2.2),

$$\begin{aligned}
&\left|\int_\Omega f_+(x,u_n)(u_n - u)\,dx\right| \\
&\quad \le C\left(\int_\Omega \exp\left(\frac{32\pi^2}{\beta^2}|u_n|^2\right)dx\right)^{\frac{1}{2}}|u_n - u|_2 \\
&\quad \le C\left(\int_\Omega \exp\left(\frac{32\pi^2}{\beta^2}\|u_n\|^2\left|\frac{u_n}{\|u_n\|}\right|^2\right)dx\right)^{\frac{1}{2}}|u_n - u|_2 \\
&\quad \le C|u_n - u|_2 \to 0.
\end{aligned}$$

Similar to the last proof of Theorem 1.1, we have $u_n \to u$ in E, which means that I_+ satisfies $(C)_{c^*}$. Thus, from the strong maximum principle, we obtain that the functional I_+ has a positive critical point u_1, *i.e.*, u_1 is a positive solution of problem (1). Similarly, we also obtain a negative solution u_2 for problem (1). □

Proof of Theorem 1.4 Combining the proof of Theorem 1.2 and Theorem 1.3, we easily prove it. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors read and approved the final manuscript.

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